Effective Asymptotics for Combinatorial Systems

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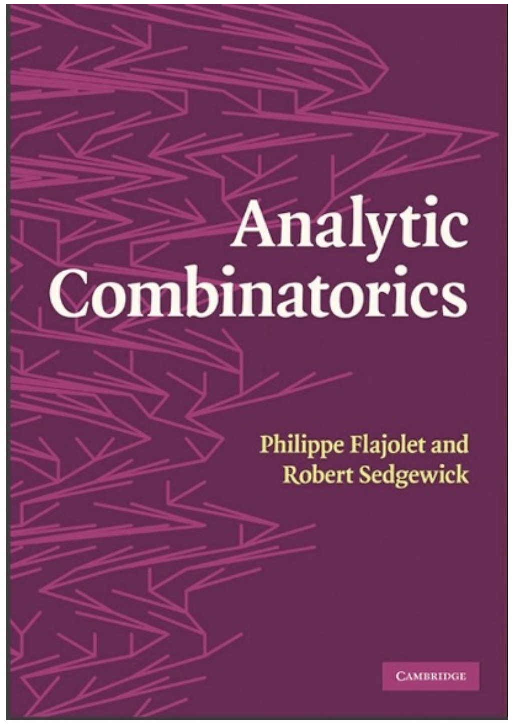
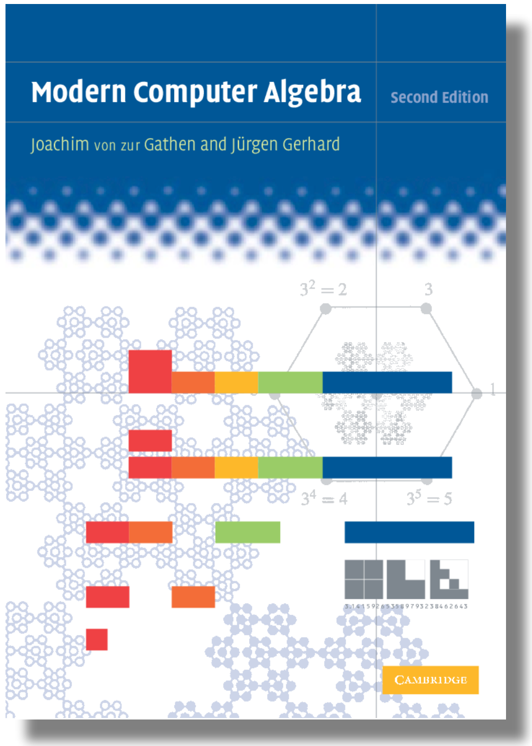
Joint work with Carine Pivoteau

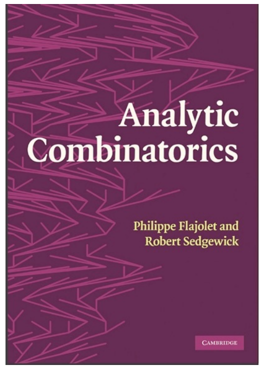
Dedicated to the memory of Philippe Flajolet

AofA Munich, June 2026

I. Introduction



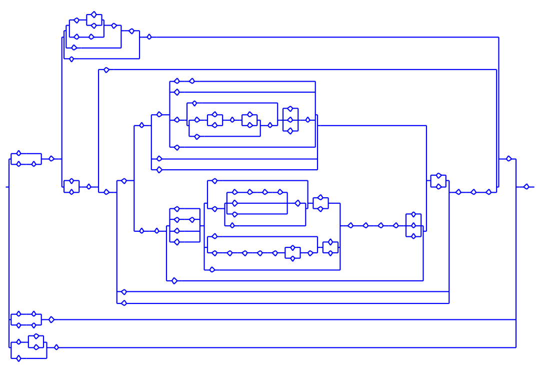




Analytic Combinatorics (Outline)

Def. (Exponential) generating function of (f_n) : $F(z) := \sum_{n=0}^{\infty} f_n \frac{z^n}{n!}$

Combinatorial objects → **Specification** → **GF equations** (f_n : #objects of size n) → **Dominant singularities** → **Local behaviour** → **Asymptotics**

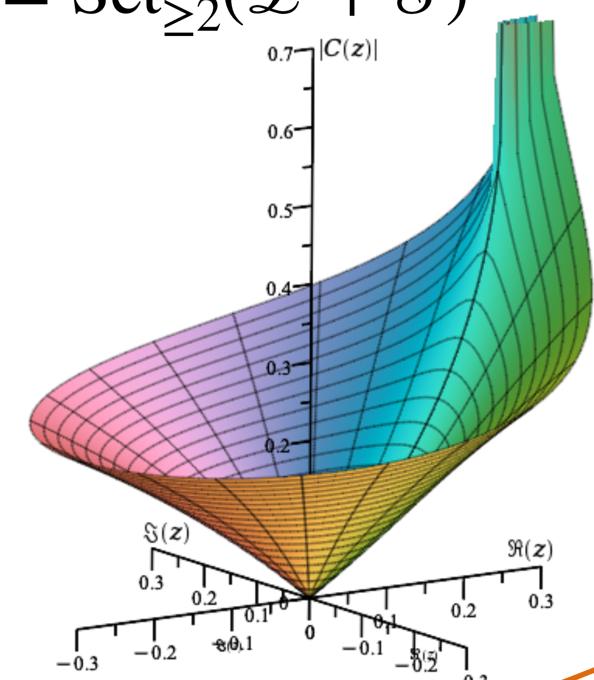


Specification:

$$\begin{cases} \mathscr{C} = \mathscr{Z} + \mathscr{S} + \mathscr{P} \\ \mathscr{S} = \mathrm{Seq}_{\geq 2}(\mathscr{Z} + \mathscr{P}) \\ \mathscr{P} = \mathrm{Set}_{\geq 2}(\mathscr{Z} + \mathscr{S}) \end{cases}$$

GF equations:

$$\begin{cases} C(z) = z + S(z) + P(z) \\ S(z) = \dfrac{(z + P(z))^2}{1 - z - P(z)} \\ P(z) = e^{z+S(z)} - 1 - z - S(z) \end{cases}$$



Dominant singularities:

$$\rho = \ln \frac{\sqrt{5}+1}{2} - \sqrt{5} + 2 \in (0.245, 0.246)$$

If you can specify it, you can analyze it. *(P. Flajolet)*

Local behaviour:

$$C(z) = C(\rho) - \frac{\sqrt{25 + 15\sqrt{5}}}{5}\sqrt{\rho - z} + O(\rho - z)$$

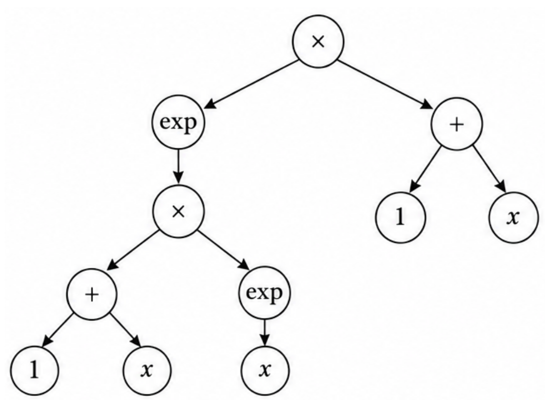
Our goal: a program that can `analyze it'

Asymptotics:

$$\frac{C_n}{n!} \sim \frac{\sqrt{25 + 15\sqrt{5}}}{10\sqrt{\pi}\, n^{3/2} \rho^{n-1/2}}$$

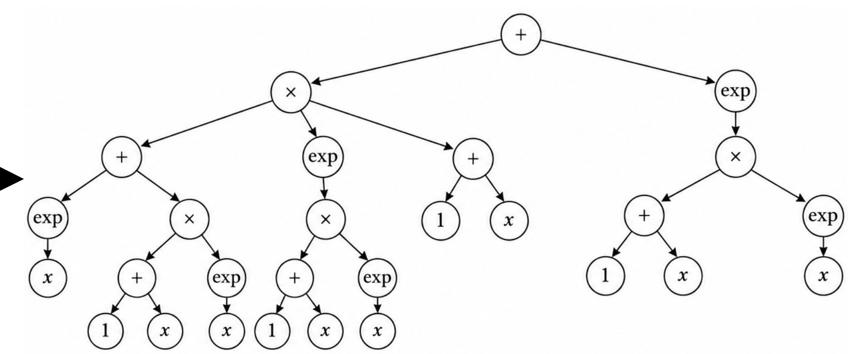
Relation to AofA

An old example:



d/dx

Complexity?



$$f = e^{(1+x)e^x}(1+x)$$

$$f' = \left(e^x + (1+x)\,e^x\right)e^{(1+x)e^x}(1+x) + e^{(1+x)e^x}$$

Principle: $B(z,u) = \sum_{t\in\mathscr{T}} u^{|t'|}\frac{z^{|t|}}{|t|!}$ ⟶ $\dfrac{[z^n]\left(\frac{\partial B(z,u)}{\partial u}\Big|_{u=1}\right)}{[z^n]B(z,1)}$

average over all trees of size n

ΛΥΩ

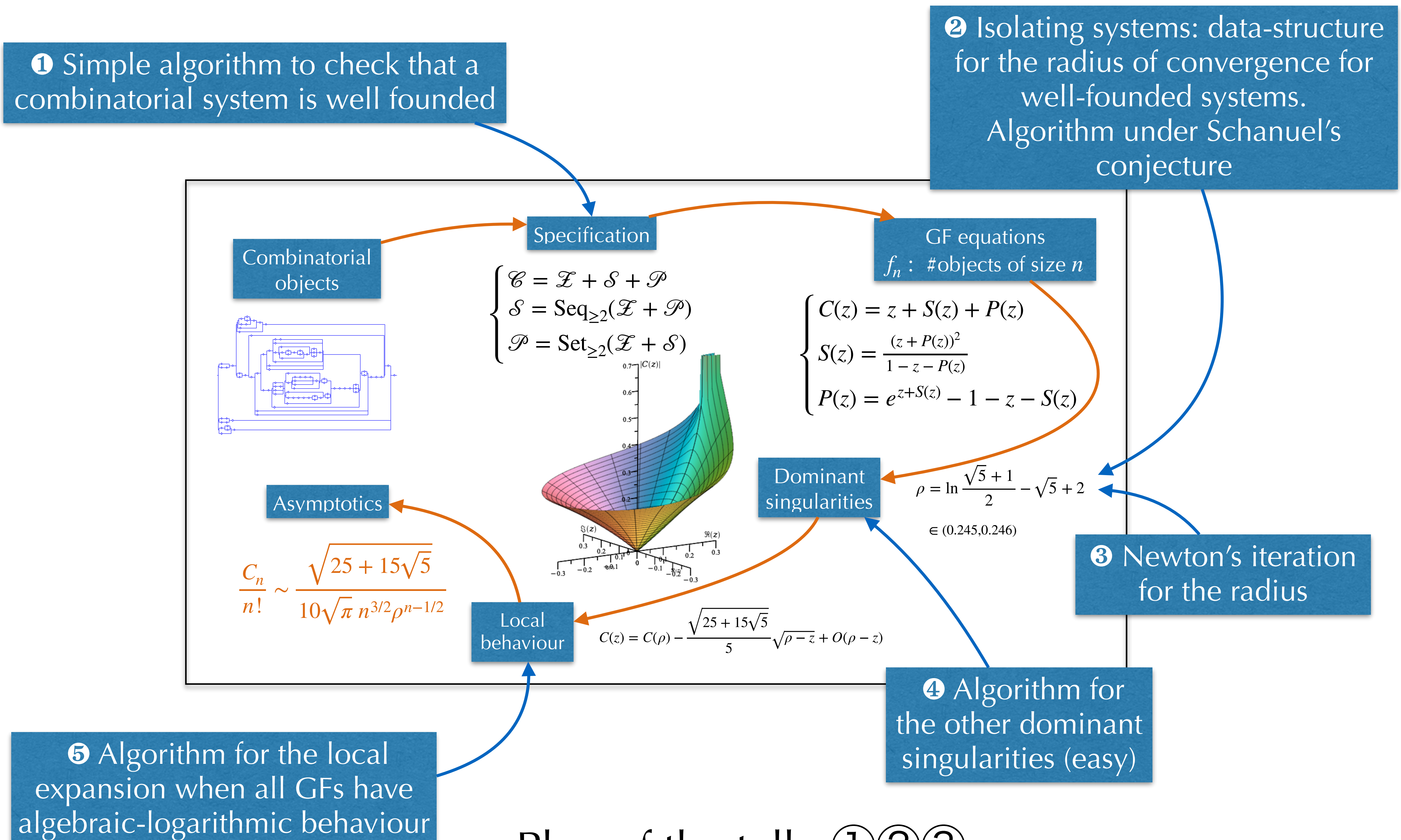
```
                 1/2   1/2   3/2
               Pi     3     n
av_tau_diff_n := --------------- + O(n)
                       6
```

Restricted to systems with explicit GF solution

AofA93

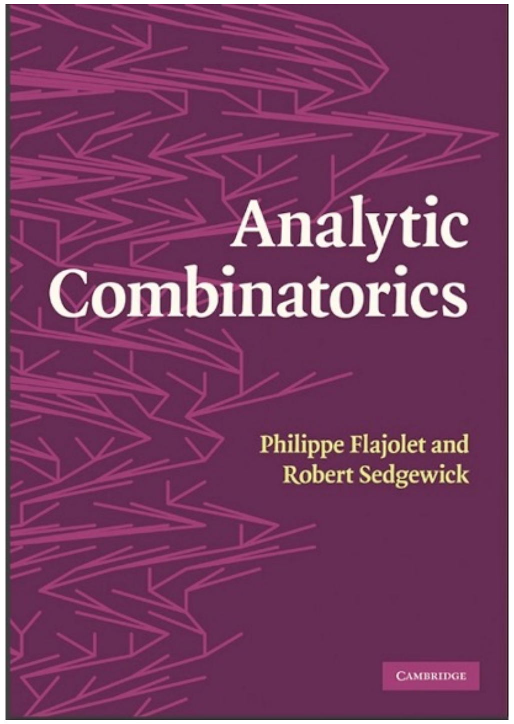
[Flajolet-S.-Zimmermann 1991; Mishna 2003]

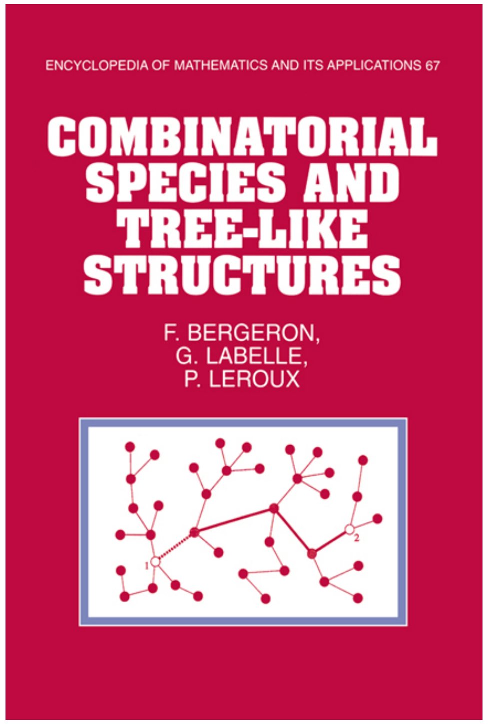
This Work



Plan of the talk: ①②③

II. Well-founded Systems



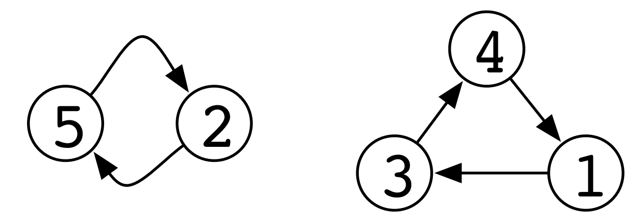


Combinatorial Operations

Language: $0, 1, \mathcal{Z}, +, \times,$ Set, Seq, Cyc and composition

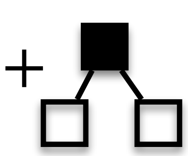
Basic rules: $0 + \mathcal{Y} = \mathcal{Y} + 0 = \mathcal{Y}$ $\quad$ $0 \times \mathcal{Y} = \mathcal{Y} \times 0 = 0,\ 1 \times \mathcal{Y} = \mathcal{Y} \times 1 = \mathcal{Y}$

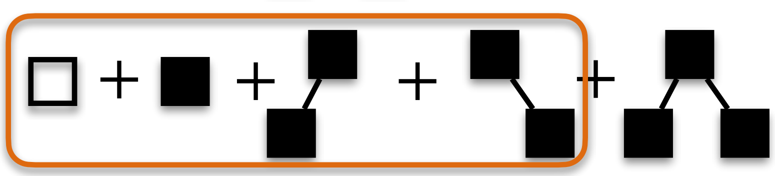
Ex.: $\mathrm{Perm} = \mathrm{Set}(\mathrm{Cyc}(\mathcal{Z}))$ permutations

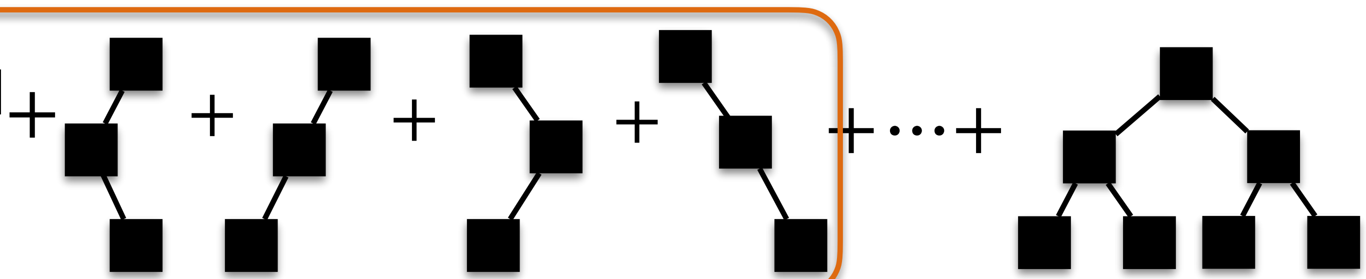


Ex.: first iterations of $\mathcal{Y}^{[n+1]} = 1 + \mathcal{Z} \times \mathcal{Y}^{[n]} \times \mathcal{Y}^{[n]}$ with $\mathcal{Y}^{[0]} = 0$:

$\mathcal{Y}^{[1]} = 1 = \square$

$\mathcal{Y}^{[2]} = 1 + \mathcal{Z} = \square +$  $= \square + \blacksquare$

$\mathcal{Y}^{[3]} = 1 + \mathcal{Z} \times (1 + \mathcal{Z})^2 =$ 

$\mathcal{Y}^{[4]} = \mathcal{Y}^{[3]} +$ 

$(\mathcal{Y}^{[n]})$ converges to binary trees (wrt size)

Combinatorial Systems

Def. The system $\boldsymbol{\mathcal{Y}} = \boldsymbol{\mathcal{H}}(\mathcal{Z}, \boldsymbol{\mathcal{Y}})$ is well founded if

$$\boldsymbol{\mathcal{Y}}^{[n+1]} = \boldsymbol{\mathcal{H}}(\mathcal{Z}, \boldsymbol{\mathcal{Y}}^{[n]}), \quad \boldsymbol{\mathcal{Y}}^{[0]} = \mathbf{0}$$

is defined for all n and converges wrt size.

Set(1) and Cyc(1) are not defined

Quiz: well founded or not?

- ✔ $\mathcal{Y} = 1 + \mathcal{Z} \times \mathcal{Y} \times \mathcal{Y}$ $\quad 1, 1$
- ✘ $\mathcal{Y} = \mathcal{Z} + \mathcal{Y}$ $\quad \mathcal{Z}, 2\mathcal{Z}$
- ✔ $\mathcal{Y} = \mathcal{Z} \times \mathcal{Y}$ $\quad 0, 0$
- ✔ $\mathcal{Y} = \mathcal{Z} \times \mathrm{Set}(\mathcal{Y})$ $\quad \mathcal{Z}, \mathcal{Z}$
- ✔ $\mathcal{Y} = \mathrm{Set}(\mathcal{Z} \times \mathcal{Y})$ $\quad 1, 1$

✔ $\begin{cases} \mathcal{Y}_1 = 1 + \mathcal{Z} \times \mathcal{Y}_1 \\ \mathcal{Y}_2 = \mathcal{Y}_1 + \mathcal{Z} \times \mathcal{Y}_2 \times \mathcal{Y}_2 \end{cases}$ $\quad \binom{1}{0}, \binom{1}{1}, \binom{1}{1}$

✘ $\begin{cases} \mathcal{Y}_1 = 1 + \mathcal{Z} \times \mathcal{Y}_1 \\ \mathcal{Y}_2 = \mathcal{Y}_1 + \mathcal{Y}_2 \times \mathcal{Y}_2 \end{cases}$ $\quad \binom{1}{0}, \binom{1}{1}, \binom{1}{2}$

Thm. $\boldsymbol{\mathcal{Y}} = \boldsymbol{\mathcal{H}}(\mathcal{Z}, \boldsymbol{\mathcal{Y}})$ of dim m is well founded iff the leading terms (smallest structures) of $\boldsymbol{\mathcal{Y}}^{[m]}$ and $\boldsymbol{\mathcal{Y}}^{[m+1]}$ are equal.

Algorithm

Algorithm WellFoundedAndLeadingTerm:

Input: $\mathcal{Y} = \mathcal{H}(\mathcal{Z}, \mathcal{Y})$ a specification in normal form with m equations

Output: Vector of pairs (c, v) corresponding to the vector of leading terms $c\mathcal{Z}^v$ of $\mathcal{Y}$; FAIL if and only if the system is not well founded

```
W_{1:m} := ((0, ∞), ..., (0, ∞))
repeat m+1 times
    V := W
    for i = 1 to m do
        if H_i = 1 then W_i := (1, 0)
        if H_i = Z then W_i := (1, 1)
        if H_i = SET(Y_j) or SEQ(Y_j) then W_i := (1, 0) if V_{j,2} ≠ 0, otherwise FAIL
        if H_i = CYC(Y_j) then W_i := V_j if V_{j,2} ≠ 0, otherwise FAIL
        if H_i = Y_{j_1} × ··· × Y_{j_k} then W_i := (V_{j_1,1} ··· V_{j_k,1}, V_{j_1,2} + ··· + V_{j_k,2})
        if H_i = Y_{j_1} + ··· + Y_{j_k} then
            W_{i,2} := min(V_{j_1,2}, ..., V_{j_k,2}), W_{i,1} := Σ_{ℓ∈[1,k], V_{j_ℓ,2}=W_{i,2}} V_{j_ℓ,1}
    if V = W then return V                    ◁ the system is well founded
if V ≠ W then FAIL
return V
```

0 coordinates are detected and can be removed → 0-free system

Improvement of [Zimmermann 1991]

Generating Functions

$$\mathscr{Y} = \mathscr{H}(\mathscr{Z}, \mathscr{Y}) \quad \mapsto \quad \boldsymbol{Y} = \boldsymbol{H}(z, \boldsymbol{Y})$$

$0, 1, \mathscr{Z}, +, \times$	$0, 1, z, +, \times$
$\mathscr{F}(\mathscr{G}(\mathscr{Z}))$	$F(G(z))$
$\mathrm{Seq}(\mathscr{Z})$	$(1-z)^{-1}$
$\mathrm{Cyc}(\mathscr{Z})$	$\ln \frac{1}{1-z}$
$\mathrm{Set}(\mathscr{Z})$	$\exp(z)$

Def. $\boldsymbol{H}(z, \boldsymbol{Y}) \in \mathbb{R}_{\geq 0}[[z, \boldsymbol{Y}]]$ analytic at $\boldsymbol{0}$, converges in $\mathrm{Dom}(\boldsymbol{H}) \subset \mathbb{C}^{m+1}$.

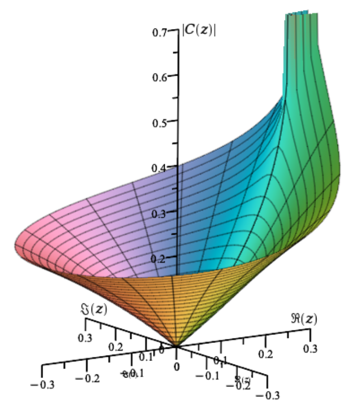
$\boldsymbol{Y} = \boldsymbol{H}(z, \boldsymbol{Y})$ well founded if:

1. $\boldsymbol{U}^{[0]} := \boldsymbol{0}, \ldots, \boldsymbol{U}^{[m+1]} := \boldsymbol{H}(0, \boldsymbol{U}^{[m]})$ belong to $\mathrm{Dom}(\boldsymbol{H})$;
2. the Jacobian matrix $\partial \boldsymbol{H}/\partial \boldsymbol{Y}$ is nilpotent at $(0, \boldsymbol{U}^{[m]})$.

Prop. $\mathscr{Y} = \mathscr{H}(\mathscr{Z}, \mathscr{Y})$ well founded and 0-free $\Rightarrow \boldsymbol{Y} = \boldsymbol{H}(z, \boldsymbol{Y})$ well founded.

Def. $\boldsymbol{Y} = \boldsymbol{H}(z, \boldsymbol{Y})$ well founded.
The generating function solution is the unique solution analytic around 0 s.t. $\boldsymbol{Y}(0) = \boldsymbol{U}^{[m]}$.

Implicit function thm



Extends defs. by [Drmota, Bell-Burris-Yeats, Pivoteau-S.-Soria, Banderier-Drmota]

III. Radius of Convergence of GF Solutions of Well-Founded Systems

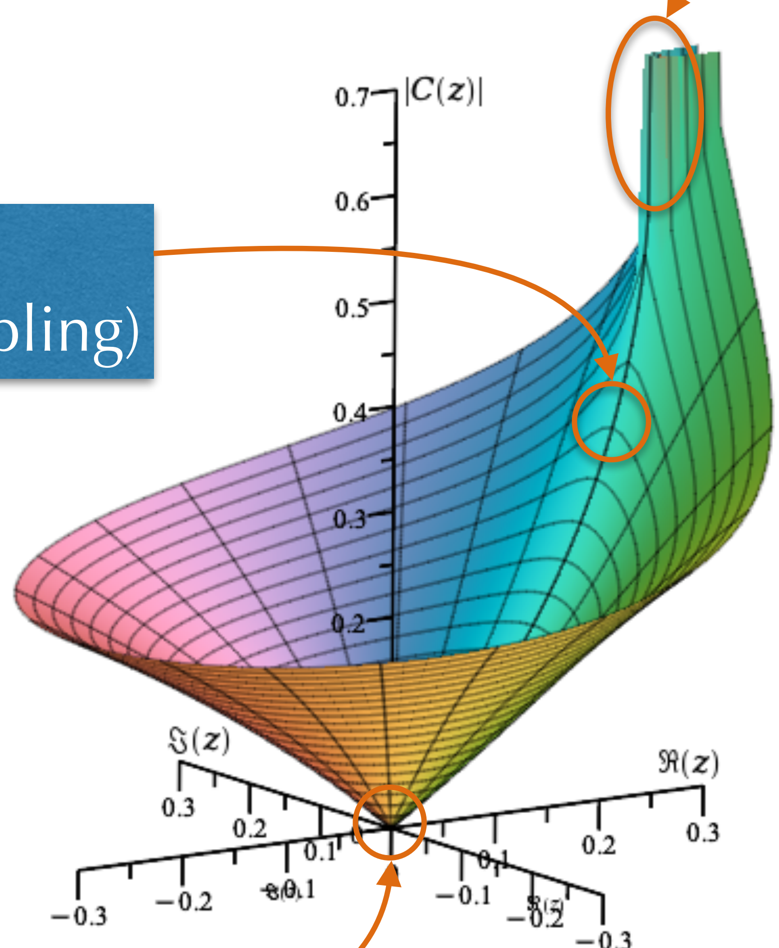
Radius of Convergence

$$A(z) = \sum_{n\geq 0} a_n z^n$$

Thm [Pringsheim] $\forall n, a_n \geq 0 \Rightarrow$ singularity at radius of convergence.

wanted here

also here
(Boltzmann sampling)



GF known here

Notation: ρ radius of convergence of the GF solution

Aim: isolating system for ρ
System of equations and intervals with a unique solution, ρ one of the coordinates

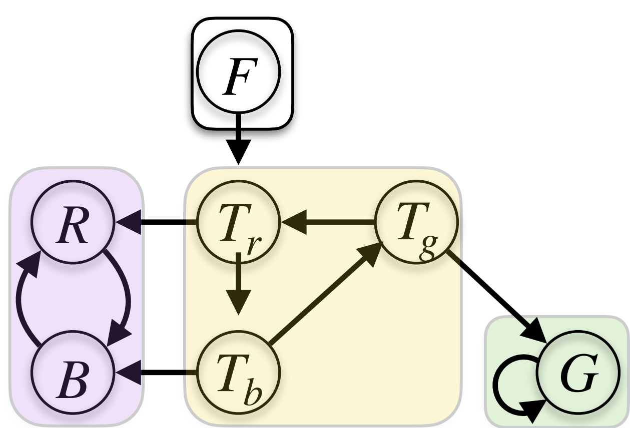
data-structure

Irreducible Components, Dominant Eigenvalue

$$
\begin{array}{ll}
H_1 & F(z) = e^{T_r(z)} \\
H_2 & T_r(z) = R(z)\,T_b(z)^2 + z \\
 & T_b(z) = B(z)\,T_g(z)^2 \\
 & T_g(z) = G(z)\,T_r(z)^2 \\
H_3 & R(z) = z^3 + \dfrac{z}{1-B(z)} \\
 & B(z) = \dfrac{z}{1-R(z)} \\
H_4 & G(z) = G(z)^2 + z
\end{array}
$$

$$
\frac{\partial \boldsymbol{H}}{\partial \boldsymbol{Y}} = \begin{pmatrix}
0 & e^{T_r} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2RT_b & 0 & T_b^2 & 0 & 0 \\
0 & 0 & 0 & 2BT_g & 0 & T_g & 0 \\
0 & 2GT_r & 0 & 0 & 0 & 0 & T_r^2 \\
0 & 0 & 0 & 0 & 0 & \dfrac{z}{(1-B)^2} & 0 \\
0 & 0 & 0 & 0 & \dfrac{z}{(1-R)^2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2G
\end{pmatrix}
$$

Def. $\Lambda_{\boldsymbol{H}}(a, \boldsymbol{b})$ dominant eigenvalue of $\partial \boldsymbol{H}/\partial \boldsymbol{Y}(a, \boldsymbol{b})$.



Strongly connected subgraphs
↔ *well-founded* irreducible
components $H_1, \ldots, H_4$

Prop. $\boldsymbol{Y} = \boldsymbol{H}(z, \boldsymbol{Y})$ well founded; $\boldsymbol{Y}(z)$ GF soln.

$$0 \le a < \rho \Rightarrow 0 \le \Lambda_{\boldsymbol{H}}(a, \boldsymbol{Y}(a)) < 1.$$

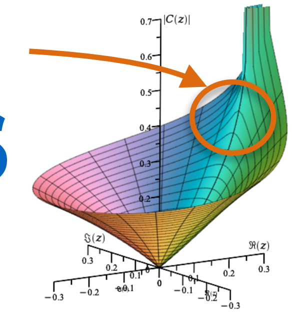
Proof. Take $\boldsymbol{H}$ irreducible; chain rule & multiply by left eigenvector

$$\underbrace{\boldsymbol{v} \cdot \boldsymbol{Y}'(a)}_{>0} = \Lambda_{\boldsymbol{H}}(a, \boldsymbol{Y}(a))\,\underbrace{\boldsymbol{v} \cdot \boldsymbol{Y}'(a)}_{>0} + \underbrace{\boldsymbol{v} \cdot \frac{\partial \boldsymbol{H}}{\partial z}(a, \boldsymbol{Y}(a))}_{>0}$$

Perron-Frobenius

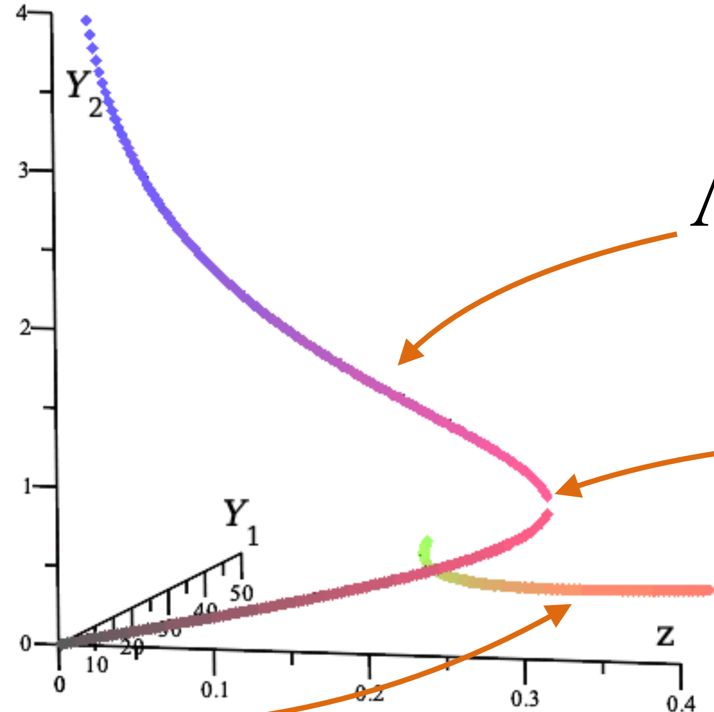
[Bell-Burris-Yeats 2010]

Nonnegative Solutions of GF Systems



$$\begin{cases} Y_1 = z e^{Y_1 Y_2} \\ Y_2 = z + z \dfrac{1 + Y_1 Y_2}{1 - Y_1^2} \end{cases}$$

Out of Dom($\boldsymbol{H}$)



$\Lambda(z, Y_1, Y_2) > 1$

$(\rho, Y_1(\rho), Y_2(\rho))$

An extension of the previous result helps recognize values of $\boldsymbol{Y}(z)$.

Thm. $\boldsymbol{Y} = \boldsymbol{H}(z, \boldsymbol{Y})$ well founded, $(a, \boldsymbol{b}) \in \mathbb{R}_{\geq 0}^{m+1} \cap \mathrm{Dom}(\boldsymbol{H})$ satisfies $\boldsymbol{b} = \boldsymbol{H}(a, \boldsymbol{b})$. Then, $a \leq \rho$ and $(\Lambda_{\boldsymbol{H}}(a, \boldsymbol{b}) \leq 1 \Leftrightarrow \boldsymbol{b} = \boldsymbol{Y}(a))$.

Branch following and elimination avoided

Consequence: $(\rho, \boldsymbol{Y}(\rho))$ is either on the boundary of Dom($\boldsymbol{H}$) or a solution to $\Sigma := \{\boldsymbol{Y} = \boldsymbol{H}(z, \boldsymbol{Y}), \Lambda_{\boldsymbol{H}}(z, \boldsymbol{Y}) = 1\}$.

Implicit fcn thm

Radius by Irreducible Components

H_1: $F(z) = \mathrm{e}^{T_r(z)}$

H_2:
$T_r(z) = R(z)\,T_b(z)^2 + z$
$T_b(z) = B(z)\,T_g(z)^2$
$T_g(z) = G(z)\,T_r(z)^2$

H_3:
$R(z) = z^3 + \frac{z}{1 - B(z)}$
$B(z) = \frac{z}{1 - R(z)}$

H_4: $G(z) = G(z)^2 + z$

$$H_4 : \Lambda_{H_4} = 1 \rightarrow \rho_4 = 1/4$$

$$H_3 : \Lambda_{H_3} = 1 \rightarrow \rho_3 \in [.246,.247]\,(< \rho_4)$$

$R(\rho_3) \in (0.50,0.51), B(\rho_3) \in (0.49,0.51)$
$\rightarrow$ inside $\mathrm{Dom}(H_3)$

$$H_2 : \Lambda_{H_2}(\rho_3, \boldsymbol{T}(\rho_3)) \approx .13 < 1 \rightarrow \rho_2 = \rho_3$$

on the boundary of $\mathrm{Dom}(H_2)$

$$H_1 : \rho_1 = \rho_3 = \rho$$

Computations: next part

Isolating system for ρ found:

$$H_3 \cup \{\det(I - \frac{\partial H_3}{\partial(R, B)}) = 0, 0 < z < .247\}$$

Algorithm for Irreducible Components

Numerical Algorithm Irr-Radius: Radius of convergence (recursive case)

Input: $\mathcal{Y} = \mathcal{H}(\mathcal{Z}, \mathcal{U}, \mathcal{Y})$ an irreducible constructive component;
$\mathcal{U} = \mathcal{G}(\mathcal{Z}, \mathcal{U})$ a constructive system defining $\mathcal{U}$;
$r_{\boldsymbol{U}}$ an isolating system for the radius of convergence of the generating function $\boldsymbol{U}$ of $\mathcal{U}$.

From the previous systems

Output: an isolating system for ρ the radius of convergence of the generating function $(\boldsymbol{Y}, \boldsymbol{U})$ of $(\mathcal{Y}, \mathcal{U})$.

```
Σ := GFSystem(𝒰 = 𝒢(𝒵,𝒰), 𝒴 = ℋ(𝒵,𝒰,𝒴))
Σ' := Σ ∪ {det(Id − ∂H/∂Y) = 0}                         ◁ characteristic system
if H is linear then Σ' := Σ' \ {Y = H(z,U,Y)}            ◁ Σ' does not depend on Y
if Σ' has a solution (a,U,B) with a < r_U and Λ_H(a,B) = 1 then   ◁ decide existence
  | Refining the intervals in Σ' and r_U,                ◁ from the isolating intervals
  |    compute I(a,U,B) isolating intervals for a        ◁ solve system
  | return Σ ∪ I(a,U,B)
else return r_U              ◁ the component does not reduce the radius of convergence
```

difficult

Existential Problems

We have to solve problems of the form

$$(\mathscr{P}) \quad \exists (x_1, \ldots, x_n) \in \mathbb{R}^n, F_1(\boldsymbol{x}) = 0 \wedge \cdots \wedge F_n(\boldsymbol{x}) = 0 \wedge F_{n+1}(\boldsymbol{x}) > 0 \wedge \cdots \wedge F_{n+k}(\boldsymbol{x}) > 0$$

1. F_i polynomials in $\mathbb{Q}[x_1, \ldots, x_n]$
   $\to (\mathscr{P})$ decidable

Costly in theory and practice

Seq: $A(1-B) - 1 = 0,\ 0 \le B < 1$

Set: $A - e^B = 0,$
Cyc: $e^A(1-B) - 1 = 0,\ 0 \le B < 1$

2. F_i polynomials in $\mathbb{Q}[x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n}]$
   $\to (\mathscr{P})$ decidable assuming

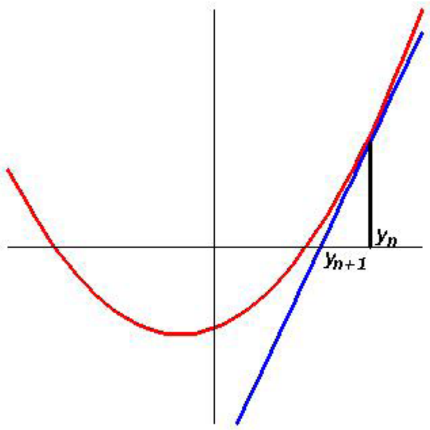
Purely theoretical

Schanuel's conjecture: if $\alpha_1, \ldots, \alpha_n$ in $\mathbb{R}$ are linearly independent over $\mathbb{Q}$, then $\operatorname{trdeg}_{\mathbb{Q}}(\alpha_1, \ldots, \alpha_n, e^{\alpha_1}, \ldots, e^{\alpha_n}) \ge n$.

Thm. $\boldsymbol{Y} = \boldsymbol{H}(z, \boldsymbol{Y})$ well founded; ρ radius of convergence of the GF soln. An isolating system for ρ can be computed, assuming SC.

[Tarski 1940, Macintyre & Wilkie 1996]

IV. Numerical Tools



Newton's Iteration

For a square system $\boldsymbol{F}(\boldsymbol{Y}) = \mathbf{0}$, the iteration is

$$\boldsymbol{Y}^{[n+1]} = \boldsymbol{Y}^{[n]} - \left(\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{Y}}\right)^{-1} \cdot \boldsymbol{F}(\boldsymbol{Y}^{[n]})$$

Prop. $\Sigma := \{\boldsymbol{Y} = \boldsymbol{H}(z, \boldsymbol{Y})\}$ well founded, ρ radius of GF soln $\boldsymbol{Y}$

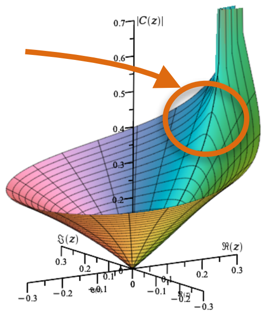
1. if $(\rho, \boldsymbol{Y}(\rho)) \in \mathrm{Dom}(\boldsymbol{H})$, let $\Sigma' = \Sigma \cup \{\det(\mathrm{Id} - \frac{\partial \boldsymbol{H}}{\partial \boldsymbol{Y}}(z, \boldsymbol{Y}) = 0\}$;
2. if $\boldsymbol{Y}_k(\rho) > 1$, let $\Sigma' = \Sigma \cup \{Y_k = 1\}$.

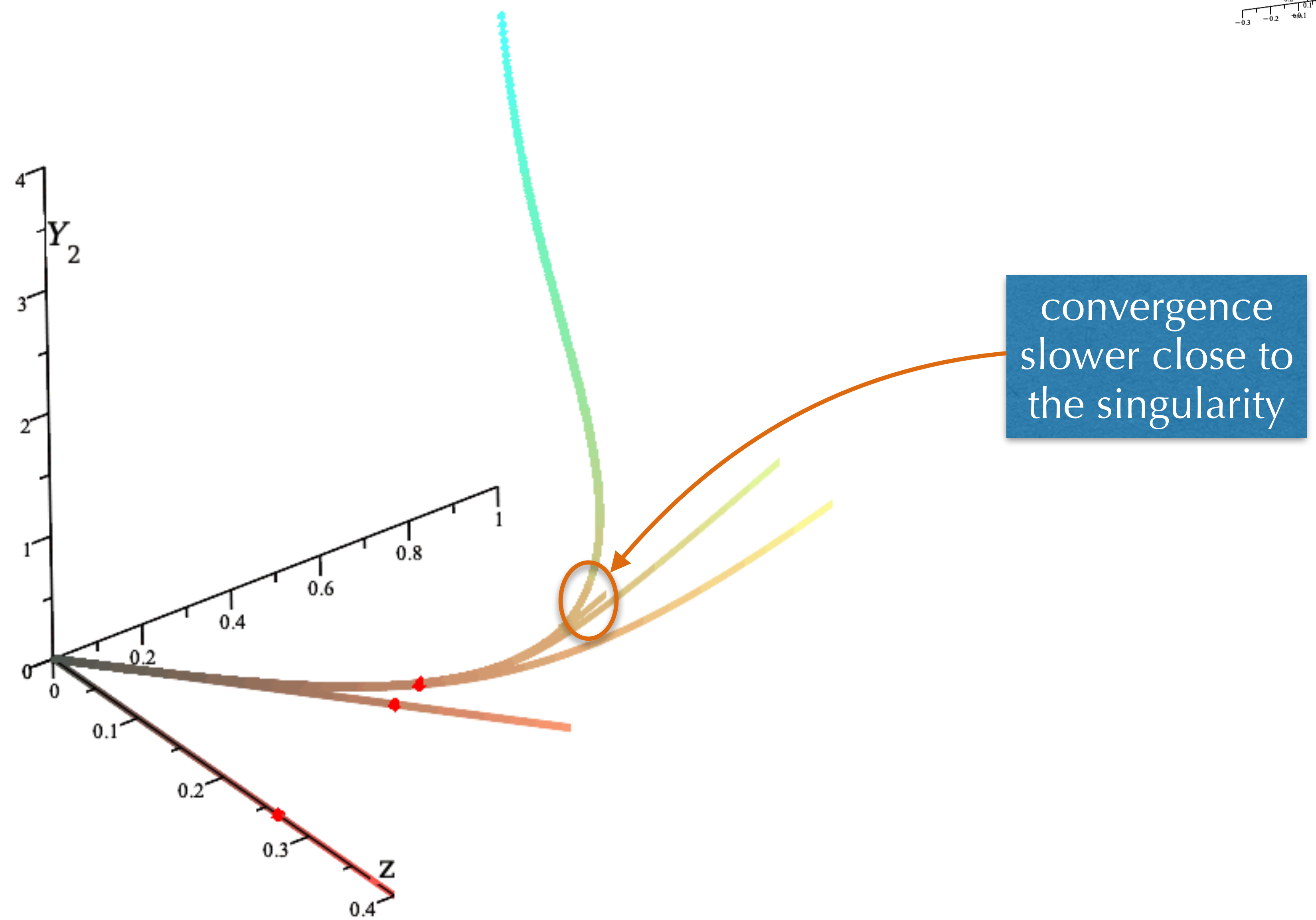
boundary of $\mathrm{Dom}(\boldsymbol{H})$

Then Σ' has a solution; Newton's iteration for Σ' converges quadratically to it when started in a small enough neighbourhood of it.

Where should we start?

Values of Generating Functions





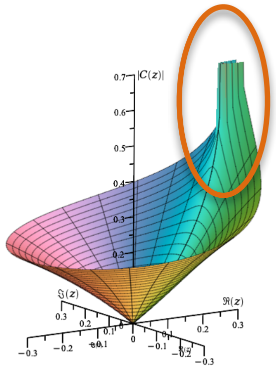
$$\begin{cases} Y_1 = z e^{Y_1 Y_2} \\ Y_2 = z + z \dfrac{1 + Y_1 Y_2}{1 - Y_1^2} \end{cases}$$

Prop. $\Sigma := \{\boldsymbol{Y} = \boldsymbol{H}(z, \boldsymbol{Y})\}$ well founded. Newton's iteration started at 0 converges to $\boldsymbol{Y}(a)$, for $0 \leq a \leq \rho$.

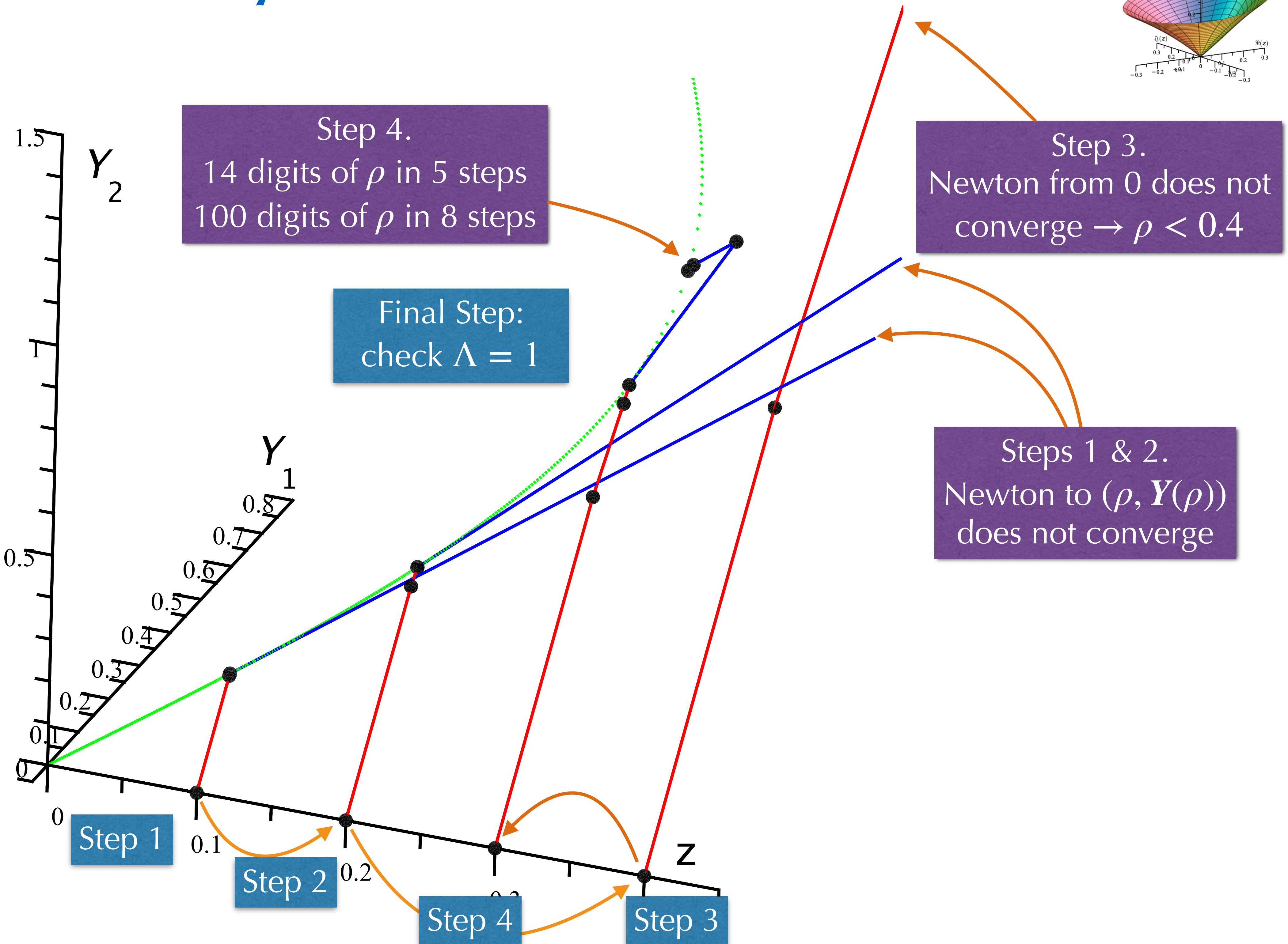
New: a posteriori bounds available

[Pivoteau-S.-Soria 2012]

Dichotomy with Both Iterations



$$\begin{cases} Y_1 = ze^{Y_1Y_2} \\ Y_2 = z + z\dfrac{1+Y_1Y_2}{1-Y_1^2} \end{cases}$$



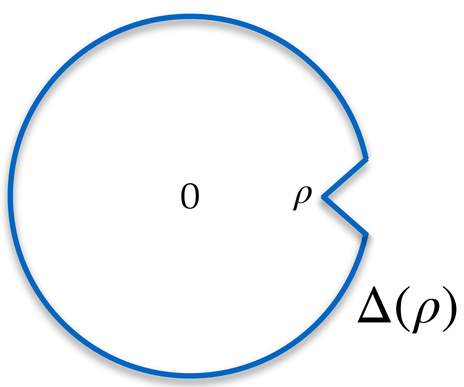
→ Isolating system

Difficult decisions limited to cases of coincidence to 100s of digits

V. Singularity Analysis of Well-Founded Systems

Transfer Theorems

If $A(z)$ is analytic in $\Delta(\rho)$ and satisfies



$$A(z) \underset{z\to\rho}{\sim} c\left(1-\frac{z}{\rho}\right)^{\alpha}\log^{m}\frac{1}{1-z/\rho}$$

Singularity of *algebraic-logarithmic* type

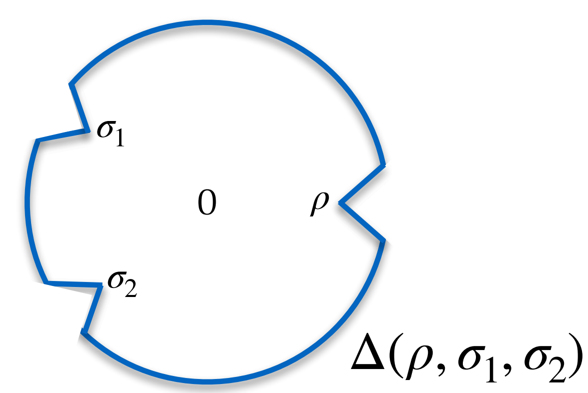
$(\alpha \notin \mathbb{N})$

then

$$a_n \underset{n\to\infty}{\sim} c\rho^{-n}\frac{n^{-\alpha-1}}{\Gamma(-\alpha)}\log^{m} n$$

full asymptotic expansion available

If $A(z)$ is analytic in $\Delta(\rho, \sigma_1, \ldots, \sigma_k)$ and satisfies then the result is obtained by adding contributions from each dominant singularity.



[Flajolet-Odlyzko1990]

Classification of Local Behaviours

Thm. $Y(z)$ a GF from a well-founded system; ρ its radius of convergence. As $z \to \rho$, either:

algebraic-logarithmic type

1. $Y(z) = c_0 + c_1(1 - z/\rho)^{1/r} + c_2(1 - z/\rho)^{2/r} + \cdots$ (r a power of 2)
2. $Y(z) \sim C(1 - z/\rho)^{\alpha} \log^k \dfrac{1}{1 - z/\rho}$ $\alpha \le 0, k \in \mathbb{N}$ $(\alpha, k) \neq (0,0)$

super-polynomial growth

3. $\log Y(z) \ge \ln^2(1 - z/\rho)$.

Algorithm (and proof) by induction on the connected components. No new decision needed.

Ex.: $\mathcal{Y} = \mathrm{Set}(\mathcal{Z} \times \mathrm{Cyc}(2\mathcal{Z}^2) \times \mathrm{Set}(\mathrm{Seq}(\mathcal{Z}^3)))$ $\rho = 1/\sqrt{2}$

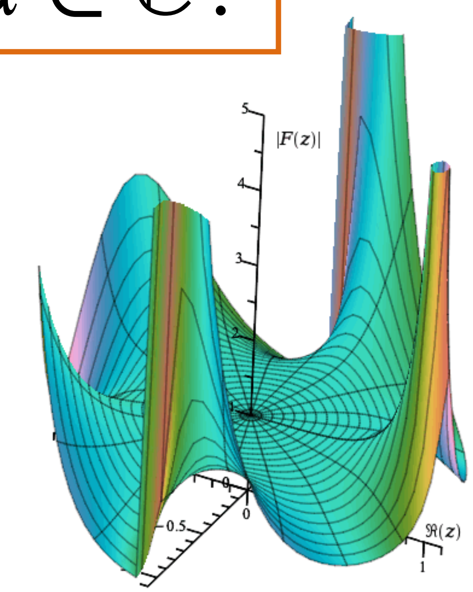
$$Y(z) \sim 2^{\alpha}(1 - z/\rho)^{\alpha},\ \alpha = -\frac{\sqrt{2}}{2}\exp\left(\frac{8 + 2\sqrt{2}}{7}\right) \mapsto Y_n \sim \frac{2^{\alpha}}{\Gamma(-\alpha)} 2^{n/2} n^{-\alpha-1}$$

Extends parts of [Flajolet-S.-Zimmermann 1991; Drmota 1997; Banderier-Drmota 2015]

Other Dominant Singularities

Thm. $\boldsymbol{Y} = \boldsymbol{H}(z, \boldsymbol{Y})$ well founded.

1. The dominant singularities of the GF solution are of the form $\sigma = \rho \exp(2i\pi k/q)$; q can be computed from $\boldsymbol{H}$.
2. If alg.-log. type at ρ, then also at σ, with exponent $\alpha \in \mathbb{C}$.



$\exp\left(z \ln\left(\frac{1}{1-z^4}\right)\right)$

Ex. $\mathcal{F} = \mathrm{Set}(\mathcal{Z} \times \mathrm{Cyc}(\mathcal{Z}^4))$

$$
\begin{aligned}
F(z) &= \frac{1/4}{1-z} - \frac{1}{4}\ln\frac{1}{1-z} + c + O((1-z)\ln^2(1-z)) && z \to 1 \\
&= 4(1+z) + O((1+z)^2 \ln(1+z)) && z \to -1 \\
&= 4^{-i}(1-z/i)^{-i} + O((1-z/i)^{-i+1}\ln(1-z/i)) && z \to i
\end{aligned}
$$

Adding contributions (and polishing by hand) gives

$$
F_n = \frac{1}{4} + \left(2\sqrt{\frac{\sinh(\pi)}{\pi}}\cos\left(\frac{\pi n}{2} - \ln(n/4) + \arg\Gamma(i)\right) - \frac{1}{4}\right)\frac{1}{n} + O\left(\frac{\ln^2 n}{n^2}\right)
$$

Conclusion

$$\begin{cases} \mathscr{C} = \mathscr{Z} + \mathscr{S} + \mathscr{P} \\ \mathscr{S} = \mathrm{Seq}_{\geq 2}(\mathscr{Z} + \mathscr{P}) \\ \mathscr{P} = \mathrm{Set}_{\geq 2}(\mathscr{Z} + \mathscr{S}) \end{cases}$$

$$\frac{C_n}{n!} \sim \frac{\sqrt{25 + 15\sqrt{5}}}{10\sqrt{\pi}\, n^{3/2} \rho^{n-1/2}}$$

If you can specify it, ~~you~~ can analyze it.

a program?

An almost complete algorithmic
chain for well-founded systems

Next steps:

1. corresponding implementation (we have worksheets);
2. super-polynomial but less than exponential growth (below Hayman's class);
3. ordinary generating functions with Set, Cyc, at least in easy cases.

Thank you!