Algorithmic Tools for the Asymptotics of Linear Recurrences

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Problem

$$p_0(n)a_{n+k} + \cdots + p_k(n)a_n = 0, \quad 0 \notin p_0(\mathbb{N}), \quad p_i \in \mathbb{Z}[n]$$

Aim: given $a_0, \ldots, a_{k-1}$, predict the behaviour of a_n as $n \to \infty$.

Simplified version: ``compute'', when they exist,

$\rho, \alpha, m, c \neq 0$ such that $a_n \sim c\rho^n n^\alpha \log^m n$

local
(at ∞)

Singularity Analysis

sequence

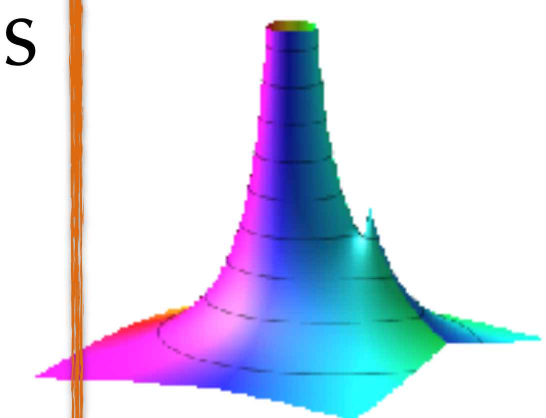
$$(a_n) \mapsto A(z) := \sum_{n \geq 0} a_n z^n$$

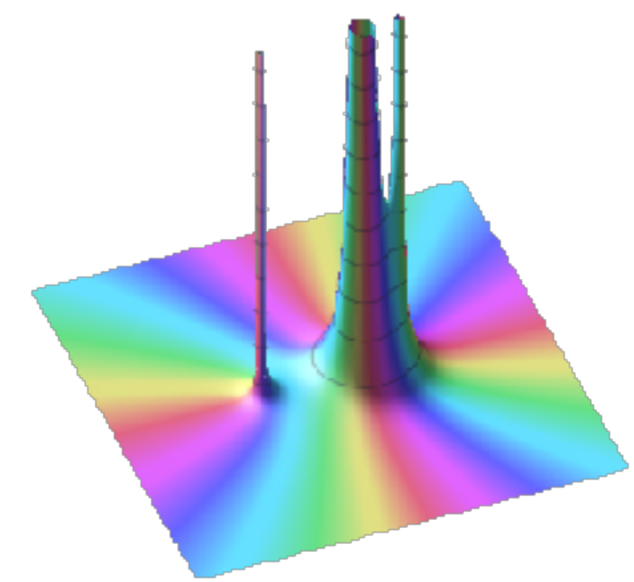
Generating function captures some structure

$$a_n = \frac{1}{2\pi i} \oint \frac{A(z)}{z^{n+1}}\, dz$$

A 3-Step Method:

1. Locate dominant singularities
   a. singularities; b. dominant ones
2. Compute local behaviour
3. Translate into asymptotics





$$A(z) \underset{z \to \rho}{\sim} c \left(1 - \frac{z}{\rho}\right)^{\alpha} \log^m \frac{1}{1 - \frac{z}{\rho}} \qquad a_n \underset{n \to \infty}{\sim} c\, \rho^{-n} \frac{n^{-\alpha-1}}{\Gamma(-\alpha)} \log^m n$$

$(\alpha \notin \mathbb{N})$

full asymptotic expansion available

[Flajolet-Odlyzko1990]

Next: computation of ρ, α, m, c

P-recursivity & D-finiteness

$$(a_n) \mapsto A(z) := \sum_{n\geq 0} a_n z^n$$

(a_n) P-recursive $\Longleftrightarrow$ **A(z) D-finite**

$$p_0(n)a_{n+k} + \cdots + p_k(n)a_n = 0 \qquad q_0(z)A^{(\ell)}(z) + \cdots + q_\ell(z)A(z) = 0$$

Classical properties of LDEs:

1. singularities satisfy $q_0(\rho) = 0$;
2. one can compute a basis of formal solutions at (regular) singular points, of the form

$$\left(1 - \frac{z}{\rho}\right)^{\alpha} \log^m\left(\frac{1}{1-\frac{z}{\rho}}\right)(1 + \cdots), \quad \alpha \in \overline{\mathbb{Q}}, m \in \mathbb{N}.$$

convergent

local

(at ρ)

More recently (M. Mezzarobba's `ore_algebra_analytic`):

certified analytic continuation ($\rightarrow$ c numerically).

Semi-decision

Asymptotics of P-recursive *Combinatorial* Sequences

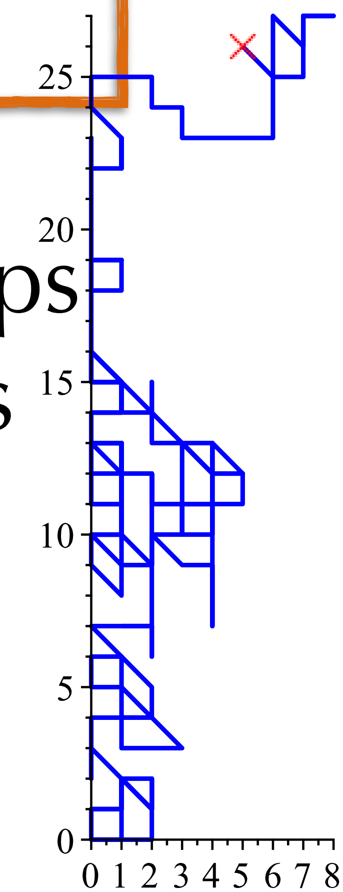
Thm. [Katz70,Chudnovsky85,André00]

$a_0+a_1z+\dots$ D-finite, a_i integers, radius in $(0,\infty)$, then its singular points are regular with **rational** exponents

$$a_n \sim \sum_{\substack{(\lambda,\alpha,k)\in \text{ finite set}\\ \text{in } \overline{\mathbb{Q}}\times\mathbb{Q}\times\mathbb{N}}} \lambda^{-n} n^{\alpha} \log^k(n)\, f_{\lambda,\alpha,k}\left(\frac{1}{n}\right).$$

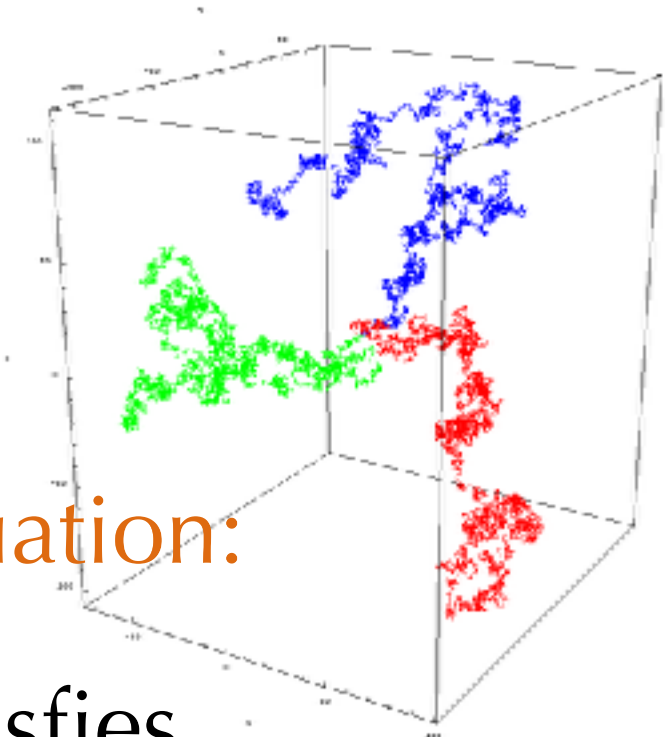
Ex. The number a_n of walks from the origin taking n steps *{N,S,E,W,NW}* and staying in the first quadrant behaves like $C\lambda^{-n}n^{\alpha}$ with $\alpha \notin \mathbb{Q}$ → not D-finite.

$$\alpha = -1 + \frac{\pi}{\arccos(u)}, \quad 8u^3 - 8u^2 + 6u - 1 = 0, \quad u > 0.$$



[Bostan-Raschel-S.14]

Ex: Pólya's 3D Random Walk



Start from the origin in $\mathbf{Z}^d$;
move one step along one of the axes; repeat.
What is the probability p_d of returning to 0?

Numerical approximation by analytic continuation:

1. $u_n :=$ **P**(3D-walk returns to 0 in 2n steps) satisfies

$$(2n+3)(2n+1)(n+1)u_n - 2(2n+3)(10n^2+30n+23)u_{n+1} + 36(n+2)^3 u_{n+2} = 0$$

2. $a_n := \sum_{k=0}^{n} u_k \to c := \frac{1}{1-p_3}$ converges slowly (1 is a singularity)

3. Given $a_0, a_1, a_2,$ MM's code produces 100 digits of c, c_2, c_3 s.t.

$$A(z) \approx c\left(\frac{1}{1-z} + \cdots\right) + c_2\left(\frac{1}{\sqrt{1-z}} + \cdots\right) + c_3\,(1 + \cdots)$$ in 0.4 sec.

$$c = \frac{\sqrt{6}}{32\pi^3}\Gamma\left(\frac{1}{24}\right)\Gamma\left(\frac{5}{24}\right)\Gamma\left(\frac{7}{24}\right)\Gamma\left(\frac{11}{24}\right)$$ not accessible to the algorithms presented here.

[Watson39;Glasser-Zucker77;Koutschan *et alii* 13,16]

Example: Apéry's Sequences

$$a_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2, \qquad b_n = a_n \sum_{k=1}^{n} \frac{1}{k^3} + \sum_{k=1}^{n} \sum_{m=1}^{k} \frac{(-1)^m \binom{n}{k}^2 \binom{n+k}{k}^2}{2m^3 \binom{n}{m} \binom{n+m}{m}}$$

and $c_n = b_n - \zeta(3) a_n$ have generating functions that satisfy

vanishes at 0,

$\alpha = 17 - 12\sqrt{2} \simeq 0.03,$

$\beta = 17 + 12\sqrt{2} \simeq 34.$

$$z^2(z^2 - 34z + 1)y''' + \cdots + (z-5)y = 0$$

In the neighborhood of α, all solutions behave like

$$\text{analytic} - \mu\sqrt{\alpha - z}(1 + (\alpha - z)\text{analytic}).$$

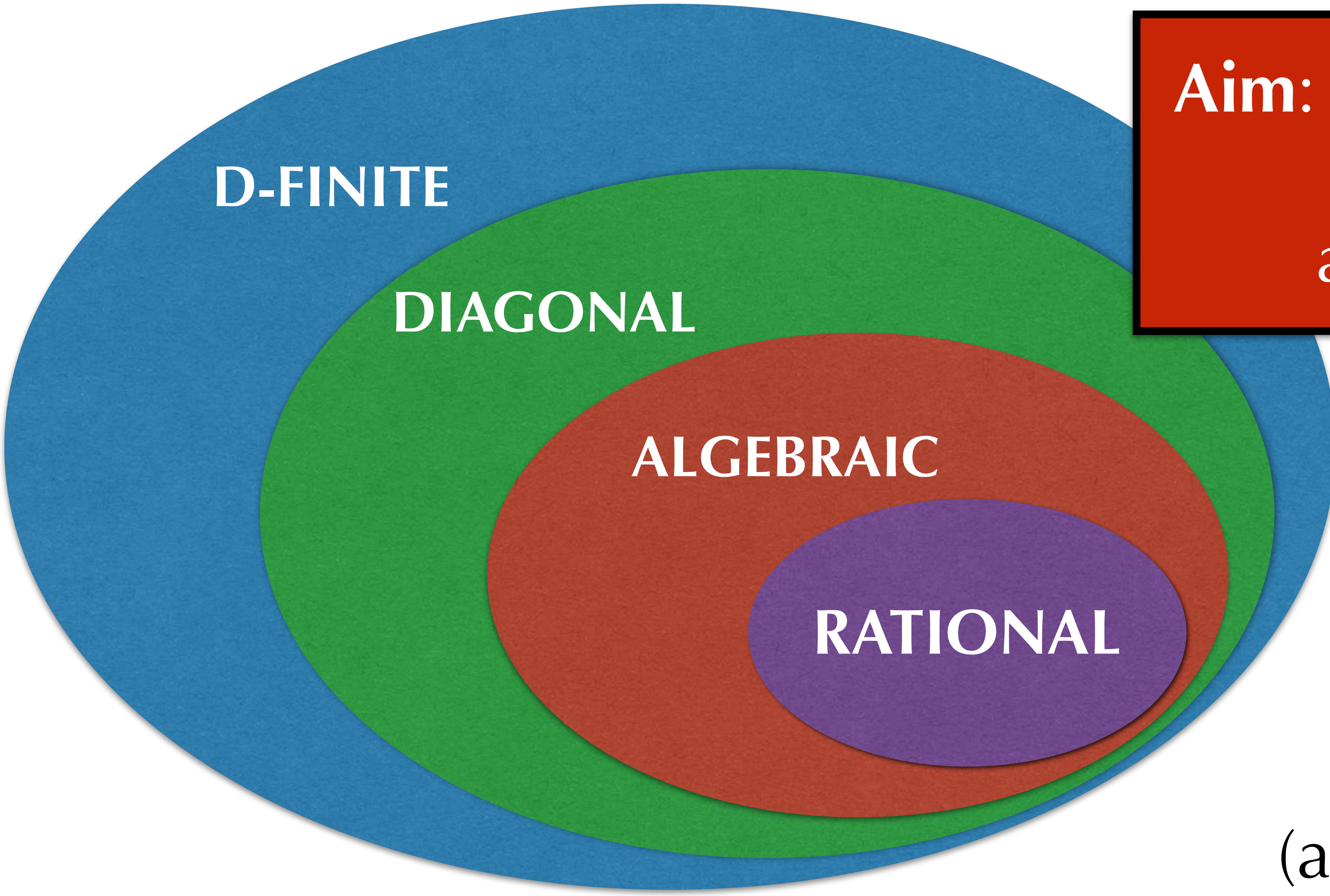
Mezzarobba's code gives $\mu_a \simeq 4.55, \quad \mu_b \simeq 5.46, \quad \mu_c \simeq 0.$

Slightly more work gives $\mu_c = 0$, then $c_n \approx \beta^{-n}$

and eventually, a proof that $\zeta(3)$ is irrational.

[Apéry1978]

Univariate Generating Functions



Aim: asymptotics of the coefficients, automatically.

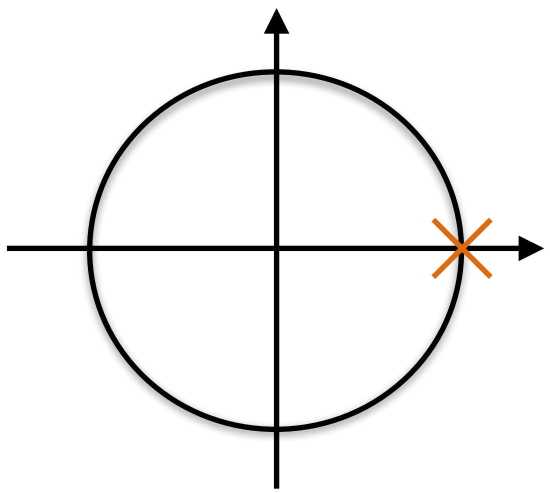
More structure
→
more complete algorithm
(analytic continuation easier)

Def diagonal: later.

Christol's conjecture: All differentially finite power series with integer coefficients and radius of convergence in $(0,\infty)$ are diagonals.

Pringsheim's Theorem

$$(a_n) \mapsto A(z) := \sum_{n \geq 0} a_n z^n$$



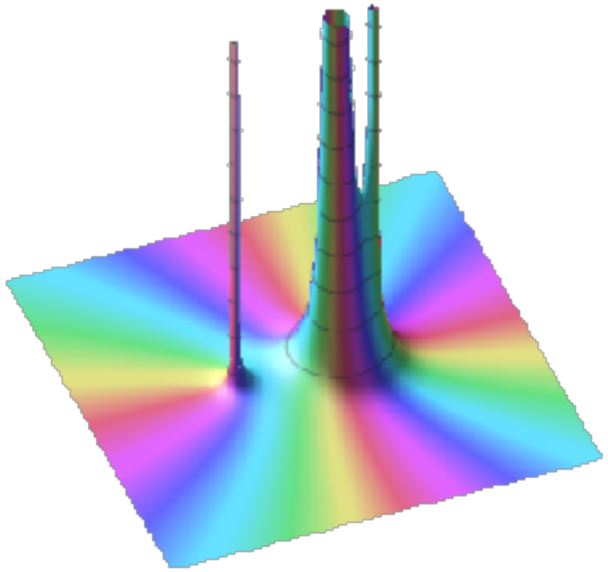
Useful property [Pringsheim Borel]

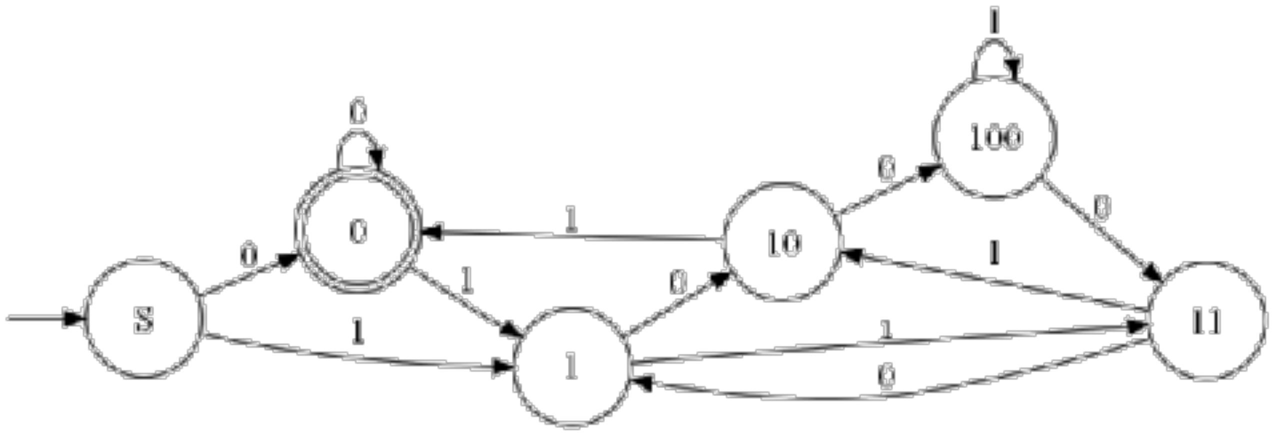
$a_n \geq 0$ for all $n \Longrightarrow$ real positive dominant singularity.

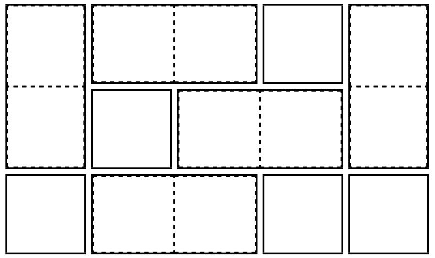
A couple of years after the publication of my Thesis, Edmund Landau wrote to me to say that the German mathematician Pringsheim thought he had discovered the theorem [...] ``But'', he added, ``of course, this result is yours''. [...] what would have happened if I had not been alive to receive his letter? He would not have been undeceived and the discovery would have been thought to be mine—until other readers would have restored it, not to Pringsheim but to E. Borel.

J. Hadamard (1954)

I. Rational Generating Functions (Linear Recurrences with Constant Coefficients)

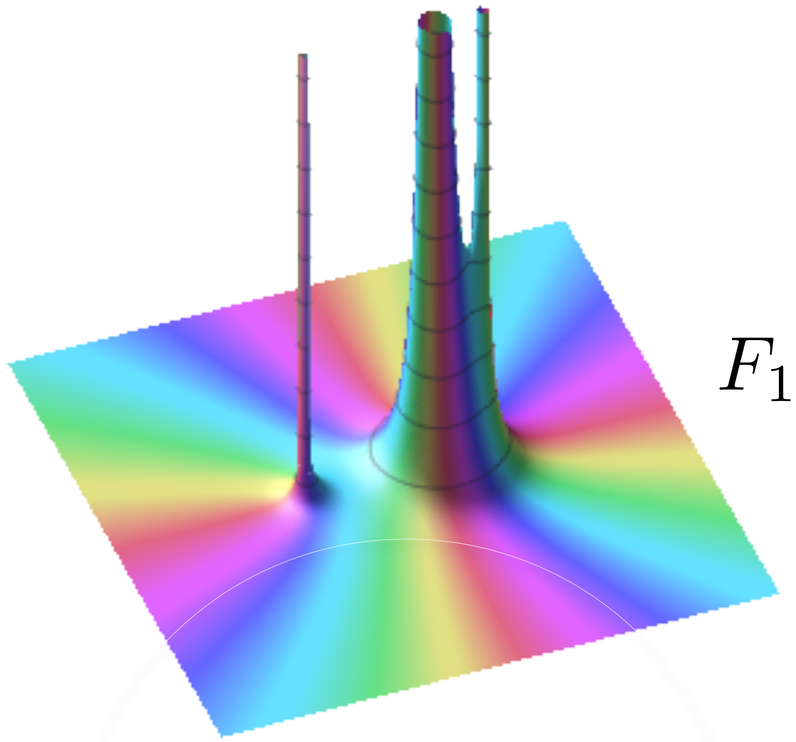




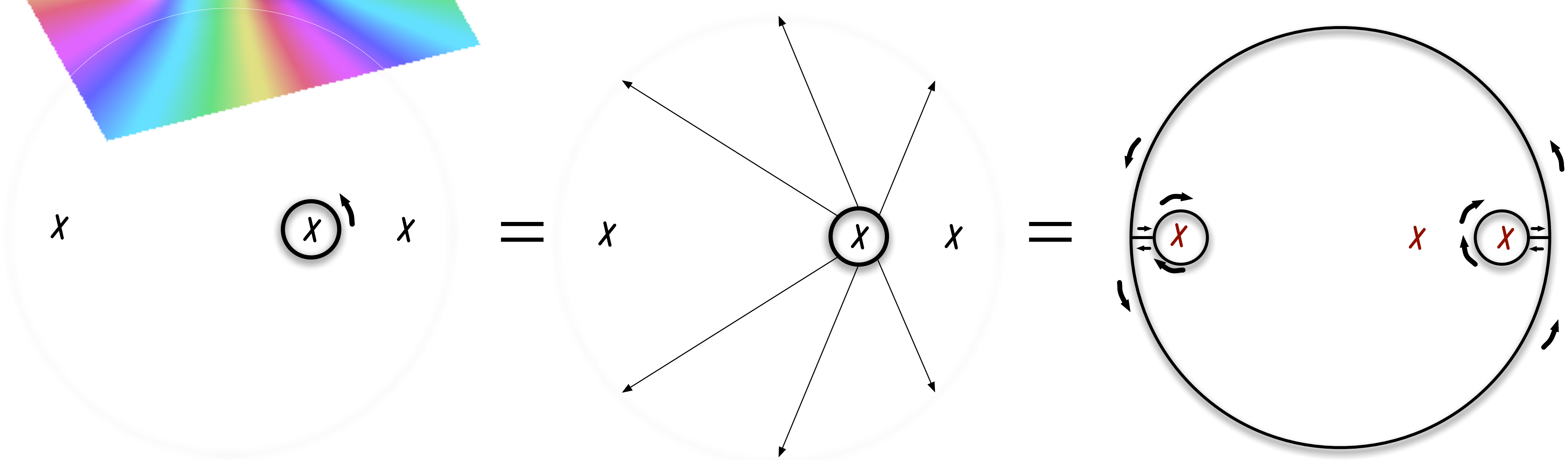


Fibonacci Numbers

$$a_n = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z^{n+1}}\,dz$$



$$F_1 = 1 = \frac{1}{2\pi i}\oint \frac{1}{1-z-z^2}\,\frac{dz}{z^2}$$



$$F_n = \frac{\phi^{-n-1}}{1+2\phi} + \frac{\overline{\phi}^{-n-1}}{1+2\overline{\phi}}$$

As n increases, the smallest singularities dominate.

Conway's sequence

1,11,21,1211,111221,…

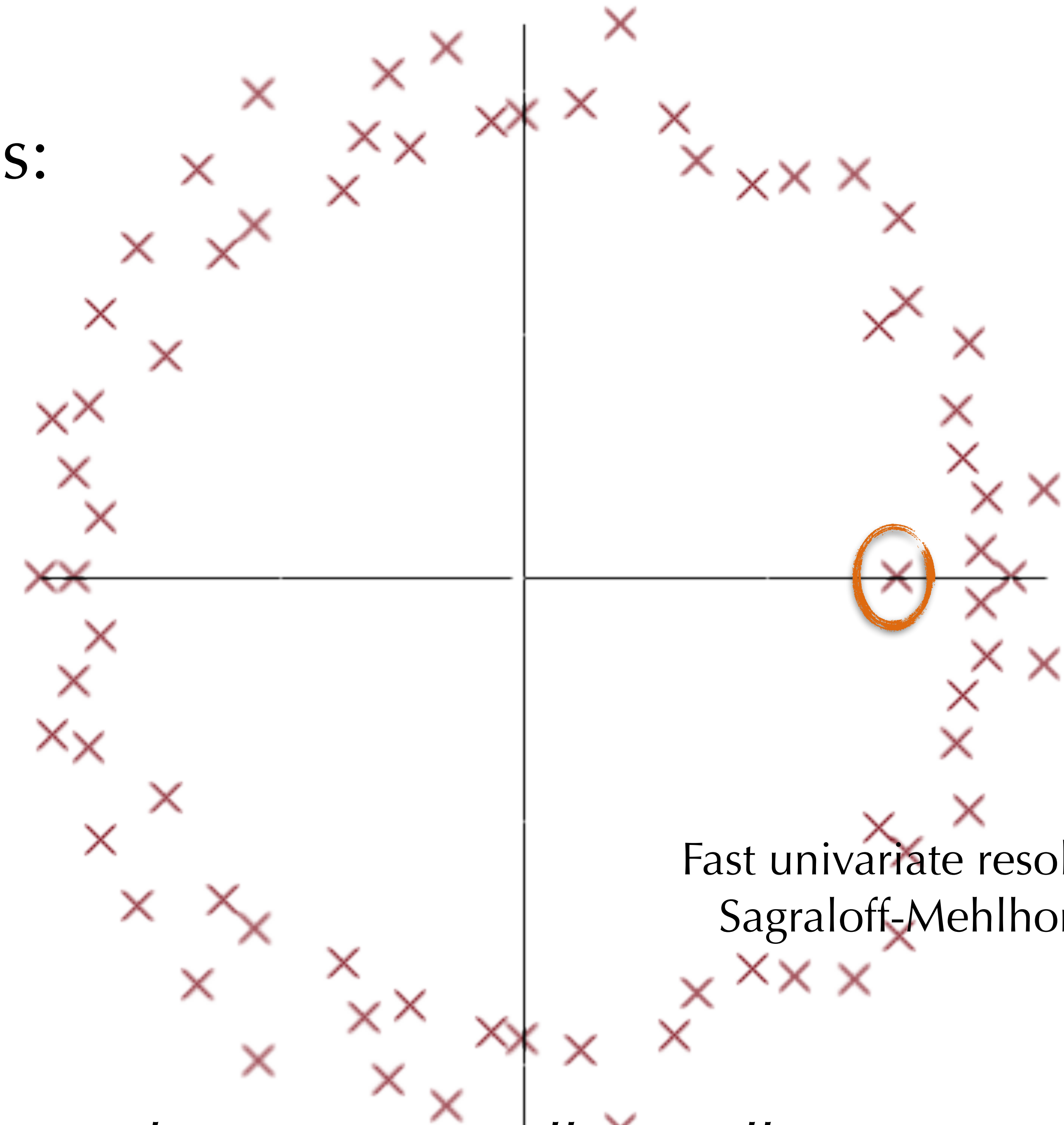
Generating function for lengths:

$$f(z)=P(z)/Q(z)$$

with deg Q=72.

Smallest singularity:

$\rho \simeq 0.7671198507$



Fast univariate resolution:
Sagraloff-Mehlhorn16

$$\boxed{\ell_n \simeq 2.04216\, \rho^{-n}}$$

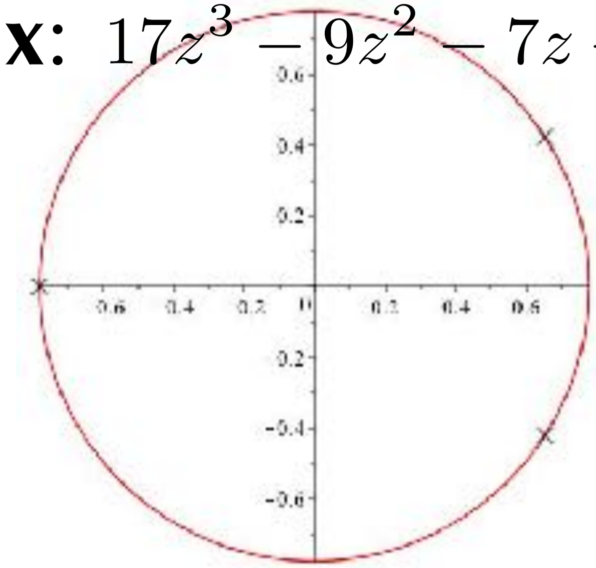
$c=\rho^{-1}\,\mathrm{Res}(f,\rho)$
algebraic

remainder exponentially small

[Conway 1987]

Singularity Analysis for Rational Functions

Ex: $17z^3 - 9z^2 - 7z + 8$



dist 10^{-5}

A 3-Step Method:

1. Locate dominant singularities ?
   a. singularities; b. dominant ones
2. Compute local behaviour ✓
3. Translate into asymptotics ✓

$$P(X) = a_0 X^n + \cdots + a_n \in \mathbb{Z}[X], \quad |a_i| \leq H, a_0 \neq 0$$

$$P(\alpha) = P(\beta) = 0$$

Prop. [Mahler64] $\alpha \neq \beta \Longrightarrow |\alpha - \beta| > C_n H^{-n+1}$

Cor. $|\alpha| \neq |\beta| \Longrightarrow ||\alpha| - |\beta|| > D_n H^{-n(n^2+2n-1)/2}$

Known to be tight only for n=2 & 3

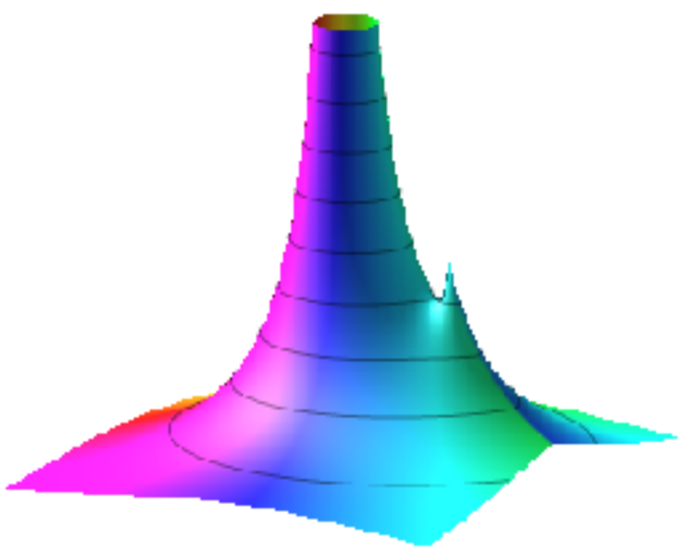
Numerical resolution with sufficient precision

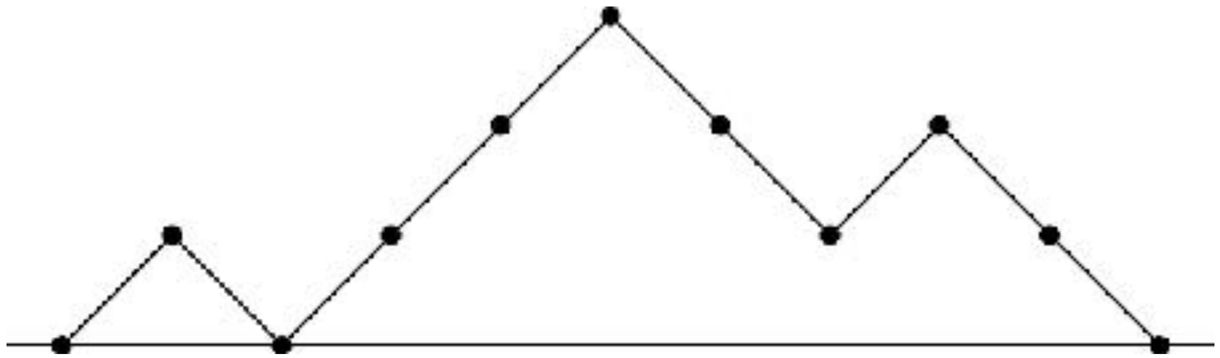
finds the dominant roots in polynomial complexity

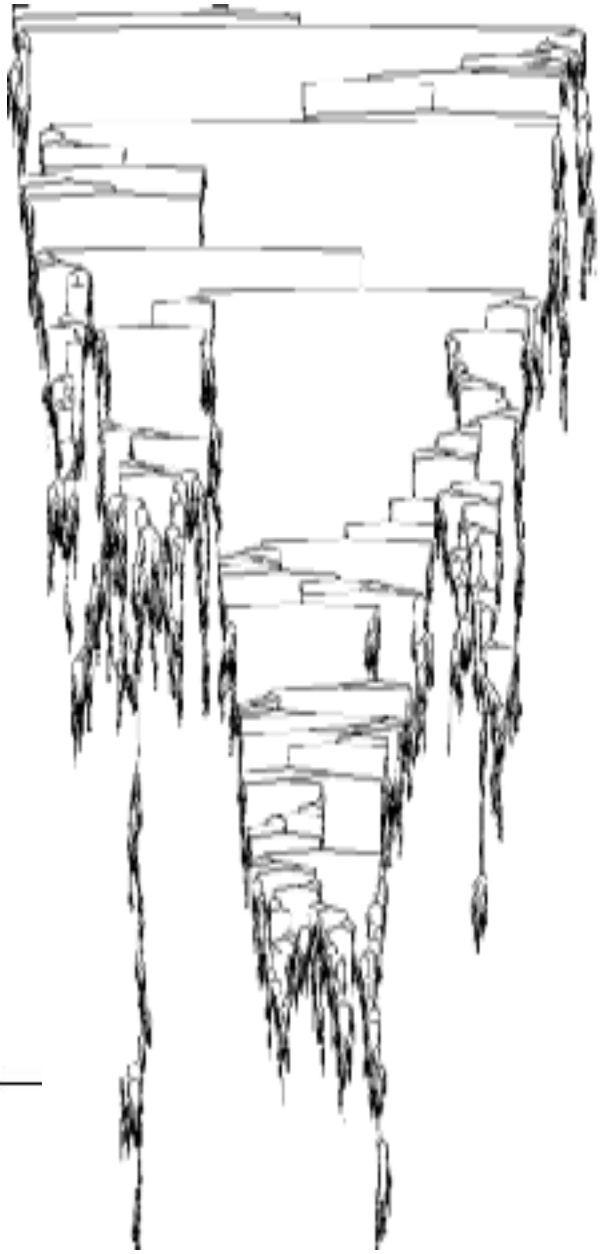
II. Algebraic Generating Functions

$$P(z, F(z)) = 0$$

with $P(z, y) \in \mathbb{Z}[z, y] \setminus \{0\}$







Algebraic Generating Functions

$$P(z, y(z)) = 0$$

1a. Location of possible singularities
Implicit Function Theorem:

$$P(z, y(z)) = \frac{\partial P}{\partial y}(z, y(z)) = 0 \quad \text{(discriminant)}$$

Numerical resolution with sufficient precision + algebraic manipulations

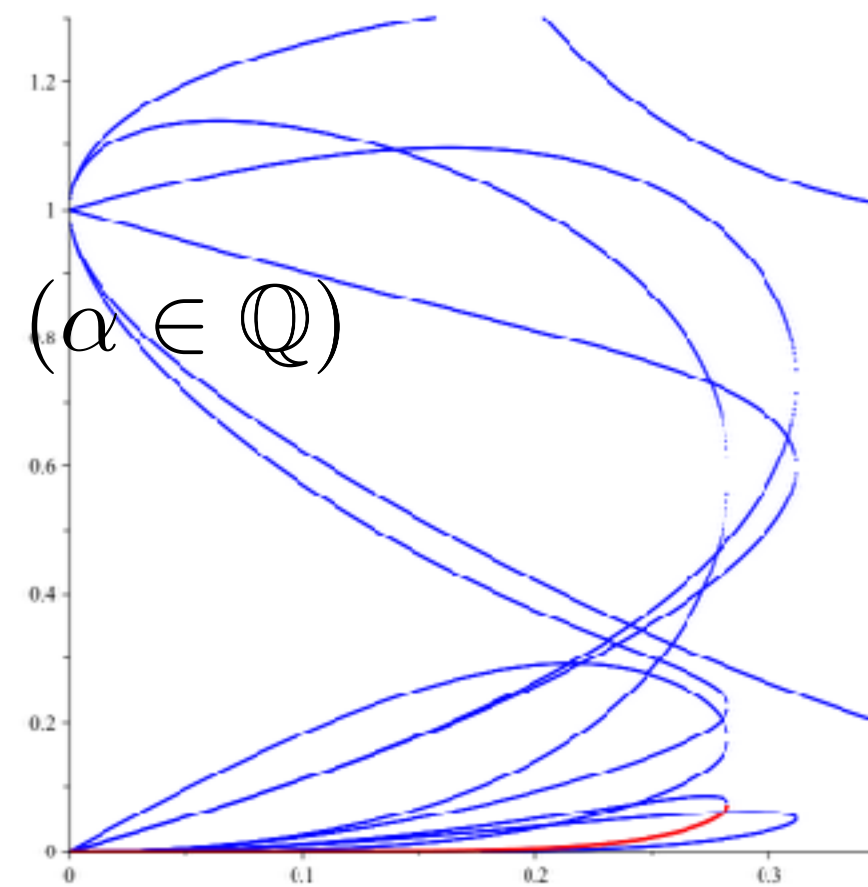
1b. Analytic continuation
finds the dominant ones

2. Local behaviour (Puiseux): $(1 - z/\rho)^{\alpha}$, $(\alpha \in \mathbb{Q})$

3. Translation: easy:

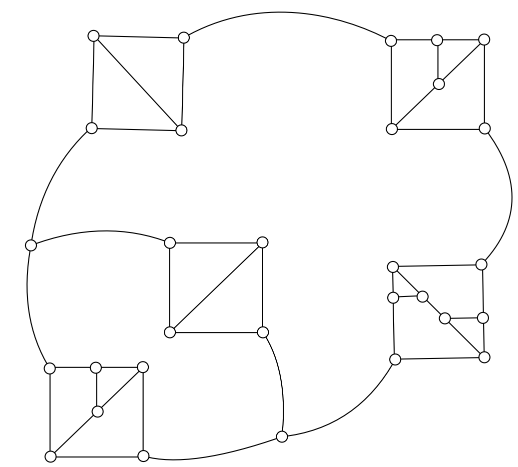
$$a_n \underset{n\to\infty}{\sim} c\,\rho^{-n} \frac{n^{-\alpha-1}}{\Gamma(-\alpha)}$$

with c, ρ algebraic, α rational.



3-regular 2-connected Planar Graphs

$$U = 2G_3 + T + 2U^2 = \frac{T}{(1-U)^3}, T = z(1+B)^3, B = \frac{G_3 + B^2}{1+B} + z\left(B + \frac{1}{2}B^2\right)$$



define power series $U(z), G_3(z), T(z), B(z)$.

The aim is to compute the asymptotic behaviour of $[z^n]B(z)$.

1. Eliminating U,T,G_3 gives $P = 16B^6z^2 + \cdots + z^2(z^2 + 11z - 1)$.
2. The discriminant has degree 20, but *only one root in (0,1]*:
   $\rho \approx .102$ root of $54z^3 + 324z^2 - 4265z + 432$.
3. At $z = \rho$, P has *only 1 (double) real positive* root: $B(\rho)$
4. Computing more terms gives
   $$B(z) = B(\rho) + c_1\left(1 - \frac{z}{\rho}\right) \pm c\left(1 - \frac{z}{\rho}\right)^{3/2} + \cdots$$ with an explicit c
5. Conclusion:
   $$[z^n]B(z) \sim \frac{3c}{4\sqrt{\pi}} n^{-5/2} \rho^{-n}.$$

Analytic continuation exploiting the combinatorial origin.

Singularity Analysis of Algebraic Series

Prop. [Abel1827;Cockle1861;Harley1862;Tannery1875]
Algebraic series are D-finite.

Exact analytic continuation for singularity analysis via LDE:

A. Compute a LDE starting from P;

B. For all roots of disc(P), sorted by increasing modulus,

   1. compute exactly the local branches;
   2. match with numerical continuation (MM's code);
   3. if a singular behaviour is encountered, return it.

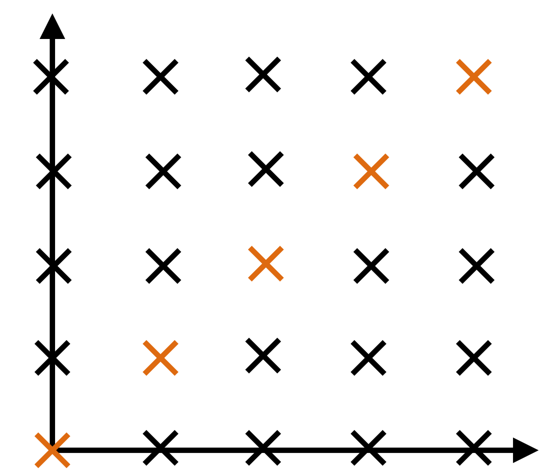
III. Diagonals

Definition

in this talk

If $F(\boldsymbol{z}) = \dfrac{G(\boldsymbol{z})}{H(\boldsymbol{z})}$ is a multivariate rational function with Taylor expansion

$$F(\boldsymbol{z}) = \sum_{\boldsymbol{i} \in \mathbb{N}^n} c_{\boldsymbol{i}} \boldsymbol{z}^{\boldsymbol{i}},$$



its **diagonal** is $\Delta F(t) = \sum_{k \in \mathbb{N}} c_{k,k,\ldots,k} t^k.$

$$\binom{2k}{k} : \qquad \frac{1}{1-x-y} = 1 + x + y + 2xy + x^2 + y^2 + \cdots + 6x^2y^2 + \cdots$$

$$\frac{1}{k+1}\binom{2k}{k} : \qquad \frac{1-2x}{(1-x-y)(1-x)} = 1 + y + 1xy - x^2 + y^2 + \cdots + 2x^2y^2 + \cdots$$

Apéry's a_k : $\dfrac{1}{1-t(1+x)(1+y)(1+z)(1+y+z+yz+xyz)} = 1 + \cdots + 5xyzt + \cdots$

Diagonals & Multiple Binomial Sums

Ex. $S_n = \sum_{r\geq 0}\sum_{s\geq 0}(-1)^{n+r+s}\binom{n}{r}\binom{n}{s}\binom{n+s}{s}\binom{n+r}{r}\binom{2n-r-s}{n}$

Thm. Diagonals ≡ binomial sums with 1 free index.

defined properly

```
> BinomSums[sumtores](S,u): (…)
```

$$\frac{1}{1-t(1+u_1)(1+u_2)(1-u_1u_3)(1-u_2u_3)}$$

has for diagonal the generating function of S_n

[Bostan-Lairez-S.17]

Multiple Binomial Sums

over a field $\mathbb{K}$

Sequences constructed from

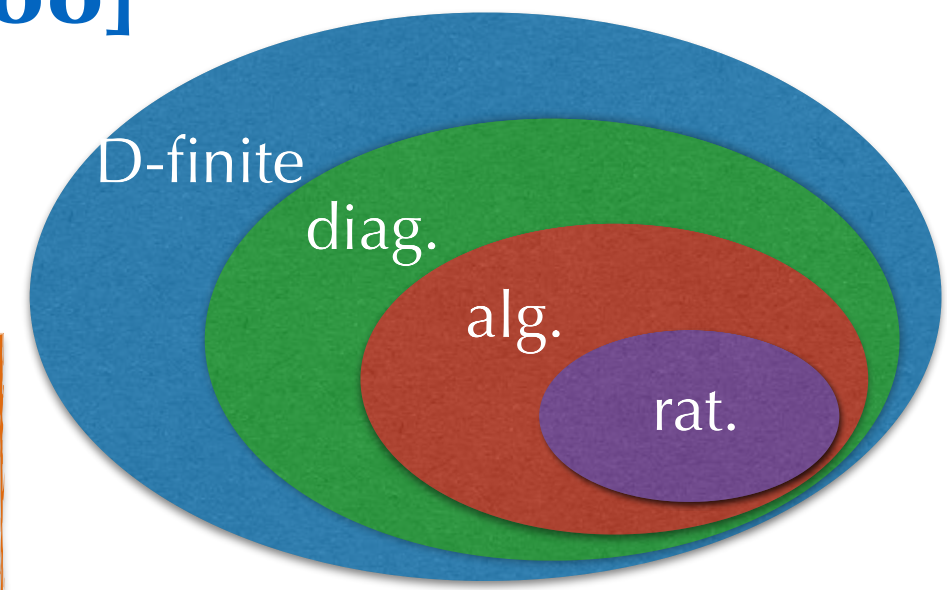
- Kronecker's δ: $n \mapsto \delta_n$;
- geometric sequences $n \mapsto C^n$, $C \in \mathbb{K}$;
- the binomial sequence $(n,k) \mapsto \binom{n}{k}$;

using algebra operations and

- affine changes of indices $(u_{\underline{n}}) \mapsto (u_{\lambda(\underline{n})})$;
- indefinite summation $(u_{\underline{n},k}) \mapsto \left(\sum_{k=0}^{m} u_{\underline{n},k}\right)$.

Diagonals are Differentially Finite
[Christol84,Lipshitz88]

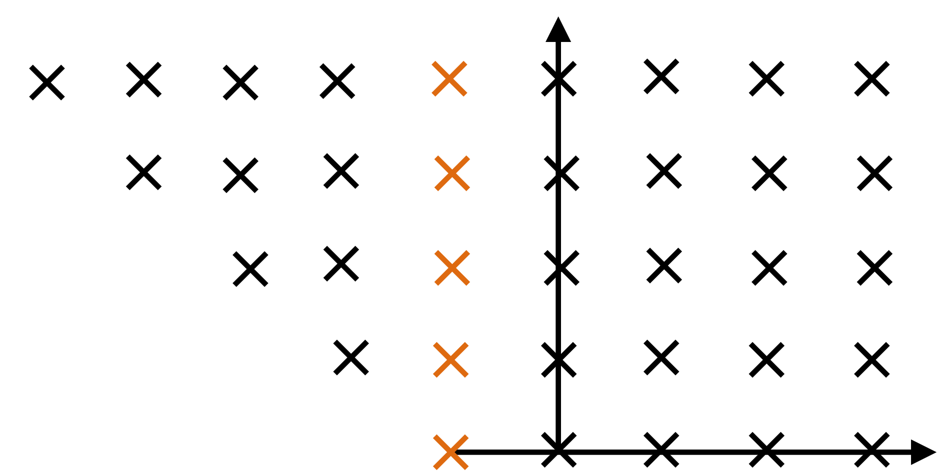
$$a_n(z)y^{(n)}(z) + \cdots + a_0(z)y(z) = 0,$$



Thm. If F has degree *d* in *n* variables,
ΔF satisfies a LDE with
order $\approx d^n$, coeffs of degree $d^{O(n)}$.

+ algo in $\tilde{O}(d^{8n})$ ops.

→ asymptotics from that LDE



Starting point:

$$F(\underline{z}) = \frac{G(\underline{z})}{H(\underline{z})} \Rightarrow \Delta F = \left(\frac{1}{2\pi i}\right)^{n-1} \oint F\left(z_1, \ldots, z_{n-1}, \frac{t}{z_1 \cdots z_{n-1}}\right) \frac{dz_1 \cdots dz_{n-1}}{z_1 \cdots z_{n-1}}.$$

[Bostan-Lairez-S.13,Lairez16]

LDE for Integrals: Griffiths-Dwork Method

$$\mathsf{I}(t) = \oint \frac{\mathsf{P}(t,\underline{x})}{\mathsf{Q}^m(t,\underline{x})}\, d\underline{x}$$

Q square-free
Int. over a cycle
where Q≠0.

Basic idea:

1. While m>1, reduce modulo $\mathsf{J} := \langle \partial_1 \mathsf{Q}, \dots, \partial_n \mathsf{Q} \rangle$
   and integrate by parts

$$\frac{P}{Q^m} = \frac{r + v_1\partial_1 Q + \cdots + v_n \partial_n Q}{Q^m} = \frac{r}{Q^m} + \frac{\tilde{P}}{Q^{m-1}} + \text{derivatives}$$

2. Apply to I,I′,I″,… until a linear dependency is found.

Thm. If P/Q has degree *d* in *n* variables, I(t) satisfies a LDE with order ≈d^n, coeffs of degree $d^{O(n)}$.

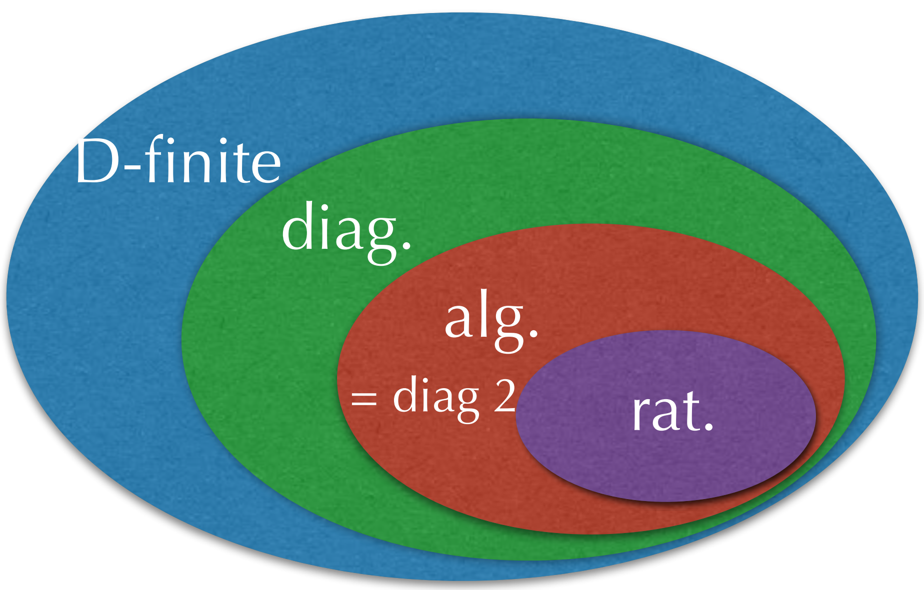
+Algo in $\tilde{O}(d^{8n})$

Diagonals:

J becomes $\langle z_1\partial_1 H - z_n\partial_n H, \dots, z_{n-1}\partial_{n-1} H - z_n \partial_n H \rangle$.

[Griffiths70;Christol84;Bostan-Lairez-S.13;Lairez16]

Bivariate Diagonals are Algebraic
[Pólya21,Furstenberg67]



Thm. F=A(x,y)/B(x,y), deg≤d in x and y, then ΔF cancels a polynomial of degree ≲ 4^d in y and t.

$$\Delta \frac{x}{1-x^2-y^3} \quad \text{satisfies}$$

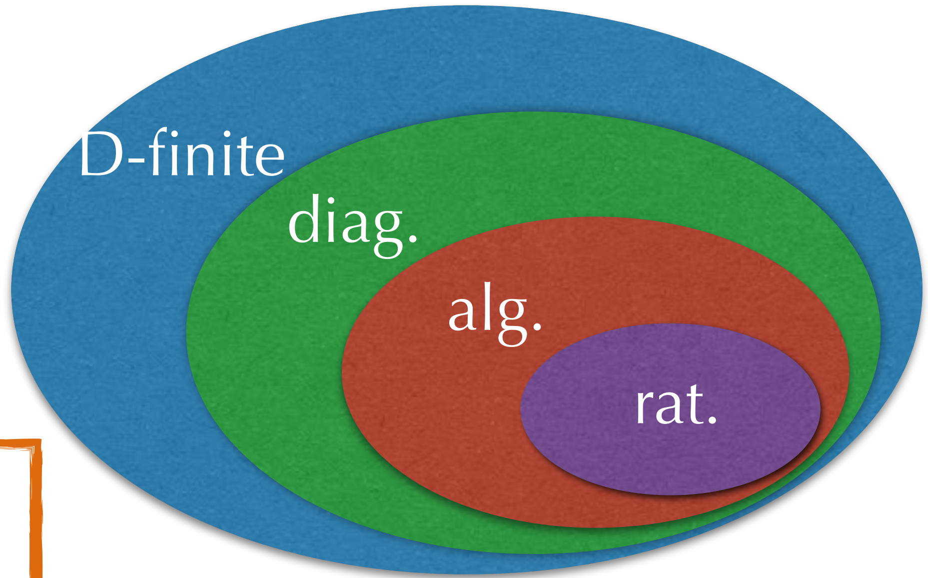
$$\begin{aligned}&\left(3125\,t^6-108\right)^3 y^{10}+81\left(3125\,t^6-108\right)^2 y^8\\&+50t^3\left(3125\,t^6-108\right)^2 y^7+\left(6834375\,t^6-236196\right)y^6\\&-t^3\left(34375\,t^6-3888\right)\left(3125\,t^6-108\right)y^5\\&+\left(-7812500\,t^{12}+270000\,t^6+19683\right)y^4\\&-54\,t^3\left(6250\,t^6-891\right)y^3+50\,t^6\left(21875\,t^6-2106\right)y^2\\&-t^3\left(50\,t^2+9\right)\left(2500\,t^4-450\,t^2+81\right)y\\&-t^6\left(3125\,t^6-1458\right)=0\end{aligned}$$

+ quasi-optimal algorithm.

→ *the differential equation is often better.*

[Bostan-Dumont-S.17]

Summary of this part



Prop. Algebraic series are the diagonals of bivariate rational functions.
Diagonals are D-finite; they are closed under sum, product, Hadamard product; their coefficients are multiple binomial sums (and conversely).

Christol's conjecture: All D- finite power series with integer coefficients and radius of convergence in $(0,\infty)$ are diagonals.

$${}_3F_2\left(\left.\begin{matrix}\frac{1}{9},\frac{4}{9},\frac{5}{9}\\ \frac{1}{3},1\end{matrix}\right|3^6z\right)\stackrel{?}{=}\Delta(?)$$

All these properties are effective,
with good bounds and complexity.

→ asymptotics from the LDE

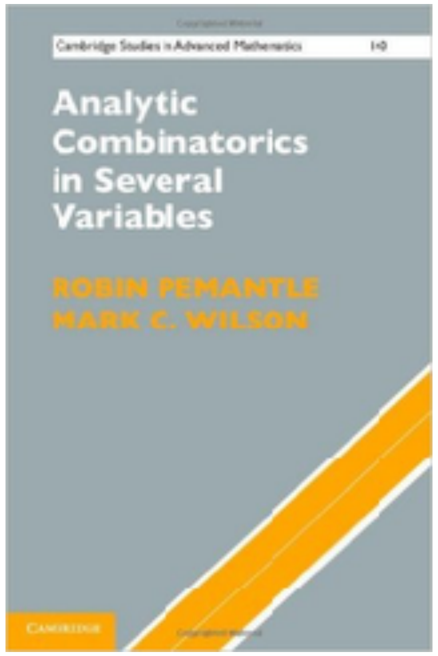
[Pólya21,Furstenberg67,Christol84,BostanLairezS.13,Lairez16,BostanDumontS.17]

IV. Analytic Combinatorics in Several Variables, with Computer Algebra

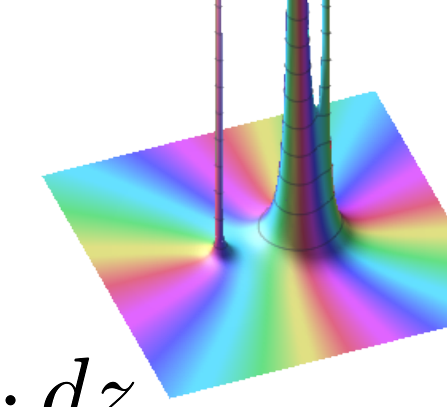
Wanted: complete algorithms, good complexity, more cases with `explicit' c.

Solution:

1. restrict to simplest class;
2. avoid amoebas and
   deal only with polynomial systems;
3. control all degrees & sizes.



Coefficients of Diagonals



$$F(\underline{z}) = \frac{G(\underline{z})}{H(\underline{z})} \qquad c_{k,\ldots,k} = \left(\frac{1}{2\pi i}\right)^n \int_T \frac{G(\underline{z})}{H(\underline{z})} \frac{dz_1 \cdots dz_n}{(z_1 \cdots z_n)^{k+1}}$$

Critical points: minimize $z_1 \cdots z_n$ on $\mathcal{V} = \{\underline{z} \mid H(\underline{z}) = 0\}$

J from G-D method

$$\operatorname{rank}\begin{pmatrix} \frac{\partial H}{\partial z_1} & \cdots & \frac{\partial H}{\partial z_n} \\ \frac{\partial(z_1 \cdots z_n)}{\partial z_1} & \cdots & \frac{\partial(z_1 \cdots z_n)}{\partial z_n} \end{pmatrix} \leq 1 \qquad \text{i.e.} \qquad z_1 \frac{\partial H}{\partial z_1} = \cdots = z_n \frac{\partial H}{\partial z_n}$$

Minimal ones: on the boundary of the domain of convergence.

A 3-step method

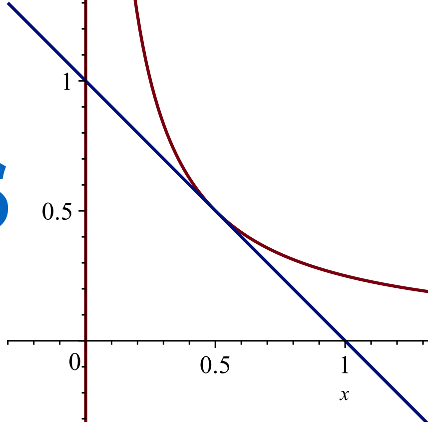
1a. locate the critical points (**algebraic** condition);
1b. find the minimal ones (**semi-algebraic** condition);
2. translate (easy in simple cases).

Analytic continuation from the rational function

Def. $F(z_1,\ldots,z_n)$ is **combinatorial** if every coefficient is ≥ 0.

Prop. [PemantleWilson] In the combinatorial case, one of the minimal critical points has positive real coordinates.

Ex.: Central Binomial Coefficients

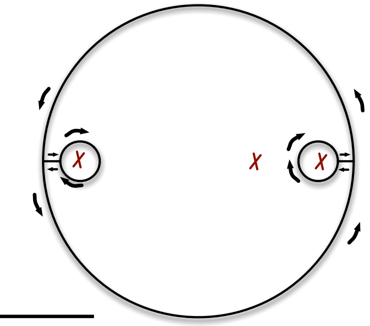


$$\binom{2k}{k}: \qquad \frac{1}{1-x-y} = 1 + x + y + 2xy + x^2 + y^2 + \cdots + 6x^2y^2 + \cdots$$

(1). Critical points: $1 - x - y = 0, x = y \Longrightarrow x = y = 1/2.$

(2). Minimal ones. Easy. In general, this is the difficult step.

(3). Analysis close to the minimal critical point:



$$a_k = \frac{1}{(2\pi i)^2} \iint \frac{1}{1-x-y} \frac{dx\,dy}{(xy)^{k+1}} \approx \frac{1}{2\pi i} \int \frac{dx}{(x(1-x))^{k+1}}$$

$$\approx \frac{4^{k+1}}{2\pi i} \int \exp(4(k+1)(x-1/2)^2)\,dx \approx \frac{4^k}{\sqrt{k\pi}}.$$

residue

saddle-point approx

Kronecker Representation for the Critical Points

Algebraic part: ``compute'' the solutions of the system

$$H(\underline{z}) = 0 \qquad z_1 \frac{\partial H}{\partial z_1} = \cdots = z_n \frac{\partial H}{\partial z_n}$$

If $\quad \deg(H) = d, \quad \max \operatorname{coeff}(H) \le 2^h \quad D := d^n$

Under genericity assumptions, a probabilistic algorithm running in $\tilde{O}(hD^3)$ bit ops finds:

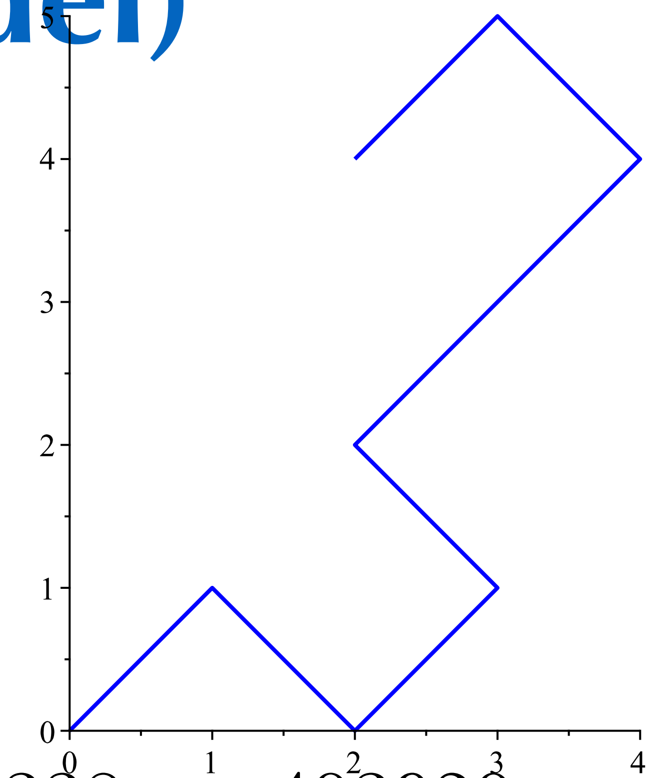
$$\left.\begin{aligned} P(u) &= 0 \\ P'(u)z_1 - Q_1(u) &= 0 \\ &\vdots \\ P'(u)z_n - Q_n(u) &= 0 \end{aligned}\right\} \quad \begin{aligned} &\text{Degree } \le D \\ &\text{Height } \le \tilde{O}(hD) \end{aligned}$$

History and Background: see Castro, Pardo, Hägele, and Morais (2001)

System reduced to a univariate polynomial.

[Giusti-Lecerf-S.01;Schost02;SafeySchost16]

Example (Lattice Path Model)



The number of walks from the origin taking steps *{NW,NE,SE,SW}* and staying in the first quadrant is

$$\Delta F, \quad F(x,y,t) = \frac{(1+x)(1+y)}{1-t(1+x^2+y^2+x^2y^2)}$$

Kronecker representation of the critical points:

$$P(u) = 4u^4 + 52u^3 - 4339u^2 + 9338u + 403920$$

$$Q_x(u) = 336u^2 + 344u - 105898$$

$$Q_y(u) = -160u^2 + 2824u - 48982$$

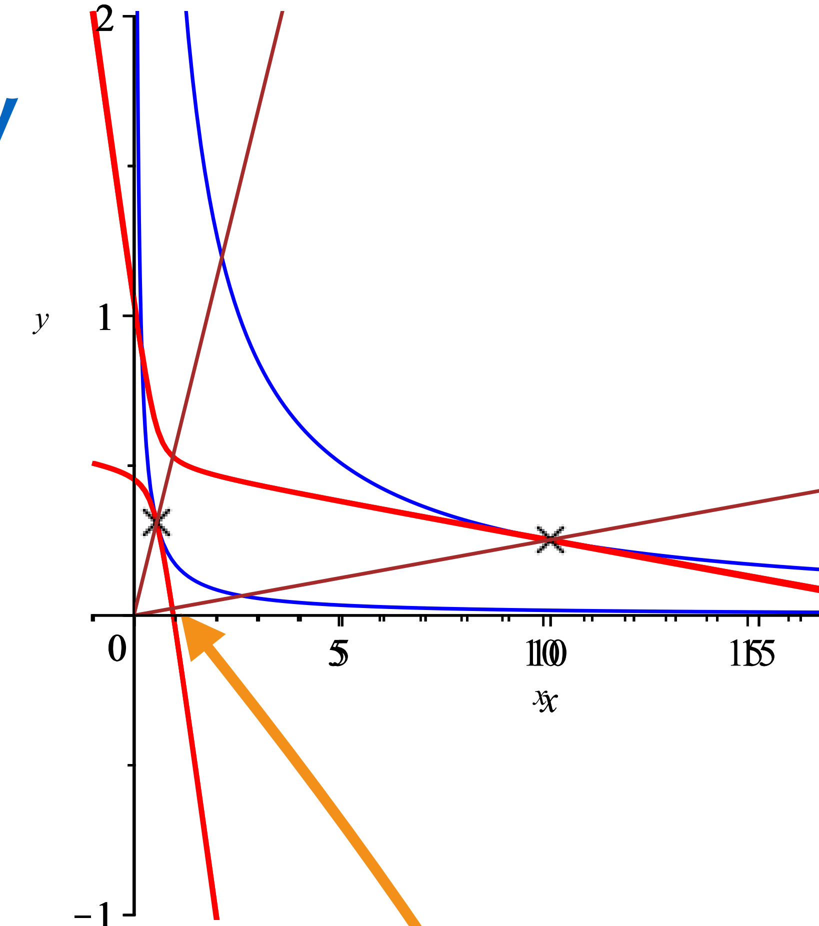
$$Q_t(u) = 4u^3 + 39u^2 - 4339u/2 + 4669/2$$

ie, they are given by:

$$P(u) = 0, \quad x = \frac{Q_x(u)}{P'(u)}, \quad y = \frac{Q_y(u)}{P'(u)}, \quad t = \frac{Q_t(u)}{P'(u)}$$

Which one of these 4 is minimal?

Testing Minimality



$$F = \frac{1}{H} = \frac{1}{(1-x-y)(20-x-40y)-1}$$

Critical point equation $x\frac{\partial H}{\partial x} = y\frac{\partial H}{\partial y}$:

$$x(2x+41y-21) = y(41x+80y-60)$$

→ 4 critical points, 2 of which are real:

$$(x_1, y_1) = (0.2528, 9.9971), \quad (x_2, y_2) = (0.30998, 0.54823)$$

Add $H(tx, ty) = 0$ and compute a Kronecker representation:

$$P(u) = 0, \quad x = \frac{Q_x(u)}{P'(u)}, \quad y = \frac{Q_y(u)}{P'(u)}, \quad t = \frac{Q_t(u)}{P'(u)}$$

Solve numerically and keep the real positive sols:

$$(0.31, 0.55, 0.99), (0.31, 0.55, 1.71), (0.25, 9.99, 0.09), (0.25, 0.99, 0.99)$$

(x_1, y_1) is not minimal, (x_2, y_2) is.

Algorithm and Complexity

Thm. If $F(\underline{z})$ is combinatorial, then under regularity conditions, the points contributing to dominant diagonal asymptotics can be determined in $\tilde{O}(hd^5D^4)$ bit operations. Each contribution has the form

$$A_k = \left(T^{-k}k^{(1-n)/2}(2\pi)^{(1-n)/2}\right)(C + O(1/k))$$

T, C can be found to $2^{-\kappa}$ precision in $\tilde{O}(h(dD)^3 + D\kappa)$ bit ops.

explicit algebraic numbers

half-integer

This result covers the easiest cases.
All conditions hold generically and can be checked within the same complexity, except combinatoriality.

[Melczer-S.16]

Example: Apéry's sequence

$$\frac{1}{1-t(1+x)(1+y)(1+z)(1+y+z+yz+xyz)} = 1 + \cdots + 5xyzt + \cdots$$

Kronecker representation of the critical points:

$$P(u) = u^2 - 366u - 17711$$

$$x = \frac{2u-1006}{P'(u)}, \quad y = z = -\frac{320}{P'(u)}, \quad t = -\frac{164u+7108}{P'(u)}$$

There are two real critical points, and one is positive. After testing minimality, one has proved asymptotics

```
> A, U := DiagonalAsymptotics(numer(F),denom(F),[t,x,y,z], u, k):
> evala(allvalues(subs(u=U[1],A)));
```

$$\frac{\left(17+12\sqrt{2}\right)^k \sqrt{2}\sqrt{24+17\sqrt{2}}}{8\,k^{3/2}\pi^{3/2}}$$

Example: Restricted Words in Factors

$$F(x,y)=\frac{1-x^3y^6+x^3y^4+x^2y^4+x^2y^3}{1-x-y+x^2y^3-x^3y^3-x^4y^4-x^3y^6+x^4y^6}$$

words over {0,1} without 10101101 or 1110101

```
> A,U:=DiagonalAsymptotics(numer(F),denom(F),indets(F),u,k,true,u-T,T):
> A;
```

$$\left(\left(\frac{84u^{20}+240u^{19}-285u^{18}-1548u^{17}-2125u^{16}-1408u^{15}+255u^{14}+756u^{13}+2509u^{12}+2856u^{11}-605u^{10}+2020u^{9}+1233u^{8}-1760u^{7}+924u^{6}-492u^{5}-675u^{4}+632u^{3}-249u^{2}+24u+16}{-12u^{20}+30u^{19}+258u^{18}+500u^{17}+440u^{16}-102u^{15}-378u^{14}-1544u^{13}-2142u^{12}-550u^{11}-2222u^{10}-1644u^{9}+2860u^{8}-1848u^{7}+1230u^{6}+2160u^{5}-2686u^{4}+1494u^{3}-228u^{2}-320u+84}\right)^{k}\right.$$

$$\sqrt{2}\sqrt{\frac{84u^{20}+240u^{19}-285u^{18}-1548u^{17}-2125u^{16}-1408u^{15}+255u^{14}+756u^{13}+2509u^{12}+2856u^{11}+605u^{10}-2020u^{9}+1233u^{8}-1760u^{7}+924u^{6}-492u^{5}-675u^{4}+632u^{3}-249u^{2}+24u+16}{-162u^{18}-612u^{17}-902u^{16}-616u^{15}+254u^{14}+548u^{13}+2054u^{12}+2156u^{11}+898u^{10}-2268u^{9}+2462u^{8}-2088u^{7}+1312u^{6}-540u^{5}-1410u^{4}+1188u^{3}-290u^{2}+32}}$$

$$\left.\left(12u^{20}+36u^{19}-21u^{18}-170u^{17}-255u^{16}-190u^{15}-19u^{14}+46u^{13}+461u^{12}+628u^{11}+133u^{10}+374u^{9}+161u^{8}-384u^{7}+146u^{6}-138u^{5}-285u^{4}-40u^{3}+91u^{2}-30u+32\right)\right)\Big/\left(2\sqrt{k}\sqrt{\pi}\left(84u^{20}\right.\right.$$

$$\left.\left.+240u^{19}-285u^{18}-1548u^{17}-2125u^{16}-1408u^{15}+255u^{14}+756u^{13}+2509u^{12}-2856u^{11}-605u^{10}+2020u^{9}-1233u^{8}-1760u^{7}+924u^{6}-492u^{5}-675u^{4}+632u^{3}-249u^{2}+24u+16\right)\right)$$

```
> U;
```

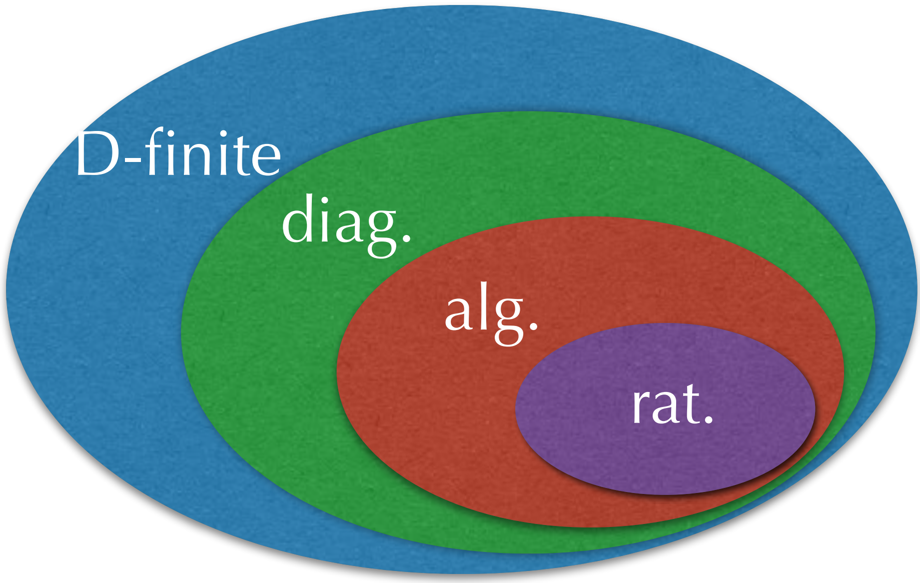
$$[\mathit{RootOf}(4\_Z^{21}+12\_Z^{20}-15\_Z^{19}-86\_Z^{18}-125\_Z^{17}-88\_Z^{16}+17\_Z^{15}+54\_Z^{14}+193\_Z^{13}+238\_Z^{12}+55\_Z^{11}+202\_Z^{10}+137\_Z^{9}-220\_Z^{8}+132\_Z^{7}-82\_Z^{6}-135\_Z^{5}+158\_Z^{4}-83\_Z^{3}+12\_Z^{2}$$

$$+16\_Z-4,0.25574184)]$$

```
> evalf(subs(u=U[1],A));
```

$$\frac{0.60294599\;3.9101932^{k}}{\sqrt{k}}$$

Summary & Conclusion



- In many cases, LDE + certified analytic continuation works.
- Diagonals are a nice and important class of generating functions for which we now have many good algorithms.
- ACSV can be made effective (at least in simple cases) and recovers explicit constants.
- Complexity issues become clearer.

Work in progress: extend beyond some of the assumptions

The End