Calcul formel & Analyse d'algorithmes

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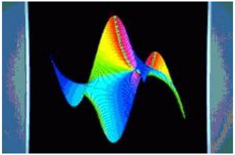


January 26, 2006

I Introduction

ΛγΩ Demonstration (I)

type exponent = **sequence**(bit);
    bit = zero | one;
    zero,one = **atom**(1);

procedure binpow(c:exponent);
case c **of**
    () : nil;
    (zero,c1) : **begin** squaring; binpow(c1) **end**;
    (one,c1) : **begin** squaring; multiply; binpow(c1) **end**
end;

measure multiply : k1;
    squaring : k2;
to_analyze : binpow;

ΛΥΩ Demonstration (II)

```
% luo
                  Lambda-Upsilon-Omega
                     (Version 1.3)
Initializing maple ...
For help about LUO, type help "";;

#analyze "binpow";;
>>> ALGEBRAIC ANALYZER ...
                exponent(z) = Q(2 z)

                      - 2 k2 z Q(2 z) - k1 z Q(2 z)
     tau_binpow(z) = - -----------------------------
                                 1 - 2 z

>>> ANALYTIC ANALYZER ...
Number of exponent of size n is:
                                           ln(2)
           /   1  \(- ln(2))       exp(n)
          (|------|         ) + (O(-----------))
           \exp(n)/                       n

Total cost on random inputs of size n:

                        /   1  \(- ln(2))             ln(2)
     ((k2 + 1/2 k1) n |------|         ) + (O(exp(n)     ))
                        \exp(n)/

Average cost on random inputs of size n:

                ((k2 + 1/2 k1) n) + (O(1))
```

Principle

Combinatorial Structure

↓

Generating Function

↓

Asymptotic Estimate

Computer Algebra

Program

↓

Generating Function

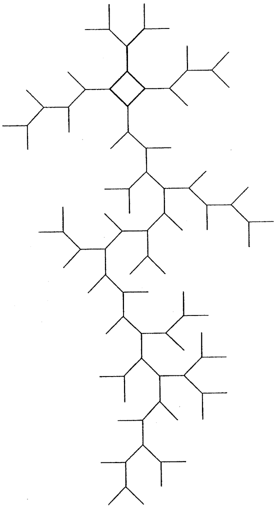
↓

Asymptotic Estimate

Examples of Automatic Analysis of Algorithms

- Pollard's ρ algorithm;
- Automatic differentiation;
- Quicksort and quickselect;
- Patterns in DNA;
- Various parameters in various structures. . .

More at http://algo.inria.fr



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Found 1 combinatorial structures for 1,3,19,193

Structure 1	
Specification	[S, {C = Prod(B,Z), S = Set(C), B = Sequence(C)}, labelled]
First terms in the sequence	[1, 1, 3, 19, 193, 2721, 49171, 1084483, 28245729, 848456353, 28875761731, 1098127402131, 46150226651233, 2124008553358849, 106246577894593683, 5739439214861417731, 332993721039856822081, 20651350143685984386753, 1363322103204314826347779, 95453198574445723828731283, 7064900016612187878152462721]
Generating function	exp(1/2-1/2 *(1-4 *x)^(1/2))
Recurrence	{f(0) = 1, f(1) = 1, -f(n) +(-4 *n-2) *f(n +1) +f(n +2)}
Asymptotics of the coefficients	1/4/Pi^(1/2) *exp(1/2)/n^(3/2) *4^n
References	EIS A001517 (up to a possible factor of n!)
ECS number	131

II Combinatorics

Language

Context-free grammars (Union, Prod, Sequence), plus Set, Powerset, Cycle. Origins: [Pólya37, Joyal81,...]

Labelled and unlabelled universes.

Example

Binary trees	B=Union(Z,Prod(B,B))
Mappings	M=Set(Cycle(Tree)), Tree=Prod(Z,Set(Tree))
Permutations	P=Set(Cycle(Z))
Children rounds	R=Set(Prod(Z,Cycle(Z)))
Integer partitions	P=Set(Sequence(Z))
Set partitions	P=Set(Set(Z,card>0))
Irreducible polynomials mod p	P=Set(Irred), P=Sequence(Coeff).

Aim: complete libraries

Complexity Descriptors [FlSt87]

Definition (P program operating over $a \in \mathcal{A}$, $\tau P(a)$ its cost)

$$\tau P(z) := \sum_{a \in \mathcal{A}} \frac{\tau P(a)}{(|a|!)} z^{|a|}.$$

Translation Rules

	$\mathcal{A}$	$P(a)$	$\tau P(z)$
Constant		c	$A(z)\tau c$
Sequence		$Q(a); R(a)$	$\tau Q(z) + \tau R(z)$
Branching	$\mathcal{B} \cup \mathcal{C}$	if $a \in \mathcal{B}$ then $Q(a)$	
		else $R(a)$	$\tau Q(z) + \tau R(z)$
Loop	$\mathcal{B} \times \mathcal{C}$	$a =: (b, c); Q(b)$	$C(z)\tau Q(z)$

Proof (loop in labelled case)

$$\tau P(z) = \sum_{(b,c)\in\mathcal{B}\times\mathcal{C}} \tau Q(b)\frac{z^{|b|+|c|}}{(|b|+|c|)!}$$

$$= \sum_{\substack{b\in\mathcal{B}\\ c\in\mathcal{C}}} \binom{|b|+|c|}{|b|} \tau Q(b)\frac{z^{|b|}z^{|c|}}{(|b|+|c|)!}$$

Relabelling!

$$= \sum_{\substack{b\in\mathcal{B}\\ c\in\mathcal{C}}} \tau Q(b)\frac{z^{|b|}}{|b|!}\frac{z^{|c|}}{|c|!}.$$

Proof (loop in labelled case)

$$\begin{aligned}\tau P(z) &= \sum_{(b,c)\in\mathcal{B}\times\mathcal{C}} \tau Q(b)\frac{z^{|b|+|c|}}{(|b|+|c|)!}\\ &= \sum_{\substack{b\in\mathcal{B}\\ c\in\mathcal{C}}} \binom{|b|+|c|}{|b|}\tau Q(b)\frac{z^{|b|}z^{|c|}}{(|b|+|c|)!} \qquad \text{Relabelling!}\\ &= \sum_{\substack{b\in\mathcal{B}\\ c\in\mathcal{C}}} \tau Q(b)\frac{z^{|b|}}{|b|!}\frac{z^{|c|}}{|c|!}.\end{aligned}$$

Exercise

$$\ell \in \text{Sequence}(\mathcal{A}) \mapsto \text{for } f \in \ell \text{ do } Q(f) \text{ od} \Longrightarrow \tau P(z) = \frac{\tau Q(z)}{(1-A(z))^2}.$$

Example: formal differentiation

Input: $\mathcal{E} = 0|1|x|\mathrm{Prod}(+,\mathcal{E},\mathcal{E})|\mathrm{Prod}(\times,\mathcal{E},\mathcal{E})|\mathrm{Prod}(\exp,\mathcal{E})$.

Cost: size of result

casetype e of	$\tau P(z) =$
0 :0;	z
1 :0;	$+z$
x :1;	$+z$
$u+v$:$u'+v'$;	$+2zE(z)\tau P(z)$
$u\times v$:$u'v+uv'$;	$+2zE(z)zE'(z)+3zE^2(z)$ *copy*
$\exp(u)$: $\exp(u)\times u'$.	$+2zE(z)+z\tau P(z)+z^2E'(z)$

Average cost: $\dfrac{[z^n]\tau P(z)}{[z^n]E(z)}$.

Attribute Grammars

Grammar & functions over objects it defines.
[Knuth68; Delest, Fédou, Dutour (94–98)]

Construction	attribute defn	meaning
C=Union(A,B)	$F(C) = c + \text{Union}(\sum F_i(A), \sum F_j(B))$	if
C=Prod(A,B)	$F(C) = c + \sum F_i(A) + \sum F_j(B)$	records
C=Φ(A)	$F(C) = c + \text{it}(\sum F_i(A))$	iteration

$\Phi \in \{\text{Sequence}, \text{Set}, \text{Cycle}\}$.

Generalizes $\Lambda\Upsilon\Omega$ [Mishna03]

Attributes

Example

Number of cycles in a permutation

$P = \text{Set}(\text{Cycle}(Z))$	cycle(P)=it(1)

Area under Dyck paths

$P = \text{Sequence}(M)$	area(P)=it(area(M))
$M = \text{Prod}(U, P, D)$	area(M)=area(P)+size(P)+1

Path Length in Binary Mappings

M=Set(Cycle(Prod(Z,B)))	pl(M)=it(it(pl(B)))
B=Prod(Z,Set(B,card={0,2}))	pl(B)=it(size(B)+pl(B))

Attributes and Multivariate Generating Functions

$$C(z_1, z_2, \dots, z_k) = \sum_{c \in C} \frac{z_1^{|c|}}{(|c|!)} z_2^{F_2(c)} \cdots z_k^{F_k(c)}$$

Average value of attribute F_i:

$$\frac{[z_1^n] \left. \frac{\partial C}{\partial z_i} \right|_{z_i=1}}{[z_1^n] C(z_1, 1, \dots, 1)}.$$

Also higher moments.

Theorem (FeDu98,Mishna03)

"Decorated" translation rules give equations for multivariate generating functions.

See `?combstruct,agfeqns`

Example: Path Length in Binary Trees

$$B(z,u) := \sum_{b\in B} z^{|b|} u^{\mathrm{pl}(b)}.$$

Combinatorial description:

B=Prod(Z,Set(B,card={0,2})) pl(B)=it(size(B)+pl(B))

$$B(z,u) = z^1 u^0 \left(1 + \frac{1}{2} \sum_{(\ell,r)\in B\times B} z^{|\ell|} u^{|\ell|+\mathrm{pl}(\ell)} z^{|r|} u^{|r|+\mathrm{pl}(r)}\right)$$

$$= z\left(1+\frac{1}{2}B^2(zu,u)\right).$$

More Examples

Number of cycles in a permutation

$P = \text{Set}(\text{Cycle}(Z))$ cycle(P)=it(1) $P(z,u) = \exp(u \log \frac{1}{1-z})$

Area of Dyck paths

$P = \text{Sequence}(M)$ area(P)=it(area(M)) $P(z,u) = \frac{1}{1-M(z,u)}$

$M = \text{Prod}(U,P,D)$ area(M)=area(P)+ size(P)+1 $M(z,u) = z^2 u P(zu,u)$

Path Length in Binary Mappings

M=Set(Cycle(Prod(Z,B))) pl(M)=it(it(pl(B))) $M(z,u) = \frac{1}{1-zB(zu,u)}$

B=Prod(Z, Set(B,card={0,2})) pl(B)=it(size(B)+pl(B)) $B(z,u) = z + \frac{z}{2} B^2(zu,u)$

Path Length in Binary Mappings

Starting from the functional equations:

$$M(z,u) = \frac{1}{1 - zB(zu,u)}, \qquad (1)$$

$$B(z,u) = z + \frac{z}{2}B^2(zu,u), \qquad (2)$$

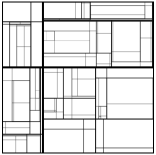
we want $M(z,1), \dfrac{\partial M}{\partial u}(z,1)$.

From (2), we get $B(z,1) = z + \frac{z}{2}B^2(z,1)$ and a linear system:

$$\frac{\partial B}{\partial z}(z,1) = 1 + \frac{1}{2}B^2(z,1) + B(z,1)\frac{\partial B}{\partial z}(z,1),$$
$$\frac{\partial B}{\partial u}(z,1) = zB(z,1)\left(z\frac{\partial B}{\partial z}(z,1) + \frac{\partial B}{\partial z}(z,1)\right),$$

whence the result. Similarly for higher moments.

Pathlength in Quadtrees



Filling rate of the pages?

Model: points distributed uniformly at random in $(0,1)^d$.

Translation in generating series:

$$\left(z(1-z)\frac{d}{dz}\right)^d (f(z) - \frac{1}{(1-z)^2}) - 2^d f(z) = 0.$$

Analysis:

1. Singularity at $z=1$, local behaviour

$$f = \frac{2}{d}\frac{1}{(1-z)^2}\log\frac{1}{1-z} + \frac{c}{(1-z)^2} + \cdots$$

2. Translation: $f_n = \frac{2}{d} n \log n + \mu_d n + O(\log n + n^{-1+2\cos(2\pi/d)})$.

[FlGoPuRo91,FlLaLaSa95]

Greene's Box Operator (I)

Labelled universe: $\mathcal{A} = \mathcal{B}^{\square} \times \mathcal{C}$

The smallest label goes in $\mathcal{C}$

Generating functions: $A(z) = \int_0^z B'(t)C(t)\,dt.$

Proof.

$$A(z) = \sum_{(b,c)\in\mathcal{B}\times\mathcal{C}} \frac{z^{|b|+|c|}}{(|b|+|c|)!}$$

$$= \sum_{\substack{b\in\mathcal{B}\\ c\in\mathcal{C}}} \binom{|b|+|c|-1}{|c|} \frac{z^{|b|}z^{|c|}}{(|b|+|c|)!} = \sum_{\substack{b\in\mathcal{B}\\ c\in\mathcal{C}}} \frac{|b|}{|b|+|c|} \frac{z^{|b|}}{|b|!} \frac{z^{|c|}}{|c|!}.$$

$\square$

Greene's Box Operator (II)

Example

Increasing trees ($B = z^{\Box} \times B \times B$). Automatic analysis of Quicksort, Quickselect.

More generally, partial orders over labels [Greene83].
Extends to Sequence, Set, Cycle
Extends to attributes [Mishna03].

Differential equations of combinatorial origin

D-finite Series

Definition

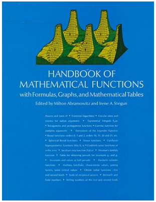
A power series $y(z)$ is differentially finite (D-finite) if there exist polynomials $a_i \in \mathbb{Q}[z]$ such that

$$a_k(z)y^{(k)}(z) + \cdots + a_0(z)y(z) = 0. \qquad \text{(E)}$$

Equivalent definition: $y(z), y'(z), y''(z), \ldots$ span a finite-dimensional vector space over $\mathbb{Q}(z)$.

Examples of functions: exp, log, sin, cos, sinh, cosh, arccos, arccosh, arcsin, arcsinh, arctan, arctanh, arccot, arccoth, arccsc, arccsch, arcsec, arcsech, ${}_pF_q$ (includes Bessel J, Y, I and K, Airy Ai and Bi and polylogarithms), Struve, Weber and Anger fcns, the large class of algebraic functions,. . .

About 60% of Abramowitz & Stegun.



Coefficients ↔ Series

Theorem

A series is D-finite if and only if its sequence of coefficients satisfies a linear recurrence.

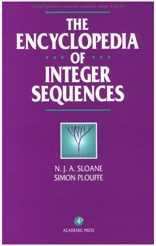
Proof. [sketch] $zD_z \leftrightarrow n$, $z^{-1} \leftrightarrow S_n$.

Corollary

1. *N first coefficients in complexity $O(N)$;*
2. *Nth coefficient in complexity $O_{\log}(N^{1/2})$ [ChCh90].*

Examples of sequences: rational sequences, hypergeometric sequences (includes $n!$, multinomials,...), classical orthogonal polynomials.

About 25% of Sloane & Plouffe.



Examples

Ex. 1. From an algorithm in number theory

kth coefficient of $P(x)^n$ (k and n large)

Computation (given n and k):

1. 1st order LDE;
2. linear recurrence of order deg P satisfied by coefficients;
3. fast evaluation.

Ex. 2. Related to a complexity analysis of Gröbner bases

Computation of $P_n(x)$ defined by

$$\sum_{n\geq 0} P_n(x)\frac{z^n}{n!} = \left(\frac{1+z}{1+z^2}\right)^x.$$

Computation (given n and x):

1. Same as above!
2. fast evaluation (without computing the polynomials).

Closure Properties

Theorem

1. *The set of D-finite series in $\mathbb{Q}[[z]]$ forms a sub-algebra of $\mathbb{Q}[[z]]$.*
2. *The set of D-finite series is closed under Hadamard product.*
3. *The set of D-finite series is closed under formal Laplace & Borel transforms (ogf $\leftrightarrow$ egf).*

Proof.

linear algebra! □

Mehler Identity for Hermite Polynomials

```
> L:=[seq(orthopoly[H](n,x),n=0..5)];
```

$$L := [1, 2x, 4x^2-2, 8x^3-12x, 16x^4-48x^2+12, 32x^5-160x^3+120x]$$

```
> gfun[listtodiffeq](L,u(z));
```

$$[\{u(0) = 1, \frac{d}{dz}u(z) + (2z - 2x)u(z)\}, \mathrm{egf}]$$

```
> deq:=%[1]:hadamardproduct(deq,subs(x=y,deq),u(z)):
> Laplace(deq,u(z),'diffeq');
```

$$\{u(0) = 1, (16z^4-8z^2+1)\frac{d}{dz}u(z)+(16z^3-16z^2xy+8zy^2-4z+8zx^2-4xy)$$

```
> dsolve(%,u(z));
```

$$u(z) = \frac{\exp\left(\frac{4z(xy-z(x^2+y^2))}{1-4z^2}\right)}{\sqrt{1-4z^2}}$$

Algebraic Series

Theorem (Tannery 1874)

1. *Algebraic power series are D-finite.*
2. *If f is D-finite and g is algebraic, $f \circ g$ is D-finite.*

Proof.

Bézout identity! □

Motzkin Numbers (Unary-Binary Trees)

$$M(z) - \left(1 + zM(z) + zM(z)^2\right) =: P(M) = 0.$$

Bézout: $AP + BP_M = 1 \Rightarrow M' = -BP_z \bmod P =: cM + d1.$

Vector space of dimension 2

```
> gfun[algeqtodiffeq](M=1+z*M+z*M^2,M(z));
```

$$-1 - z + (-3\,z + 1)\,M(z) + \left(z - 6z^2 + z^3\right)\frac{d}{dz}M(z)$$

```
> gfun[diffeqtorec](%,M(z),u(n));
```

$$\{nu(n) + (-9 - 6n)u(1+n) + (3+n)u(n+2), u(0) = 1, u(1) = 2\}$$

$\rightarrow$ fast computation.

Works for arbitrary degree.

Forests of Catalan Trees

$$Y(z) = \exp\left(\frac{1-\sqrt{1-4z}}{2}\right).$$

Same computation $\rightarrow$

$$\{(n+2)(n+1)u(n+2) = u(n) + 2(n+1)(2n+1)u(n+1),$$
$$u(0) = 1, u(1) = 1\}$$

III Asymptotics

Automated Examples

- Conway's sequence: 1, 11, 21, 1211, 111221, 312211,...

$$\ell_n \simeq 2.042160077\rho^n, \qquad \rho \simeq 1.3035772690343$$

ρ root of a polynomial of degree 71.

- Catalan numbers (binary trees): 1, 1, 2, 5, 14, 42, 132,...

$$B_n \sim \frac{1}{\sqrt{\pi}}\frac{4^n}{n^{3/2}}$$

- Cayley trees: 1, 2, 9, 64, 625, 7776, 117649, 2097152,...

$$\frac{T_n}{n!} \sim \frac{e^n}{\sqrt{2\pi}n^{3/2}}$$

- Bell numbers (set partitions): 1, 1, 2, 5, 15, 52, 203,...

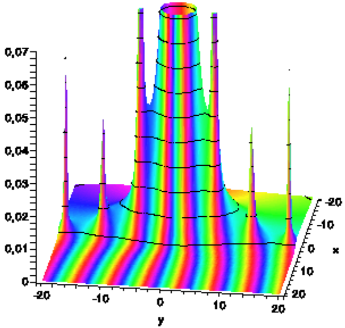
$$\log\frac{B_n}{n!} \sim -n\log\log n$$

Starting point: generating function

Cauchy's Formula

$$[z^n]f(z) = \frac{1}{2i\pi}\oint \frac{f(z)}{z^{n+1}}\,dz$$

$$[z^2]\frac{z}{e^z-1} = \frac{1}{12}$$



Singularity Analysis

$$\text{Cauchy's formula: } [z^n]f(z) = \frac{1}{2i\pi}\oint \frac{f(z)}{z^{n+1}}\,dz$$

Singularity of smallest modulus → exponential growth

Local behaviour → sub-exponential terms

Under mild conditions [FlOd84]

$$f(z) = g(z) + O(h(z)), \quad z \to \rho \Rightarrow [z^n]f(z) = [z^n]g(z) + O([z^n]h(z)),$$

A simple asymptotic scale translates

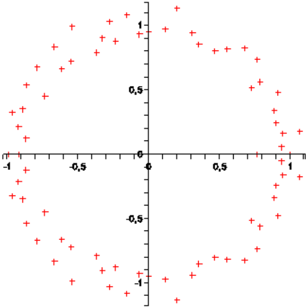
$$[z^n](1-\frac{z}{\rho})^\alpha \log^\beta \frac{1}{1-z/\rho} \sim \rho^{-n}\frac{n^{-\alpha-1}}{\Gamma(-\alpha)}\log^\beta n, \quad \alpha \notin \mathbb{N}.$$

Method:

1. Find dominant sing. Symbolic-Numeric
2. Expand locally. Symbolic Asymptotics
3. If algebraico-logarithmic, translate.

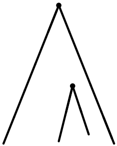
Locating Dominant Singularities

Rational series: resultants, algebraic/numerical separation of roots by moduli [GoSa96];
Algebraic series: selection of branches, connexion problem (solution: semi-numerical separation of roots plus bounds on remainders of Puiseux expansions) [ChFlSa01];



Th. [Pringsheim] Positive Taylor coefficients ⇒ one real positive dominant singularity.
Implicit functions: reduce to polynomial solving.
More general: heuristically reduce to solving. Undecidability issues.

Asymptotic Expansion



exp-log functions

$+, \times, \exp, \log |\cdot|$

Problem: indefinite cancellation.

$$e^{1/x+e^{-x}} - e^{1/x} = 1 + \frac{1}{x} + \cdots - (1 + \frac{1}{x} + \cdots) = O(x^{-K}).$$

Algorithm [RiSaShvdH96]:

1. Build a tower of diff. fields $\mathbb{R}(x) \subset \cdots \subset \mathbb{R}(t_1, \ldots, t_n)$ s.t. $t_1, \ldots, t_n$ is the “right” asymptotic scale;
2. Express fcn in $t_1, \ldots, t_n$;
3. Expand w.r.t. t_n and recurse.

Rely on zero-equivalence tests.

Algorithm (by induction)

$$I_k(x) = t_1 \prec \cdots \prec t_n$$

$$f \mapsto \sum_{\alpha \in S} \underbrace{c_\alpha(t_1, \ldots, t_{n-1})}_{\neq 0} t_n^\alpha.$$

Elementary functions (Sum, product, $\exp(f)$ with $f \to 0, \ldots$).
Equality-check between function and truncatures of its expansion.
Logarithm $\quad f \sim C t_1^{\alpha_1} \cdots t_n^{\alpha_n} (=: f_0)$.
If $\alpha_1 = 0$,

$$\log|f| = \log|C| + \sum \alpha_i \underbrace{\log|t_i|}_{\in \mathbb{R}[[t_1, \ldots, t_n]]} + \log(1 - \underbrace{\frac{f - f_0}{f_0}}_{\to 0});$$

otherwise, add $t_0 = \log t_1$ to the scale first.

Algorithm II

Exponential $f \to \infty$. Check whether

$$f \sim \alpha \log t_i, \quad i > 1, \alpha \in \mathbb{R} \setminus \{0\}.$$

If so, $\exp(f) = t_i^\alpha \exp(f - \alpha \log t_i)$ and recurse.
Otherwise, insert $\exp(f)$ in the scale.

Theorem (RiSaShvdH96)

It always works!

Example

$$f = \log\log(xe^{xe^{x}} + 1) - \exp\exp(\log\log x + \frac{1}{x}), \qquad x \to \infty$$

Example

$$f = \log\log(xe^{xe^x} + 1) - \exp\exp(\log\log x + \frac{1}{x}), \qquad x \to \infty$$

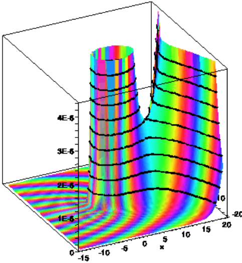
Scale $\{\log\log x, \log x, x, e^x, \exp(xe^x)\}$, i.e.,

$$f = \sum_{i \in \mathbb{N}} c_i(\log\log x, \log x, x, e^x) e^{-\alpha_i x e^x},$$

each (α_i, c_i) can be computed explicitly and expanded recursively. For instance, $\alpha_0 = 0$, ...

$$\longrightarrow f \sim -\frac{\log^2 x}{2x}.$$

Saddle-Point Method



(Functions with fast singular growth)

$$[z^n]f(z) = \frac{1}{2i\pi}\oint \underbrace{\frac{f(z)}{z^{n+1}}}_{=:\exp(h(z))} dz$$

Saddle-point equation $h'(R_n) = 0$ i.e. $R_n \frac{f'(R_n)}{f(R_n)} - 1 = n$

Change of variables $h(z) = h(\rho) - u^2$

Termwise integration

$$f_n \approx \frac{f(R_n)}{R_n^{n+1}\sqrt{2\pi h''(R_n)}}$$

Sufficient conditions: Hayman (1st order), Harris & Schoenfeld, Odlyzko & Richmond, Wyman

Hayman admissibility

A set of analytic conditions and easy-to-use sufficient conditions.

Hyp. f, g admissible, P polynomial

1. $\exp(f)$, fg and $f+P$ admissible.
2. $\mathrm{lc}(P) > 0 \Rightarrow fP$ and $P(f)$ admissible.
3. if e^P has ultimately positive coefficients, it is admissible.

Examples: Stirling's formula ($\exp(z)$), involutions ($\exp(z+z^2/2)$), set partitions ($\exp(\exp(z)-1)$),...

Example: Parts in a Partition

$$F(z,u)=e^{u(e^z-1)}=\sum S_{n,k}u^k\frac{z^n}{n!}$$

Stirling numbers of the 2nd kind

$u=1\rightarrow$ Bell numbers

Saddle-point: $Re^R-1=n$

Result: $\displaystyle\frac{B_n}{n!}=\frac{e^{e^R-1}e^{-R}}{R^{n+1/2}\sqrt{2\pi(1+Re^{-R})}}(1+\cdots)$

$\frac{\partial}{\partial u}\Big|_{u=1}\rightarrow\sum_k kS_{n,k}$

Saddle-point: $R_1e^{R_1}-1+\frac{R_1}{1-e^{-R_1}}=n$

Result: $\displaystyle\frac{\sum kS_{n,k}}{n!}=\mathrm{BIG}(R_1,n)(1+\cdots)$

Ratio: expected number of parts in a partition of size n

Variance: $\frac{\partial^2}{\partial u^2}\Big|_{u=1}$ . . . (even worse)

Problems

Expansion of the saddle-point (automation?)

$$R = \log n - \log\log n + \frac{\log\log n}{\log n} + \frac{\log\log n(\log\log n - 2)}{2\log^2 n} + \cdots.$$

Substitution in the estimate (use?)

$$\frac{B_n}{n!} = \exp\left(-n\log\log n + n\frac{\log\log n + 1}{\log n} + n\frac{(\log\log n)^2}{2\log^2 n} + \dots\right)$$

Indefinite cancellation

$$R_1 = \log n - \log\log n + \frac{\log\log n}{\log n} + \frac{\log\log n(\log\log n - 2)}{2\log^2 n} + \cdots.$$

$R_1 - R = ?$, $e^{e^{R_1}} / e^{e^R} \sim ?$, average number of parts? variance?

Asymptotic Inversion [SaSh97]

Complete algorithm for asymptotic inverses of exp-log functions.
Application:

$$\mu_n = \frac{n}{\log n}\left(1 + \frac{\log\log n}{\log n} + \frac{\log\log^2 n - \log\log n}{\log^2 n} + \dots\right)$$
$$\sigma_n = \frac{n}{\log^2 n}\left(1 + \frac{2\log\log n - 1}{\log n} + \dots\right)$$

Also, higher towers of exponentials.

IV Conclusion

Next Steps

- More automatic asymptotic analysis of functional equations;
- Automatic derivation of distributions ;
- Automatic handling of Pólya's operators;
- Computational preprocessing of algorithms into attribute grammars;
- Speed!
- ...