Introduction aux séries D-finies

II. Séries multivariées

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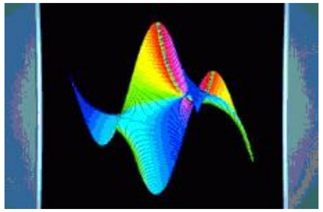

Univariate D-finite Series: Summary

1. Linear differential equations $\leftrightarrow$ Linear recurrence equations
2. Finite dimensional vector space $\rightarrow$ closure properties
3. Hypergeometric closed forms
4. Asymptotics

Multivariate:

2. generalizes
5. creative telescoping

Automatically Generated Encyclopedia of Special Functions

This site deals with special functions. Special functions include the functions *cos*, *sin*, *exp*, *log*, *Bessel*, *Airy*, ... These functions were first studied by great mathematicians (such as *Gauss*, *Euler* and *Riemann*) and have been referenced in many books (such as the *Handbook of Mathematical Functions*, by *M.Abramowitz* and *I.A.Stegun*). Usually, formulae on special functions are computed by hand.

The progress made in symbolic computation over the last 20 years makes possible to automate completely the derivation of many of the results and formulae on special functions. This opens the way to developing an Automatically Generated Encyclopedia of Special Functions.

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Multivariate Objects & Questions

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \sum_{j=0}^{k} \binom{k}{j}^3,$$

$$\sum_{n=0}^{+\infty} P_n(x) y^n = \frac{1}{\sqrt{1-2xy+y^2}}, \quad P_n(x) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k}^2 x^k$$

$$\int_0^{+\infty} x J_1(ax) I_1(ax) Y_0(x) K_0(x)\, dx = -\frac{\ln(1-a^4)}{2\pi a^2},$$

$$\oint_0 \frac{(1+2xy+4y^2)\exp\left(\frac{4x^2y^2}{1+4y^2}\right)}{y^{n+1}(1+4y^2)^{\frac{3}{2}}}\, dy = \frac{n! H_n(x)}{\lfloor n/2 \rfloor !},$$

$$\sum_{k=0}^{n} \frac{q^{k^2}}{(q;q)_k (q;q)_{n-k}} = \sum_{k=-n}^{n} \frac{(-1)^k q^{(5k^2-k)/2}}{(q;q)_{n-k}(q;q)_{n+k}}$$

$$\sum_{j=0}^{n} \sum_{i=0}^{n-j} \frac{q^{(i+j)^2+j^2}}{(q;q)_{n-i-j}(q;q)_i(q;q)_j} = \sum_{k=-n}^{n} \frac{(-1)^k q^{7/2k^2+1/2k}}{(q;q)_{n+k}(q;q)_{n-k}}.$$

I. Univariate Ore Polynomials

Operators & Commutation Rules

Notation: $\cdot \leftrightarrow$ application

Operator	Leibniz Rule	Commutation
Differential	$D_x \cdot (fg(x)) = f'(x)g(x) + f(x)D_x \cdot g(x)$	$D_x f = f' + fD_x$
Shift	$S_n \cdot (f_n g_n) = f_{n+1} S_n \cdot f_n$	$S_n f = f_{n+1} S_n$
Difference	$\Delta_n \cdot (f_n g_n) = f_{n+1}(\mathrm{Id} + \Delta_n) \cdot g_n$	$\Delta_n f = f_{n+1} + f_{n+1}\Delta_n$
q-shift	$Q_x \cdot (fg(x)) = f(qx)Q_x \cdot g(x)$	$Q_x f = f(qx)Q_x$
Mahler	$M_k \cdot (fg(x)) = f(x^k)M_k \cdot g(x)$	$M_k f = f(x^k)M_k$

Common pattern: $\partial f = \delta(f)\,\mathrm{Id} + \sigma(f)\partial$

Natural condition: $\partial(fg) = (\partial f)g$

$$\delta(fg)\,\mathrm{Id} + \sigma(fg)\partial = (\delta(f)\,\mathrm{Id} + \sigma(f)\partial)g = (\delta(f)g + \sigma(f)\delta(g))\,\mathrm{Id} + \sigma(f)\sigma(g)\partial.$$

Skew Polynomial Rings

Def. Skew polynomial ring $\mathbb{A}[\partial;\sigma,\delta]$: set of polynomials in ∂ with coefficients in the (non-commutative) integral domain $\mathbb{A}$, with commutation rule $\partial f = \delta(f) + \sigma(f)\partial$, where σ is a ring endomorphism of $\mathbb{A}$ and δ is a σ-derivation.

Def. f is D-finite if there exists $P \in \mathbb{A}[\partial;\sigma,\delta]$ such that $P \cdot f = 0$.

Thm.[Ore33] Euclidean division and extended Euclidean algorithm yield greatest common right divisor (GCRD), least common left multiple (LCLM), Bézout identity.

Example: Contiguity of Hypergeometric Series

$$F(a,b;c;z) = {}_2F_1\left(\begin{matrix} a,b \\ c \end{matrix} \middle| z\right) = \sum_{n=0}^{\infty} \underbrace{\frac{(a)_n (b)_n}{(c)_n n!}}_{u_{a,n}} z^n, \qquad (x)_n := x(x+1)\cdots(x+n-1).$$

$$\frac{u_{a,n+1}}{u_{a,n}} = \frac{(a+n)(b+n)}{(c+n)(n+1)} \quad \rightarrow \quad z(1-z)F'' + (c-(a+b+a)z)F' - abF = 0,$$

$$\frac{u_{a+1,n}}{u_{a,n}} = \frac{n}{a} + 1 \quad \rightarrow \quad S_a F := F(a+1,b;c;z) = \frac{z}{a}F' + F.$$

dim=2 $\Rightarrow S_a^2 F, S_a F, F$ linearly dependent [Gauss].

Also:

- S_a^{-1} in terms of Id, D_z via Bézout;
- relation between any three polynomials in S_a, S_b, S_c;
- generalizes to any ${}_pF_q$.

Closure Properties

Prop. Closure under $+$.

Proof. LCLM.

Prop. Closure under $\times$: if there exists polynomials A and B such that $\sigma = A(\partial)$ and $\delta = B(\partial)$, with $((A-1)\cdot w)B = (B \cdot w)(A-1)$, for all $w \in \mathbb{A}$.

Proof. Linear algebra.

II. Multivariate D-Finiteness

Ore Algebras & Their Ideals

Prop. $\mathbb{A}[\partial;\sigma,\delta]$ is an integral domain.

Def. $\mathbb{K}$ a field, $\mathbb{O}=\mathbb{K}[\partial_1;\sigma_1,\delta_1]\cdots[\partial_n;\sigma_n,\delta_n]$, such that $\partial_i\partial_j=\partial_j\partial_i$, for all i,j is a Ore algebra.

Def. Left ideal: $\mathcal{I}\subset\mathbb{O}$ such that $\mathcal{I}+\mathcal{I}=\mathbb{O}\mathcal{I}=\mathcal{I}$.

Def. A left ideal $\mathcal{I}\subset\mathbb{O}$ is D-finite if the quotient $\mathbb{O}/\mathcal{I}$ is a *finite-dimensional* vector space over $\mathbb{K}$.

Def. f is D-finite wrt $\mathbb{O}$ if its annihilating ideal in $\mathbb{O}$.
Then $\mathbb{O}/\operatorname{Ann} f\simeq\mathbb{O}\cdot f$.

Algorithms based on noncommutative Gröbner bases [ChSa98]

Example: Jacobi Polynomials

$$\mathbb{O} = \mathbb{Q}(n, x, a, b)[S_n; S_n, 0][D_x; 1, D_x]$$

Annihilating ideal $\operatorname{Ann} P$ generated by

$$G_1 = p_{11}(n, a, b)S_n^2 + p_{12}(n, x, a, b)S_n + p_{13}(n, a, b)\mathrm{Id},$$
$$G_2 = p_{21}(n, x, a, b)S_n D_x + p_{22}(n, x, a, b)S_n + p_{23}(n, a, b)\mathrm{Id}, \qquad (p_{ij} \text{ polynomials})$$
$$G_3 = p_{31}(x, a, b)D_x^2 + p_{32}(x, a, b)D_x + p_{33}(n, a, b)\mathrm{Id}.$$

$$\Rightarrow \operatorname{Ann} P \text{ D-finite: } \dim \mathbb{O}/\operatorname{Ann} P = 3.$$

$$G_4 = p_{41}(n, x, a, b)S_a + p_{42}(n)S_n + p_{43}(n, a)\mathrm{Id},$$
$$G_5 = p_{51}(n, x, a, b)S_b + p_{52}(n)S_n + p_{53}(n, b)\mathrm{Id}.$$

In $\mathbb{O}' = \mathbb{O}[S_a; S_a, 0][S_b; S_b; 0]$, $\dim \mathbb{O}'/\operatorname{Ann} P = 3$ also.

Properties [ChSa98]

Prop. 1 Closure under $+$.

Prop. 2 Closure under $\times$: if $\sigma_i = A_i(\partial_i)$ and $\delta_i = B_i(\partial_i)$, with $((A_i - 1) \cdot w)B_i = (B_i \cdot w)(A_i - 1)$, for all $w \in \mathbb{K}$ and all i.

Prop. 3 f D-finite wrt $\mathbb{O}$, then $P \cdot f$ D-finite for any $P \in \mathbb{O}$.

Prop. 4 f D-finite wrt $\mathbb{O}$, then for any $P \in \mathbb{O}$, f satisfies an equation $\sum_{i=0}^{k} a_i P^i \cdot f = 0$, with $k \leq \dim \mathbb{O}/\operatorname{Ann} f$.

Prop. 5 $f(\mathbf{x}, \mathbf{y})$ D-finite wrt $\mathbb{K}(\mathbf{x}, \mathbf{y})[\boldsymbol{\partial}_\mathbf{x}; \boldsymbol{\sigma}_\mathbf{x}, \boldsymbol{\delta}_\mathbf{x}][\boldsymbol{\partial}_\mathbf{y}; \boldsymbol{\sigma}_\mathbf{y}, \boldsymbol{\delta}_\mathbf{y}]$, then for any $\mathbf{a} \in \mathbb{K}^m$, the specialization $f(\mathbf{x}, \mathbf{a})$ is D-finite wrt $\mathbb{K}(\mathbf{x})[\boldsymbol{\partial}_\mathbf{x}; \boldsymbol{\sigma}_\mathbf{x}, \boldsymbol{\delta}_\mathbf{x}]$.

Proof $\mathbb{O} \cdot f \oplus \mathbb{O} \cdot g$ of finite dimension.

Proof $\mathbb{O} \cdot (P \cdot f) \subset \mathbb{O} \cdot f$ which is finite-dimensional.

Proof Use Prop. 4 for each of the $\partial_{\mathbf{x}}$ and specialize the coefficients.

III. Creative Telescoping

Principle

$$F_n = \sum_k u_{n,k} = ?$$

If one knows $A(n, S_n)$ and $B(n, k, S_n, S_k)$ such that

$$(A(n, S_n) + \Delta_k B(n, k, S_n, S_k)) \cdot u_{n,k} = 0,$$

then the sum "telescopes", leading to

$$A(n, S_n) \cdot F_n = 0.$$

Example [Apéry] $F_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$ satisfies

$$(n+2)^3 F_{n+2} - \left((n+2)^3 + (n+1)^3 + 4(2n+3)^3\right) F_{n+1} + (n+1)^3 F_n = 0.$$

"Neither Cohen nor I had been able to prove [this] in the intervening two months." [Van der Poorten]

Ideally

Aim: Find annihilators of

$$I(x_1, \dots, x_{n-1}) = \partial_n^{-1}\big|_\Omega f(x_1, \dots, x_n)$$

knowing generators of Ann_f in
$\mathbb{O}_n = \mathbb{K}(x_1, \dots, x_n)[\partial_1; \sigma_1, \delta_1] \cdots [\partial_n; \sigma_n, \delta_n]$.

Crucial step: compute $(\mathbb{O}_n \,\mathrm{Ann}_f + \partial_n \mathbb{O}_n) \cap \mathbb{O}_{n-1}$ (open problem).

Idea 1: find $P \in \mathrm{Ann}_f \cap \mathbb{K}(x_1, \dots, x_{n-1})[\partial_1; \sigma_1, \delta_1] \dots [\partial_n; \sigma_n, \delta_n]$, then rewrite $P = Q\partial_n + R$ and return R. Generalization of Zeilberger's "slow" algorithm by GB [ChSa98,Chyzak98].

Idea 2: proceed by increasing orders in $\mathbb{O}_{n-1}$: generalization of Zeilberger's "fast" algorithm [Chyzak00].

Generating Series of the Jacobi Polynomials

$$\mathbb{O} = \mathbb{Q}(n,x,a,b,z)[S_n;S_n,0][D_x;1,D_x][S_a;S_a,0][S_b;S_b,0][D_z;1,D_z]$$

1. $\operatorname{Ann} P_n^{(a,b)}(x)$ generated by $(G_1,\dots,G_5,D_z)$ (dim 3);
2. $\operatorname{Ann} z^n$ generated by $(S_n - z, D_x, S_a - 1, S_b - 1, zD_z - n)$ (dim 1);
3. Closure by product (dim 3);
4. Creative telescoping:

$$q_{11}1 + q_{12}D_z + q_{13}D_x, \quad q_{21}1 + q_{22}D_z + q_{23}S_b,$$
$$q_{31}1 + q_{32}D_z + q_{33}S_a, \quad q_{41}1 + q_{42}D_z + q_{43}D_z^2 \qquad (\dim 2).$$

5. (optional) Resolution:

$$\sum_{n\geq 0} P_n^{(a,b)}(x)z^n = \frac{2^{a+b}}{R(x,z)\,(1-z+R(x,z))^a\,(1+z+R(x,z))^b},$$
$$R(x,z) = \frac{1}{\sqrt{1-2xz+z^2}}.$$

Neumann's Addition Theorem for Bessel Functions

$$J_0(z)^2 + 2\sum_{k=1}^{\infty} J_k(z)^2 = 1, \qquad J_k(z) := \left(\frac{z}{2}\right)^k \sum_{k\geq 0} \frac{(-z^2/4)^n}{n!(n+k)!}.$$

1. Bessel J_k defined by

$$z^2D_z^2 + zD_z + (z^2 - k^2), \quad S_k + D_z - k/z \qquad \text{dim } 2$$

2. Square (dim 3) by closure (lin. alg.)

$$z^2D_z^3 + 3zD_z^2 + (4z^2 - 4k^2 + 1)D_z + 4z, \quad zS_k - z + \left(k - \frac{1}{2}\right)D_z + \frac{z}{2}D_z^2.$$

3. Look for $P + (S_k - 1)Q$ with P free of k and of order 1 in D_z:

$$P = D_z, \qquad Q = \frac{k}{z} + \frac{1}{2}D_z.$$

4. Conclusion

$$D_z \sum_{k=-\infty}^{\infty} J_k^2 + [QJ_k^2]_{k=-\infty}^{\infty} = 0.$$

Applications of Creative Telescoping

Generating series: Multiplication by z^n using closure by $\times$, then definite summation.

Extraction of coefficients: $\times z^{-n-1}$, then Cauchy integral.

Diagonals: If $f(x,y) = \sum_{n,k} a_{n,k} x^n y^k$, its *diagonal*

$$\sum_n a_{n,n} x^n = \frac{1}{2i\pi} \oint f(x/s, s) \frac{ds}{s}.$$ Generalizes to more variables.

Hadamard product: $f(\mathbf{x}) \odot g(\mathbf{x})$ is a diagonal of $f(\mathbf{x})g(\mathbf{y})$.

Coefficients in Chebyshev or Neumann series:

$$\int_{-1}^{1} \frac{f(t)T_n(t)}{\sqrt{1-t^2}}\, dt, \qquad \int_0^1 f(r)J_n(r)r\, dr.$$

But termination/success not guaranteed!

IV. Holonomy

Definition

Motivation: $(\mathbb{O}_n \operatorname{Ann}_f + \partial_n \mathbb{O}_n) \cap \mathbb{O}_{n-1}$ could be $\{0\}$.

Example: $u_n = \displaystyle\sum_{k=-\infty}^{\infty} \frac{1}{n^2+k^2+1}$ is not D-finite.

Def. Weyl algebra $\mathcal{A}_n = \mathbb{K}\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n \rangle$, with $\partial_i x_i = x_i \partial_i + 1$ $(i = 1, \dots, n)$ and commutation otherwise.

Def. Let $\mathcal{I} \subset \mathcal{A}_n$ be a left ideal. The quotient $\mathcal{A}_n/\mathcal{I} = \mathcal{A}_n(1+\mathcal{I})$ is holonomic if

$$\dim \mathcal{G}_k(1+\mathcal{I}) = c_n k^n + \dots + c_0, \quad k \in \mathbb{N},$$

where $\mathcal{G}_k = \{P \in \mathcal{A}_n, \deg P \le k\}$, the degree being wrt $\mathbf{x}$ and $\boldsymbol{\partial}$.

Closure Properties

Thm. [Takayama] If $\mathcal{I} \subset \mathbb{K}(x_1, \ldots, x_n)[D_{x_1}; 1, D_{x_1}] \cdots [D_{x_n}; 1, D_{x_n}]$ is D-finite, then $\mathcal{A}_n/(\mathcal{A}_n \cap \mathcal{I})$ is holonomic.

Thm. [Bernstein] $\mathcal{A}_n/\mathcal{I}$ holonomic, then if $\ell + m > n$, the subalgebra of $\mathcal{A}_n$ generated by $x_{i_1}, \ldots, x_{i_\ell}, \partial_{j_1}, \ldots, \partial_{j_m}$ has a nonzero intersection with $\mathcal{I}$.

Proof $\binom{\ell + m + k - 1}{k} = \Theta(k^{\ell+m})$, $k \to \infty$.

Cor. Creative telescoping works for *differentially* finite series.

Thm. [Bernstein] $\mathcal{A}_n/\mathcal{I}$ holonomic $\Rightarrow \mathcal{A}_{n-1}/\left(\mathcal{A}_{n-1} \cap (\mathcal{I} + \partial_n \mathcal{A}_{n-1})\right)$ holonomic.

Cor. [Lipshitz] Closure by diagonal and Hadamard product.

Holonomy of Sequences

Def. Holonomic sequence: differentially finite generating series.

Thm. [Lipshitz] Closure under $+$, $\times$, convolution, section, ...

Def. A sequence $u_{n_1,\dots,n_d}$ is hypergeometric if it satisfies a system of d linear first-order recurrences. (D-finite and dim 1).

Thm. [Sato] Hypergeometric sequences can be written

$$\underbrace{R(n_1,\dots,n_d)}_{\text{rational}}\prod_{i=1}^{d}\underbrace{\rho_i}_{\in\mathbb{K}}{}^{n_i}\cdot\prod_{i=1}^{p}\prod_{k=0}^{\overbrace{e_i(n_1,\dots,n_d)}^{\in\mathbb{Z}n_1+\cdots+\mathbb{Z}n_d}-1}\underbrace{\psi_i}_{\in\mathbb{K}(X)}\big(e_i(n_1,\dots,n_d)-k\big).$$

Thm. [AbPe01] A hypergeometric sequence is holonomic iff $R=1$. (Sufficient condition for creative telescoping).

New [Abramov 02] Necessary and sufficient condition for hypergeometric sequences + algorithm (bivariate case).

Perspectives

Beyond multivariate D-finite symetric functions with D-finite specializations [Gessel90]: k-regular graphs, Young tableaux of fixed height, $k \times n$ Latin rectangles, permutations with longest increasing subsequence of length k. Algorithms in [ChMiSa02].

Foundations Understand left + right. Understand problems of non-minimality.

Efficiency Other elimination methods. Use univariate algorithms as much as possible.

Applications ESF, code generation, extension to multivariate.

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