Minicourse 2: Asymptotic Techniques for AofA

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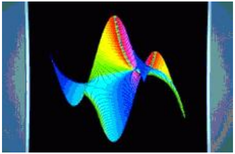


AofA'08, Maresias, Brazil
Sunday 8:30–10:30 (!)

I Introduction

Overview of the 3 Minicourses

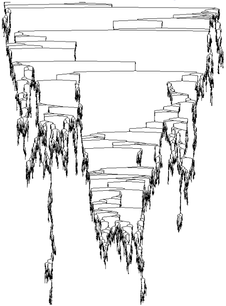
Combinatorial Structure

↓ Combinatorics (MC1)↓

Generating Functions

$$F(z) = \sum_{n \geq 0} f_n z^n$$

Example: binary trees



$$B(z) = z + B^2(z)$$

Overview of the 3 Minicourses

Combinatorial Structure

↓ Combinatorics (MC1)↓

Generating Functions

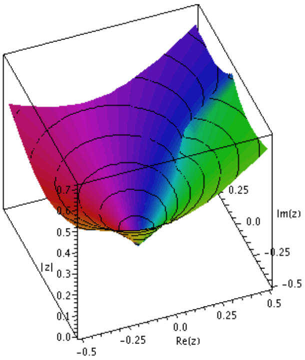
$$F(z) = \sum_{n\geq 0} f_n z^n$$

↓ Complex Analysis (MC2)↓

Asymptotics

$f_n \sim \ldots, n \to \infty.$

Example: binary trees



$$B_n \sim \frac{4^{n-1} n^{-3/2}}{\sqrt{\pi}}$$

Overview of the 3 Minicourses

Combinatorial Structure

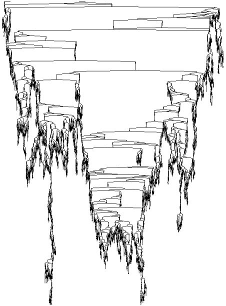
+ parameter

↓ Combinatorics (MC1)↓

Generating Functions

$$F(z,u)=\sum_{n\geq 0} f_{n,k}u^k z^n$$

Example: path length in binary trees



$$B(z,u)=\sum_{t\in T} u^{\mathrm{pl}(t)}z^{|t|}$$

$$=z+B^2(zu,u)$$

$$P(z):=\frac{\partial}{\partial u}B(z,u)\Big|_{u=1}$$

Overview of the 3 Minicourses

Combinatorial Structure
+ parameter
↓ Combinatorics (MC1)↓

Generating Functions
$$F(z) = \sum_{n\geq 0} f_n z^n$$

↓ Complex Analysis (MC2)↓

Asymptotics
$f_n \sim \dots, n \to \infty.$

Example: path length in binary trees

$$B_n = \frac{4^{n-1}n^{-3/2}}{\sqrt{\pi}}\left(1+\frac{3}{8n}+\cdots\right),$$

$$P_n = 4^{n-1}(1-\frac{1}{\sqrt{\pi n}}+\cdots),$$

$$\frac{P_n}{nB_n} = \sqrt{\pi n} - 1 + \cdots.$$

Also, variance and higher moments

Overview of the 3 Minicourses

Combinatorial Structure
$+$ parameter
$\downarrow$ Combinatorics (MC1) $\downarrow$

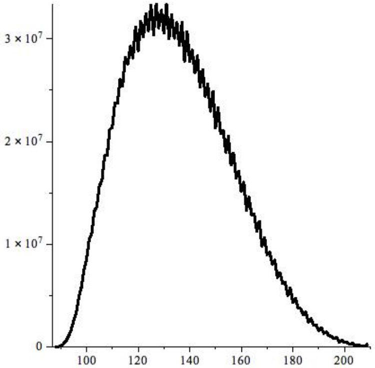
Generating Functions

$$F(z,u) = \sum_{n\geq 0} f_{n,k} u^k z^n$$

$\downarrow$ Multivariate Analysis (MC3) $\downarrow$
Distribution

$$f_{n,k} \sim \ldots, n \to \infty.$$

Example: path length in binary trees



Examples for this Course

- Conway's sequence: 1, 11, 21, 1211, 111221, 312211,...

$$\ell_n \simeq 2.042160077\rho^n, \qquad \rho \simeq 1.3035772690343$$

ρ root of a polynomial of degree 71.

- Catalan numbers (binary trees): 1, 1, 2, 5, 14, 42, 132,...

$$B_n \sim \frac{1}{\sqrt{\pi}} \frac{4^n}{n^{3/2}}$$

- Cayley trees (T=Prod(Z,Set(T))): 1, 2, 9, 64, 625, 7776,...

$$\frac{T_n}{n!} \sim \frac{e^n}{\sqrt{2\pi} n^{3/2}}$$

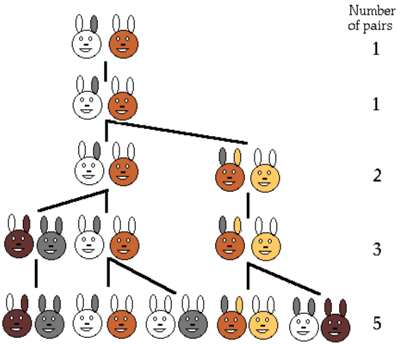
- Bell numbers (set partitions): 1, 1, 2, 5, 15, 52, 203, 877,...

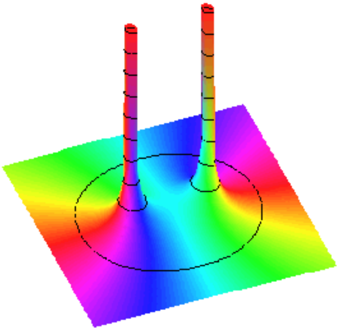
$$\log \frac{B_n}{n!} \sim -n \log \log n$$

Starting point: generating function

A Gallery of Combinatorial Pictures

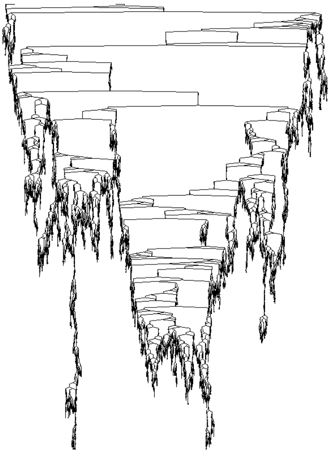
Fibonacci Numbers: $\frac{1}{1-z-z^2} = 1 + z + 2z^2 + 3z^3 + 5z^4 + \cdots$

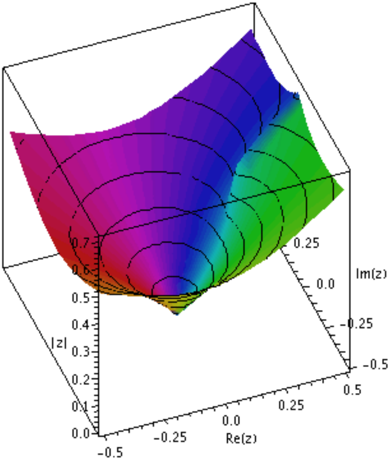




A Gallery of Combinatorial Pictures

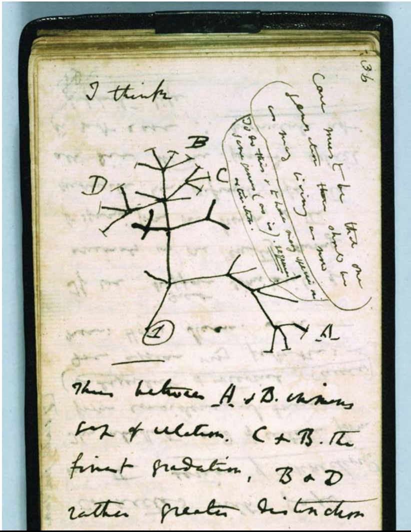
Binary Trees: $\dfrac{1-\sqrt{1-4z}}{2} = z + z^2 + 2z^3 + 5z^4 + 14z^5 + \cdots$

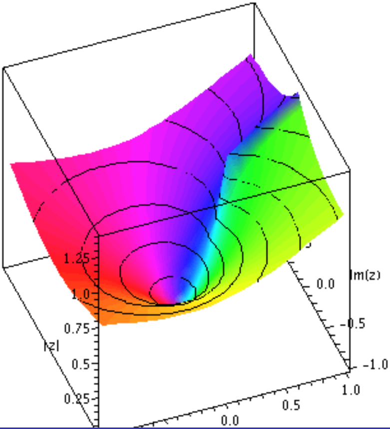




A Gallery of Combinatorial Pictures

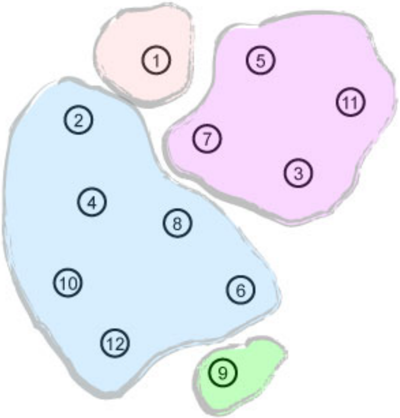
Cayley Trees: $T(z) = z\exp(T(z)) = z + 2\frac{z}{2!} + 9\frac{z}{3!} + 64\frac{z}{4!} + \cdots$

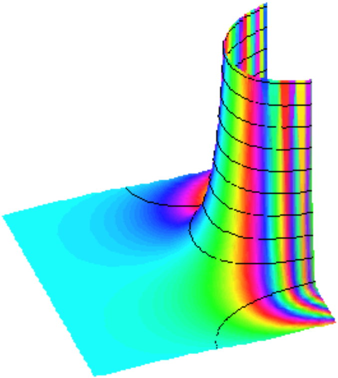




A Gallery of Combinatorial Pictures

Set Partitions: $\exp(\exp(z)-1) = 1 + 1\frac{z}{1!} + 2\frac{z^2}{2!} + 5\frac{z^3}{3!} + 15\frac{z^4}{4!} + \cdots$





II Mini-minicourse in complex analysis

Basic Definitions and Properties

Definition

$f : D \subset \mathbb{C} \to \mathbb{C}$ is analytic at x_0 if it is the sum of a power series in a disc around x_0.

Proposition

- f, g analytic at x_0, then so are $f + g$, $f \times g$ and f'.
- g analytic at x_0, f analytic at $g(x_0)$, then $f \circ g$ analytic at x_0.

Same def. and prop. in several variables.

Examples

f	analytic at 0?	why
polynomial	Yes	
$\exp(x)$	Yes	$1+x+x^2/2!+\cdots$
$\frac{1}{1-x}$	Yes	$1+x+x^2+\cdots\ (\lvert x\rvert<1)$
$\log\frac{1}{1-x}$	Yes	$x+x^2/2+x^3/3\cdots\ (\lvert x\rvert<1)$
$\frac{1-\sqrt{1-4x}}{2x}$	Yes	$1+\cdots+\frac{1}{k+1}\binom{2k}{k}x^k+\cdots\ (\lvert x\rvert<1/4);$
$\frac{1}{x}$	No	infinite at 0
$\log x$	No	derivative not analytic at 0
$\sqrt{x}$	No	derivative infinite at 0

Combinatorial Generating Functions I

Proposition (Labeled)

The labeled structures obtained by iterative use of

$$\textsc{Seq},\ \textsc{Cyc},\ \textsc{Set},\ +,\ \times \text{ starting with } 1, \mathcal{Z}$$

have exponential generating series that are analytic at 0.

Recall Translation Table (MC1)

A + B	$A(z)+B(z)$
A × B	$A(z)\times B(z)$
$\textsc{Seq}$(C)	$\frac{1}{1-C(z)}$
$\textsc{Cyc}$(C)	$\log\frac{1}{1-C(z)}$
$\textsc{Set}$(C)	$\exp(C(z))$

Proof by induction.

$+, \times$, and composition with $\frac{1}{1-x}$, $\log\frac{1}{1-x}$, $\exp(x)$. □

Combinatorial Generating Functions II

Proposition (Unlabeled)

The unlabeled structures obtained by iterative use of

$\textsc{Seq}, \textsc{Cyc}, \textsc{PSet}, \textsc{MSet}, +, \times$ *starting with* $1, \mathcal{Z}$

have ordinary generating series that are analytic at 0.

Proof by induction.

Recall Translation Table (MC1)

A + B	$A(z)+B(z)$	easy
A × B	$A(z)\times B(z)$	easy
$\textsc{Seq}$(C)	$\frac{1}{1-C(z)}$	easy
$\textsc{PSet}$(C)	$\exp(C(z)-\frac12 C(z^2)+\frac13 C(z^3)-\cdots)$	?
$\textsc{MSet}$(C)	$\exp(C(z)+\frac12 C(z^2)+\frac13 C(z^3)+\cdots)$	?
$\textsc{Cyc}$(C)	$\sum_{k\geq 1}\frac{\phi(k)}{k}\log\frac{1}{1-C(z^k)}$	?

Combinatorial Generating Functions II

Proposition (Unlabeled)

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$$\textsc{Seq}, \textsc{Cyc}, \textsc{PSet}, \textsc{MSet}, +, \times \text{ starting with } 1, \mathcal{Z}$$

have ordinary generating series that are analytic at 0.

Proof by induction.

- $\textsc{MSet}(C)$: by induction, there exists $K > 0$, $\rho \in (0,1)$, s.t. $|C(z)| < K|z|$ for $|z| < \rho$. Then
  $C(z) + \frac{1}{2}C(z^2) + \frac{1}{3}C(z^3) + \cdots < K \log \frac{1}{1-|z|}, \quad |z| < \rho.$
  Uniform convergence $\Rightarrow$ limit analytic (Weierstrass).
- $\textsc{PSet}$, $\textsc{Cyc}$: similar.

□

Analytic Continuation & Singularities

Definition

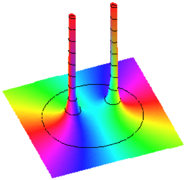
Analytic on a region (= connected, open, $\neq \emptyset$): at each point.

Proposition

$R \subset S$ regions. f analytic in R. There is at most one analytic function in S equal to f on R (the analytic continuation of f to S).

Definition

- Singularity: a point that cannot be reached by analytic continuation;
- Polar singularity α: isolated singularity and $(z-\alpha)^m f$ analytic for some $m \in \mathbb{N}$;
- residue at a pole: coefficient of $(z-\alpha)^{-1}$;
- f meromorphic in R: only polar singularities.



Combinatorial Examples

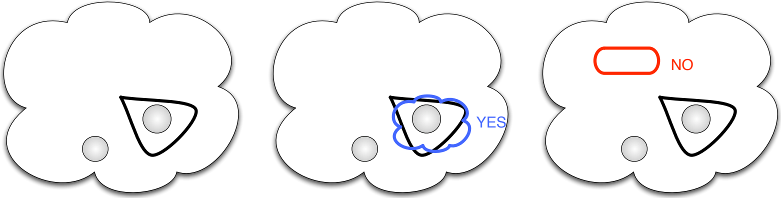
Structure	GF	Sings	Mero. in $\mathbb{C}$
Set	$\exp(z)$	none	Yes
Set Partitions	$\exp(e^z - 1)$	none	Yes
Sequence	$\frac{1}{1-z}$	1	Yes
Bin Seq. no adj.0	$\frac{1+z}{1-z-z^2}$	$\phi, -1/\phi$	Yes
Derangements	$\frac{e^{-z}}{1-z}$	1	Yes
Rooted plane trees	$\frac{1-\sqrt{1-4z}}{2z}$	$1/4$	No
Integer partitions	$\prod_{k\geq 1} \frac{1}{1-z^k}$	roots of 1	No
Irred. pols over $\mathbb{F}_q$	$\sum_{r\geq 1} \frac{\mu(r)}{r} \ln \frac{1}{1-qz^r}$	roots of $\frac{1}{q}$	No
Exercise: Bernoulli nbs	$\frac{z}{\exp(z)-1}$	?	?

Integration of Analytic Functions

Theorem

f analytic in a region R, Γ_1 and Γ_2 two closed curves that are homotopic wrt R (= can be deformed continuously one into the other) then

$$\int_{\Gamma_1} f = \int_{\Gamma_2} f.$$

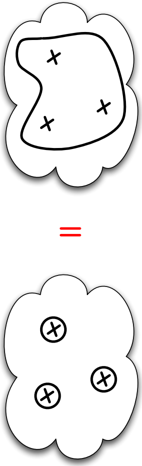


Residue Theorem: from Global to Local

Corollary

f meromorphic in a region R, Γ a closed path in $\mathbb{C}$ encircling the poles $\alpha_1, \dots, \alpha_m$ of f once in the positive sense. Then

$$\int_\Gamma f = 2\pi i \sum_j \operatorname{Res}(f; \alpha_j).$$



Proof.

- $g_j := P_j(z)/(z-\alpha_j)^{m_j}$ polar part at α_j;
- $h := f - (g_1 + \dots + g_m)$ analytic in R;
- Γ homotopic to a point in $R \Rightarrow \int_\Gamma h = 0$;
- Γ homotopic to a circle centered at α_j in $R \setminus \{\alpha_j\}$;
- $\displaystyle\int_\Gamma (z-\alpha_j)^m\,dz = i\int_0^{2\pi} r^{m+1} e^{i(m+1)\theta}\,d\theta = \begin{cases} 2\pi i & m=-1, \\ 0 & \text{otherwise.}\end{cases}$

Cauchy's Coefficient Formula

Corollary

If $f = a_0 + a_1 z + \dots$ *is analytic in* $R \ni 0$ *then*

$$a_n = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z^{n+1}}\, dz$$

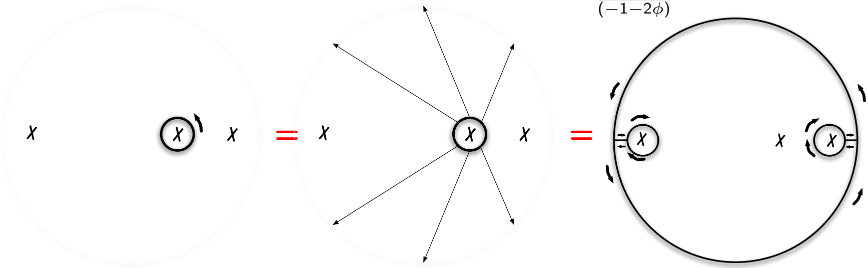
for every closed Γ *in* R *encircling 0 once in the positive sense.*

Proof.

$f(z)/z^{n+1}$ meromorphic in R, pole at 0, residue a_n. □

Coefficients of Rational Functions by Complex Integration

$$2\pi i F_n = \int_\Gamma \underbrace{\frac{z^{-n-1}}{1-z-z^2}}_{g(z)} dz = \Big(\int_{|z|=R} g - \underbrace{\int_\phi g}_{\frac{\phi^{-n-1}}{(-1-2\phi)}} - \underbrace{\int_{\overline{\phi}} g}_{\text{idem}}\Big)$$



When $|z| = R$, $|g(z)| \leq \dfrac{R^{-n-1}}{R^2 - R - 1} \Rightarrow 2\pi R\,|g(z)| \to 0, \quad R \to \infty.$

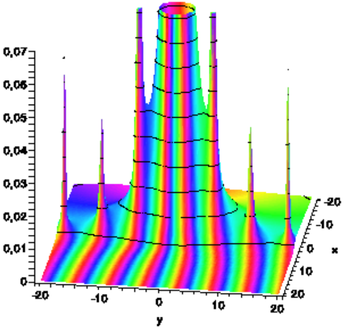
Conclusion: $F_n = \dfrac{\phi^{-n-1}}{1+2\phi} + \dfrac{\overline{\phi}^{-n-1}}{1+2\overline{\phi}}.$

III Dominant Singularity

Cauchy's Formula

$$[z^n]f(z) = \frac{1}{2\pi i} \oint \frac{f(z)}{z^{n+1}}\, dz$$

$$[z^2]\frac{z}{e^z - 1} = \frac{1}{12}$$



As n increases, the smallest singularities dominate.

Exponential Growth

Definition

Dominant singularity: singularity of minimal modulus.

Theorem

$f = a_0 + a_1 z + \cdots$ analytic at 0;
R modulus of its dominant singularities, then

$$a_n = R^{-n}\theta(n), \qquad \limsup_{n\to\infty} |\theta(n)|^{1/n} = 1.$$

Proof (Idea).

1. integrate on circle of radius $R - \epsilon \Rightarrow |a_n| \leq C(R-\epsilon)^{-n}$;
2. if $(R+\epsilon)^{-n} \leq K a_n$, then convergence on a larger disc.

□

General Principle for Asymptotics of Coefficients

$$[z^n]f(z) = \frac{1}{2\pi i}\oint \frac{f(z)}{z^{n+1}}\,dz$$

Singularity of smallest modulus $\longrightarrow$ exponential growth

Local behaviour $\longrightarrow$ sub-exponential terms

Algorithm

1. Locate dominant singularities
2. Compute local expansions
3. Transfer

Rational Functions

Dominant singularities: roots of denominator of smallest modulus.

Conway's sequence:

1, 11, 21, 1211, 111221,...

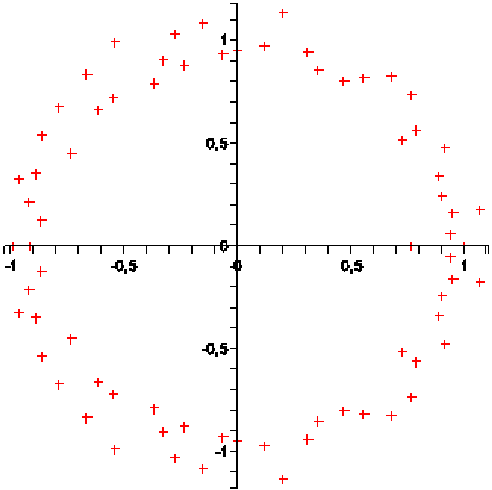
Generating function:

$f(z) = \frac{P(z)}{Q(z)}$
with $\deg Q = 72$.

$\delta(f) \simeq 0.7671198507$,

$\rho \simeq 1.3035772690343$,

$\ell_n \simeq 2.042160077\rho^n$



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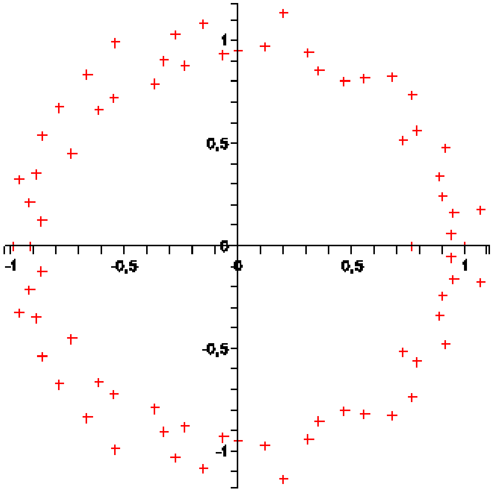
$\ell_n \simeq \underbrace{2.042160077}_{\rho \operatorname{Res}(f,\delta(f))} \rho^n$



Iterative Generating Functions

Algorithm Dominant Singularity

Function F	Dom. Sing. $\delta(F)$
$\exp(f)$	$\delta(f)$
$1/(1-f)$	$\min(\delta(f), \{z \mid f(z)=1\})$
$\log(1/(1-f))$	idem
fg, $f+g$	$\min(\delta(f), \delta(g))$
$f(z)+\frac{1}{2}f(z^2)+\frac{1}{3}f(z^3)+\cdots$	$\min(\delta(f), 1)$.

Iterative Generating Functions

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Note: f has coeffs $\geq 0 \Rightarrow \min(\delta(f), \{z \mid f(z)=1\}) \in \mathbb{R}^+$.

Iterative Generating Functions

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Pringsheim's Theorem

f analytic with nonnegative Taylor coefficients has its radius of convergence for dominant singularity.

Iterative Generating Functions

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$f(z)+\frac{1}{2}f(z^2)+\frac{1}{3}f(z^3)+\cdots$	$\min(\delta(f), 1)$.

Exercise

Dominant singularity of $\frac{1}{2}\left(1-\sqrt{1-4\log\left(\frac{1}{1-\log\frac{1}{1-z}}\right)}\right)$.

(Binary trees of cycles of cycles)

Implicit Functions

Proposition (Implicit Function Theorem)

The equation

$$\mathbf{y} = \mathbf{f}(z, \mathbf{y})$$

admits a solution $\mathbf{y} = \mathbf{g}(z)$ that is analytic at z_0 when

- $\mathbf{f}(z, \mathbf{y})$ is analytic in $1 + n$ variables at $(z_0, \mathbf{y_0}) := (z_0, \mathbf{g}(z_0))$,
- $\mathbf{f}(z_0, \mathbf{y_0}) = \mathbf{y_0}$ and $\det |I - \partial \mathbf{f}/\partial \mathbf{y}| \neq 0$ at $(z_0, \mathbf{y_0})$.

Example (Cayley Trees: $T = z \exp(T)$)

1. Generating function analytic at 0;
2. potential singularity when $1 - z\exp(T) = 0$, whence $T = 1$, whence $z = e^{-1}$.

More generally, solutions of combinatorial systems are analytic.

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Example (Cayley Trees: $T = z\exp(T)$)

1. Generating function analytic at 0;
2. potential singularity when $1 - z\exp(T) = 0$, whence $T = 1$, whence $z = e^{-1}$.

Exercises

1. Binary trees;
2. $T(z) \underset{z \to e^{-1}}{\sim} ?$.

More generally, solutions of combinatorial systems are analytic.

IV Singularity Analysis

General Principle for Asymptotics of Coefficients

$$[z^n]f(z) = \frac{1}{2\pi i} \oint \frac{f(z)}{z^{n+1}}\,dz$$

Singularity of smallest modulus → exponential growth

Local behaviour → sub-exponential terms

Algorithm

1. Locate dominant singularities
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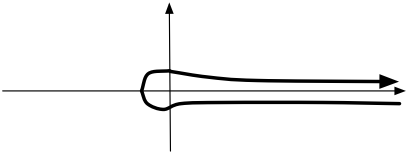
The Gamma Function

- **Def.** Euler's integral: $\Gamma(z) := \int_0^{+\infty} t^{z-1} e^{-t}\,dt$;
- **Recurrence:** $\Gamma(z+1) = z\Gamma(z)$ (integration by parts);
- **Reflection formula:** $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin(\pi z)}$;
- **Hankel's loop formula:** $\dfrac{1}{\Gamma(z)} = \dfrac{1}{2\pi i}\displaystyle\int_{(0)}^{+\infty} (-t)^{-z} e^{-t}\,dt$.



Idea for the last one:
$\int_0^{+\infty} (e^{-\pi i})^{-z} t^{-z} e^{-t}\,dt - \int_0^{+\infty} (e^{\pi i})^{-z} t^{-z} e^{-t}\,dt.$

Basic Transfer Toolkit

Singularity Analysis Theorem [Flajolet-Odlyzko]

1. If f is analytic in $\Delta(\phi, R)$, and

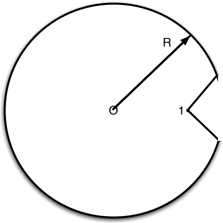
$$f(z) \underset{z\to 1}{=} O\left((1-z)^{-\alpha}\log^\beta\frac{1}{1-z}\right),$$

then $[z^n]f(z) \underset{n\to\infty}{=} O(n^{\alpha-1}\log^\beta n)$.

2. $[z^n](1-z)^{-\alpha} \underset{n\to\infty}{=} \dfrac{n^{\alpha-1}}{\Gamma(\alpha)}\left(1+\sum_{k\geq 1}\dfrac{e_k(\alpha)}{n^k}\right),$

$\alpha \in \mathbb{C}\setminus\mathbb{Z}^-$, $e_k(\alpha)$ polynomial;

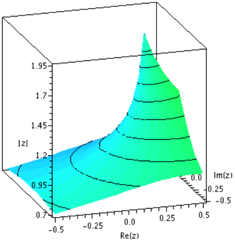
3. similar result with a $\log^\beta(1/(1-z))$.



$\Delta(\phi, R)$

Example: Binary Trees

$$B(z) = \frac{1-\sqrt{1-4z}}{2z}$$



1. Dominant singularity: $1/4$;
2. Local expansion:
   $B = 2 - 2\sqrt{1-4z} + 2(1-4z) + O((1-4z)^{3/2})$;
3. $O((1-4z)^{3/2})) \to O(4^n n^{-5/2})$;
4. $-2\sqrt{1-4z} \to \frac{4^n}{\sqrt{\pi} n^{3/2}} + \star \frac{4^n}{n^{5/2}} + \cdots$.

Conclusion: $B_n = \dfrac{4^n}{\sqrt{\pi} n^{3/2}} + O(4^n n^{-5/2})$.

Exercise

Cayley trees.

Proof of the Singularity Analysis Theorem I

Part I. Scale

2. $[z^n](1-z)^{-\alpha} \underset{n\to\infty}{=} \frac{n^{\alpha-1}}{\Gamma(\alpha)}\left(1+\sum_{k\geq 1}\frac{e_k(\alpha)}{n^k}\right),$

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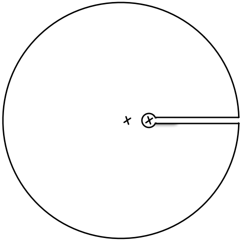

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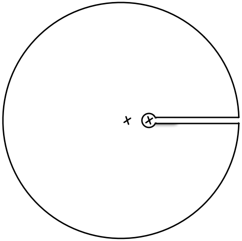

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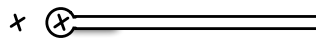

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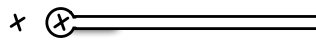

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2. Extending the rest to a full Hankel contour changes the integral by $O(R^{-n})$;
3. On this part, change variable: $z := 1 + t/n$

$$[z^n](1-z)^{-\alpha} = \frac{1}{2\pi i}\int_{(0)}^{+\infty}\left(-\frac{t}{n}\right)^{-\alpha-1}\left(1+\frac{t}{n}\right)^{-n-1} dt + O(R^{-n}).$$

Recognize 1/Γ?

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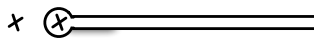

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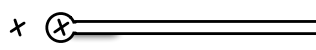

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5. Integrate termwise (+ uniform convergence).

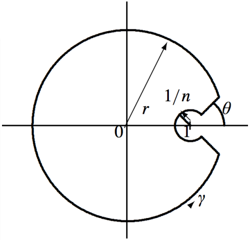
Proof of the Singularity Analysis Theorem II

Part II. $O()$

1. If f is analytic in $\Delta(\phi, R)$, and

$$f(z) \underset{z\to 1}{=} O\left((1-z)^{-\alpha}\log^\beta\frac{1}{1-z}\right),$$

then $[z^n]f(z) \underset{n\to\infty}{=} O(n^{\alpha-1}\log^\beta n)$.



inside $\Delta(\phi, R)$

Easier than previous part:

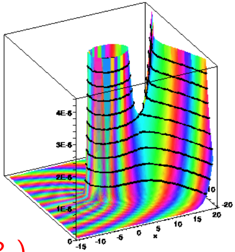
1. Outer circle: r^{-n};
2. Inner circle: use hypothesis and simple bounds;
3. Segments: the key is that $(1 + t\cos\theta/n)^{-n}$ converges to e^t, which is sufficient.

V Saddle-Point Method

Functions with Fast Singular Growth

(Functions with fast singular growth)

$$[z^n]f(z) = \frac{1}{2\pi i}\oint \underbrace{\frac{f(z)}{z^{n+1}}}_{=:\exp(h(z))} dz$$



1. **Saddle-point equation**: $h'(R_n) = 0$ i.e. $R_n \frac{f'(R_n)}{f(R_n)} - 1 = n$
2. **Change of variables:** $h(z) = h(\rho) - u^2$
3. **Termwise integration:**

$$f_n \approx \frac{f(R_n)}{R_n^{n+1}\sqrt{2\pi h''(R_n)}}$$

4. **Sufficient conditions:** Hayman (1st order), Harris & Schoenfeld, Odlyzko & Richmond, Wyman.

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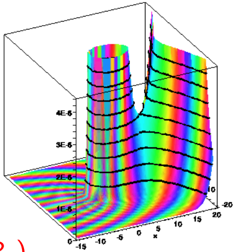

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Exercise

Stirling's formula ($f = \exp$).

4. **Sufficient conditions:** Hayman (1st order), Harris & Schoenfeld, Odlyzko & Richmond, Wyman.

Hayman admissibility

A set of analytic conditions and easy-to-use sufficient conditions.

Theorem

Hyp. *f, g admissible, P polynomial*

1. $\exp(f)$, *fg and $f + P$ admissible.*
2. $\operatorname{lc}(P) > 0 \Rightarrow$ *fP and $P(f)$ admissible.*
3. *if e^P has ultimately positive coefficients, it is admissible.*

Example

- sets ($\exp(z)$),
- involutions ($\exp(z + z^2/2)$),
- set partitions ($\exp(\exp(z) - 1)$).

VI Conclusion

Summary

- Many generating functions are analytic;
- Asymptotic information on their coefficients can be extracted from their singularities;
- Starting from bivariate generating functions gives asymptotic averages or variances of parameters;
- A lot of this can be automated.

Want More Information?