Introduction aux séries D-finies

I. Séries univariées

Bruno Salvy

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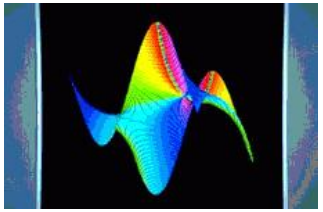

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Encyclopedia of Combinatorial Structures

Algorithms Project

INRIA

Search a combinatorial structure by:

First terms in the sequence 1,3,19,193 Submit

When searching by first terms in the sequence, the integers should be separated by commas, e.g., 1,2,3,5,8,13

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First terms in the sequence | 1,3,19,193 | Submit

Results of Search by First Terms in the Sequence

1,3,19,193

Found 1 combinatorial structures for 1,3,19,193

Structure 1	
Specification	[S, {C = Prod(B,Z), S = Set(C), B = Sequence(C)}, labelled]
First terms in the sequence	[1, 1, 3, 19, 193, 2721, 49171, 1084483, 28245729, 848456353, 28875761731, 1098127402131, 46150226651233, 2124008553358849, 106246577894593683, 5739439214861417731, 332993721039856822081, 20651350143685984386753, 1363322103204314826347779, 95453198574445723828731283, 7064900016612187878152462721]
Generating function	exp(1/2-1/2 *(1-4 *x)^(1/2))
Recurrence	{f(0) = 1, f(1) = 1, -f(n) +(-4 *n-2) *f(n +1) +f(n +2)}
Asymptotics of the coefficients	1/4/Pi^(1/2) *exp(1/2)/n^(3/2) *4^n
References	EIS A001517 (up to a possible factor of n!)
ECS number	131

Identities

$$F_n F_{n+2} - F_{n+1}^2 = (-1)^n$$ Cassini

$$\frac{1-\sqrt{1-4z}}{2z} = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{z^n}{n+1}$$ Catalan numbers

$${}_2F_1\left(\left. \begin{matrix} a, b \\ a+b+1/2 \end{matrix} \right| z \right) = {}_2F_1\left(\left. \begin{matrix} 2a, 2b \\ a+b+1/2 \end{matrix} \right| \frac{1-\sqrt{1-z}}{2} \right)$$ Legendre

$$\sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{u^n}{n!} = \frac{\exp\left(\frac{4u(xy-u(x^2+y^2))}{1-4u^2} \right)}{\sqrt{1-4u^2}}$$ Mehler

I. Definition & Closure Properties

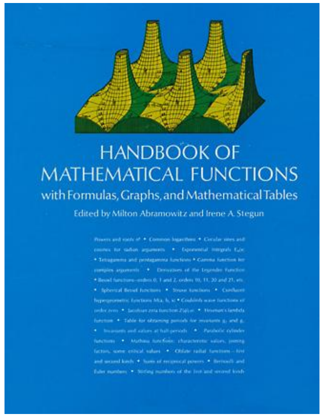
D-finite Series

Def. A power series $y(z)$ is differentially finite (D-finite) if there exist polynomials $a_i \in \mathbb{Q}[z]$ such that

$$a_k(z)y^{(k)}(z) + \cdots + a_0(z)y(z) = 0. \tag{E}$$

Equivalent definition: $y(z), y'(z), y''(z), \ldots$ span a finite-dimensional vector space over $\mathbb{Q}(z)$.

Examples of functions: exp, log, sin, cos, sinh, cosh, arccos, arccosh, arcsin, arcsinh, arctan, arctanh, arccot, arccoth, arccsc, arccsch, arcsec, arcsech, ${}_pF_q$ (includes Bessel J, Y, I and K, Airy Ai and Bi and polylogarithms), Struve, Weber and Anger fcns, the large class of algebraic functions,…



About 60% of Abramowitz & Stegun.

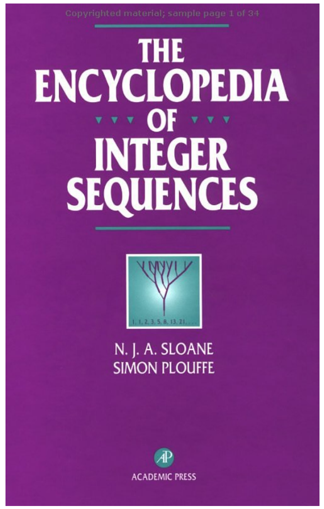
Coefficients $\leftrightarrow$ Series

Thm. A series is D-finite if and only if its sequence of coefficients satisfies a linear recurrence.

Proof (idea). $zD_z \leftrightarrow n$, $z^{-1} \leftrightarrow S_n$.

Cor 1. N first coefficients in complexity $O(N)$.

Cor 2. [ChCh90] Nth coefficient in complexity $O_{\log}(N^{1/2})$.



Examples of sequences: rational sequences, hypergeometric sequences (includes $n!$, multinomials,...), classical orthogonal polynomials.

About 25% of Sloane & Plouffe.

Examples

Ex. 1. From an algorithm in number theory

kth coefficient of $P(x)^n$ (k and n large)

Computation (given n and k):

(i) 1st order LDE;

(ii) linear recurrence of order $\deg P$ satisfied by coefficients;

(iii) fast evaluation.

Ex. 2. Related to a complexity analysis of Gröbner bases

Computation of $P_n(x)$ defined by

$$\sum_{n\geq 0} P_n(x)\frac{z^n}{n!} = \left(\frac{1+z}{1+z^2}\right)^x.$$

Computation (given n and x): Same as above!

(iii) fast evaluation (without computing the polynomials).

Generalized Hypergeometric Series

$$y(z) := {}_pF_q\left(\left.\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\right| z\right) = \sum_{n=0}^{\infty} \underbrace{\frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n n!}}_{u_n} z^n,$$

$$(x)_n := x(x+1)\cdots(x+n-1).$$

$\Leftrightarrow$ first order linear recurrences (hypergeometric sequences)

$$\frac{u_n}{u_{n-1}} = \frac{(n+a_1-1)\cdots(n+a_p-1)}{n(n+b_1-1)\cdots(n+b_q-1)} \Longrightarrow$$

$$((\theta+a_1-1)\cdots(\theta+a_p-1)z - \theta(\theta+b_1-1)\cdots(\theta+b_q-1))\, y(z) = 0,$$

$$(\theta = z\frac{d}{dz}).$$

Special cases: exp, log, polylogs, Bessel J, Y, I, K, Airy Ai & Bi,...

D-finite Series in Arithmetic

Def. For $k \in \mathbb{Z}$, modular form of weight k: f defined on $\Im z > 0$ such that $f((az+b)/(cz+d)) = (cz+d)^k f(z)$, for all matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $SL(2,\mathbb{Z})$ or one of its subgroups of finite index.

Def. Modular function: modular form of weight 0.

Thm. [19th century] Let $f(z)$ be a meromorphic modular form of weight $k > 0$ and $t(z)$ a modular function. Then the (many-valued) function $F(t)$ defined by $F(t(z)) = f(z)$ satisfies a linear differential equation of order $k+1$ with algebraic coefficients.

Ex. [Apéry] $t(z) = \left(\dfrac{\eta(z)\eta(6z)}{\eta(2z)\eta(3z)}\right)^{12}, \quad f(z) = \dfrac{(\eta(2z)\eta(3z))^7}{(\eta(z)\eta(6z))^5},$

$$F(t) = 1 + 5t + 73t^2 + \cdots = \sum_{n \geq 0} \sum_k \binom{n}{k}^2 \binom{n+k}{k}^2 t^n.$$

Closure Properties

Thm. 1 The set of D-finite series in $\mathbb{Q}[[z]]$ forms a sub-algebra of $\mathbb{Q}[[z]]$.

Proof (idea): linear algebra!

Thm. 2 The set of D-finite series is closed under Hadamard product.

Cor. The set of D-finite series is closed under formal Laplace & Borel transforms (ogf $\leftrightarrow$ egf).

Mehler Identity for Hermite Polynomials

```
> L:=[seq(orthopoly[H](n,x),n=0..5)];
```

$$L := [1, 2x, 4x^2 - 2, 8x^3 - 12x, 16x^4 - 48x^2 + 12, 32x^5 - 160x^3 + 120x]$$

```
> gfun[listtodiffeq](L,u(z));
```

$$[\{u(0) = 1, \frac{d}{dz}u(z) + (2z - 2x)u(z)\}, \mathrm{egf}]$$

```
> deq:=%[1]:hadamardproduct(deq,subs(x=y,deq),u(z)):
> Laplace(%,u(z),'diffeq');
```

$$\{u(0) = 1, (16z^4 - 8z^2 + 1)\frac{d}{dz}u(z) + (16z^3 - 16z^2xy + 8zy^2 - 4z + 8zx^2 - 4xy)u(z)\}$$

```
> dsolve(%,u(z));
```

$$u(z) = \frac{\exp\left(\frac{4z(xy - z(x^2 + y^2))}{1 - 4z^2}\right)}{\sqrt{1 - 4z^2}}$$

Algebraic Series

Thm. 1 [Tannery1874] Algebraic power series are D-finite.

Proof (idea). Bézout identity!

Thm. 2 If f is D-finite and g is algebraic, $f \circ g$ is D-finite.

Proof Same.

Motzkin Numbers (Unary-Binary Trees)

$$M(z) - \left(1 + zM(z) + zM(z)^2\right) =: P(M) = 0.$$

$$\text{Bézout:} \quad AP + BP_M = 1 \Rightarrow M' = -BP_z \bmod P =: cM + d1.$$

Vector space of dimension 2

```
> gfun[algeqtodiffeq](M=1+z*M+z*M^2,M(z));
```

$$-1 - z + (-3\,z + 1)\,M\,(z) + \left(z - 6z^2 + z^3\right)\frac{d}{dz}M\,(z)$$

```
> gfun[diffeqtorec](%,M(z),u(n));
```

$$\{nu(n) + (-9 - 6n)u(1 + n) + (3 + n)u(n + 2), u(0) = 1, u(1) = 2\}$$

$\rightarrow$ fast computation.

Works for arbitrary degree.

Forests of Catalan Trees

$$Y(z) = \exp\left(\frac{1-\sqrt{1-4z}}{2}\right).$$

Same computation $\rightarrow$

$$\{(n+2)(n+1)u(n+2) = u(n) + 2(n+1)(2n+1)u(n+1),$$
$$u\,(0) = 1, u\,(1) = 1\}$$

An Example in Asymptotics

$$\mathrm{Ai}(z) = \frac{\sqrt{z}e^{-\xi}}{\pi} \int_{-\infty}^{\infty} e^{-\xi[(u-1)(4u^2+4u+1)]}\, dv, \quad \xi = \frac{2}{3} z\sqrt{z}, \quad u = \sqrt{1 + \frac{1}{3}v^2}.$$

change variable $t^2 = (u-1)(4u^2+4u+1)$:

$$\mathrm{Ai}(z) = \frac{\sqrt{z}e^{-\xi}}{\pi} \int_{-\infty}^{\infty} e^{-\xi t^2} f(t)\, dt, \quad f(t) = \frac{dv}{dt},$$

$$\sim \frac{1}{2}\pi^{-1/2} z^{-1/4} e^{-\xi} \sum_{n=0}^{\infty} (-1)^n \xi^{-n} \frac{2^n}{3^{2n}(2n)!} \frac{\Gamma(3n+1/2)}{\Gamma(1/2)}.$$

Computation: *(i)* algebraic change of variable; *(ii)* recurrence satisfied by coefficients; *(iii)* termwise integration.

No More Closure Properties

Thm. 1 [Harris & Sibuya 85] Both f and $1/f$ are D-finite if and only if f'/f is algebraic.

Thm. 2 [Singer 86] Both f and $\exp \int f$ are D-finite if and only if f is algebraic.

Thm. 3 [Singer 86] Let g be algebraic of genus ≥ 1. Both f and $g \circ f$ are D-finite if and only if f is algebraic.

II. Interlude: Algorithms

Petkovšek's Algorithm Hyper

Input

$$a_0(n)u_{n+k} + \cdots + a_k(n)u_n = b(n), \tag{R}$$

a_i polynomials, $b = 0$ or hypergeometric or linear combination of hypergeometric sequences.

Output All solutions that are linear combinations of hypergeometric sequences, or a proof that none exists.

Example Motzkin numbers are not hypergeometric.

Derangements

$$\frac{e^{-z}}{1-z} \to \text{LDE} \to u_n = (n-1)u_{n-1} + (n-1)u_{n-2}$$

1. Hyper $\to n!$
2. Reduction of order: $u_n =: n! \sum^{n-1} v_k, \quad (n+2)v_{n+1} + v_n = 0.$
3. Conclusion: $u_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$, and it is *not* hypergeometric.

Thm. [Petkovšek92] If (R) has a solution of the form

$$h_0(n) \sum_{k_1=s_1}^{n-1} h_1(k_1) \sum_{k_2=s_2}^{k_1-1} h_2(k_2) \cdots \sum_{k_m=s_m}^{k_{m-1}-1} h_m(k_m),$$

then it is found by applying Hyper and reduction of order.

Other Algorithms

Symbolic solutions: [Kovacic, Singer, Bronstein, Ulmer, Weil] Partly implemented in most CAs.

Fast numerical evaluation: [ChCh90,vdH98] We have a prototype.

Local and asymptotic expansions: [Tournier87, van Hoeij97] Implemented in most CAs. (Maple `DEtools[formal_sol]`)

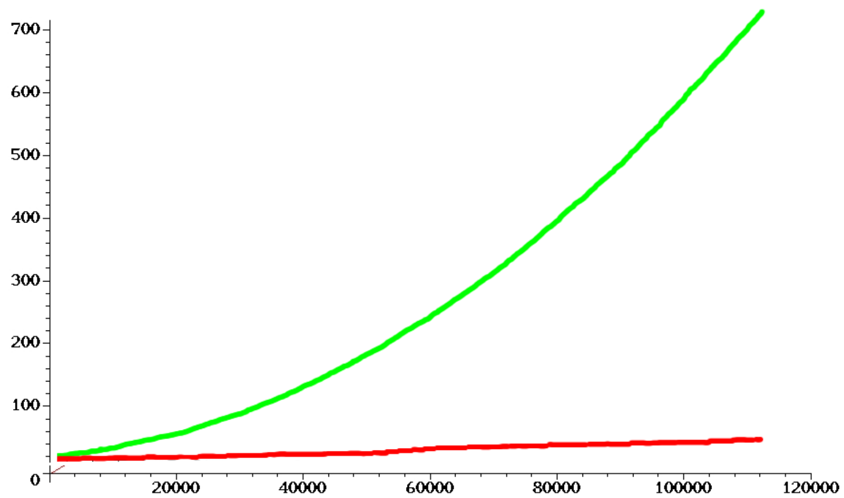
Guessing by Padé-Hermite approximants. Implemented in `gfun`.

Multiple integrals and sums: [Zeilberger90-91,ChSa98,Chyzak94-00] Implemented in `Mgfun`.

Binary Splitting

[Chudnovsky-Chudnovsky87] Hypergeometric series

$$\frac{1}{\pi} = \frac{1}{53360\sqrt{640320}} \sum_{n \geq 0} \frac{(-1)^n (6n)!(13591409 + 545140134n)}{(n!^3(3n)!(8 \cdot 100100025 \cdot 327843840)^n)}.$$



Key idea Sort computation sequence so as to take advantage of fast multiplication algorithms (operands of same sizes).

III. Analytic Behaviour

Most properties will be stated for analytic rather than polynomial coefficients.

Motivation: Singularity Analysis of Coefficients

Singularity of smallest modulus $\rightarrow$ exponential growth

Local behaviour $\rightarrow$ sub-exponential terms

Thm. [FlOd84] Under mild conditions (granted for isolated singularities)

$$f(z) = g(z) + O(h(z)), \quad z \to \rho \Rightarrow [z^n]f(z) = [z^n]g(z) + O([z^n]h(z)), \quad n \to \infty$$

A simple asymptotic scale translates

$$[z^n](1 - \frac{z}{\rho})^\alpha \log^\beta \frac{1}{1 - z/\rho} \sim \rho^{-n} \frac{n^{-\alpha-1}}{\Gamma(-\alpha)} \log^\beta n, \quad \alpha \notin \mathbb{N}.$$

Method: (i.) Find dominant singularity; (ii.) Expand locally; (iii.) Translate.

Connection problem

Location of Singularities

$$Ly := a_0(z)y^{(n)}(z) + \cdots + a_n(z)y(z) = 0 \Leftrightarrow$$

$$(E) \quad Y'(z) = A(z)Y(z), \qquad Y = \begin{pmatrix} y \\ \vdots \\ y^{(n-1)} \end{pmatrix}, \quad A = \begin{pmatrix} 1 & & 0 \\ 0 & 1 & \\ -\frac{a_n}{a_0} & \cdots & -\frac{a_1}{a_0} \end{pmatrix}.$$

Thm. [Cauchy] If $A(z)$ is analytic in a simply connected domain $R \subset \mathbb{C}$, then for any $a \in R$ and $\alpha \in \mathbb{C}^n$, (E) has a unique solution analytic in R such that $Y(a) = \alpha$.

Cor. y solution of $Ly = 0$, ρ singularity of y implies $a_0(\rho) = 0$.

Cor. If a_0 is a polynomial, the singularities are *isolated* ($\rightarrow$ singularity analysis).

Def. Singularity at ∞: $y(1/z)$ singular at 0.

Indicial Polynomial

Motivation: a polynomial whose roots are the possible (algebraic) valuations of formal power series solutions $(z-\rho)^\sigma \sum_{n\geq 0} c_n^{(\sigma)}(z-\rho)^n$.

$$\mathcal{L}y := a_k(z)y^{(k)}(z) + \cdots + a_0(z)y(z) = 0. \tag{E}$$

Def. $a_k^{-1}(z)(z-\rho)^{k-\sigma}\mathcal{L}(z-\rho)^\sigma = f(\sigma) + \cdots \in \mathbb{Q}[\sigma][[z-\rho]]$.

Prop. If the coefficients a_i's are polynomials, $\{c_n^{(0)}\}_{n\geq 0}$ satisfies

$$b_p(n)c_{n+p}^{(0)} + \cdots + b_0(n)c_n^{(0)} = 0,$$

where $f(\sigma) = b_p(\sigma - p)$ up to a constant multiple.

Matricial viewpoint: If $Y'(z) = \frac{B(z)}{z-\rho}Y(z)$, the indicial polynomial is the characteristic polynomial of $B(\rho)$.

Regular and Irregular Singular Points

Def. A singular point ρ of (E) is called regular when the indicial polynomial $f(\sigma)$ at ρ has degree k, it is called irregular otherwise.

At a singular point ρ, (E) admits a basis of formal solutions $(i = 1, \ldots, k)$:

$$\rho \text{ regular}: \quad \Psi_i(z) = (z-\rho)^{\sigma_i} \sum_{j=0}^{d_i} \log^j(z-\rho) \underbrace{\Phi_{i,j}(z-\rho),}_{\text{convergent p. s.}} \qquad f(\sigma_i) = 0.$$

$$\rho \text{ irregular}: \quad y_i(t) = \exp\Big(\underbrace{P_i(1/t)}_{\text{polynomial}} \Big) \underbrace{\Psi_i(t)}_{\text{as above}}, \qquad \underbrace{t^{\mu_i}}_{\mu_i \in \mathbb{N}^*} = (z-\rho).$$

Power series generally divergent in the irregular case.

Algorithms for everything. Maple `DEtools[formal_sol]`.

Quicksort with Median-of-3

$$C_n = \underbrace{n+1}_{\#\text{ comparisons}} + 2\sum_{k=1}^{n} \frac{(k-1)(n-k)}{\binom{n}{3}} C_{k-1}.$$

Differential equation: $\frac{1}{24}(1-z)^2 C^{(3)}(z) - \frac{1}{2}C'(z) = \frac{1}{(1-z)^3}$.

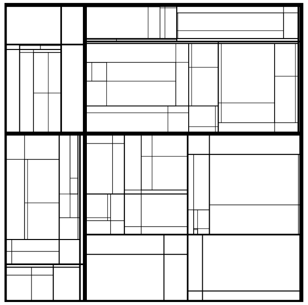
Indicial Polynomial: $\frac{\sigma(\sigma-1)(\sigma-2)}{24} - \frac{\sigma}{2}$, roots $-2, 0, 5$.

Homogeneous part: $C_1 \sim (1-z)^{-2}, C_2 \sim 1, C_3 \sim (1-z)^5$, $z \to 1$.

Inhomogeneous part: $C = -\frac{12}{7}\frac{\ln(1-z)}{(1-z)^2} + O((1-z)^{-2})$.

Singularity analysis: $C_n = \frac{12}{7} n \log n + O(n)$.

Pathlength in Quadtrees



Filling rate of the pages?

Model: points distributed uniformly at random in $(0,1)^d$.

Translation in generating series:

$$\left(z(1-z)\frac{d}{dz}\right)^d (f(z) - \frac{1}{(1-z)^2}) - 2^d f(z) = 0.$$

Local behaviour at $z = 1$:

$$f = \frac{2}{d}\frac{1}{(1-z)^2}\log\frac{1}{1-z} + \frac{c}{(1-z)^2} + \cdots$$

Translation: $f_n = \frac{2}{d} n \log n + \mu_d n + O(\log n + n^{-1+2\cos(2\pi/d)})$.

[FlGoPuRo91,FlLaLaSa95]

$\zeta(3)$ is irrational [Apéry78]

$$a_n := \sum_k \binom{n}{k}^2 \binom{n+k}{k}^2, \quad b_n := a_n \sum_{k=1}^{n} \frac{1}{k^3} + \sum_{k=1}^{n} \sum_{m=1}^{k} \frac{(-1)^{m+1} \binom{n}{k}^2 \binom{n+k}{k}^2}{2m^3 \binom{n}{m} \binom{n+m}{m}}.$$

1. $m^3 \binom{n}{m} \binom{n+m}{m} \geq n^2 \Rightarrow \lim_{n\to\infty} \frac{b_n}{a_n} = \zeta(3).$

2. $a_n \in \mathbb{N}^\star$, $d_n^3 b_n \in \mathbb{Z}$, where $d_n := \mathrm{lcm}(1, \ldots, n)$:

$$\frac{()^2 ()^2}{2m^3 () ()} = \frac{\binom{n}{k} \binom{n+k}{k} \binom{n-m}{n-k} \binom{n+k}{k-m}}{2m^3 \binom{k}{m}^2} \quad \text{and} \quad m \binom{k}{m} \Big| \, d_k.$$

3. Both a_n and b_n satisfy [creative telescoping]

$$(n+1)^3 u_{n+1} = (34n^3 + 51n^2 + 27n + 5) u_n - n^3 u_{n-1}, \quad n \geq 1.$$

4. [Singularity analysis] $a_n\zeta(3) - b_n \sim C\dfrac{\alpha_\pm^n}{n^{3/2}}$, $\alpha_\pm = 17 \pm 12\sqrt{2}$ and $\lim = 0 \Rightarrow \alpha_-$

5. $0 < \zeta(3) - \dfrac{b_n}{a_n} = \displaystyle\sum_{k\geq n+1} \dfrac{b_k}{a_k} - \dfrac{b_{k-1}}{a_{k-1}}$: $b_k a_{k-1} - b_{k-1}a_k = \dfrac{6}{k^3}$ [closure]

6. Conclusion: $0 < \underbrace{a_n d_n^3}_{\in\mathbb{N}}\zeta(3) - \underbrace{d_n^3 b_n}_{\in\mathbb{N}} \simeq C\alpha_-^n e^{3n} \to 0.$

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