Linear differential and recurrence equations viewed as data-structures

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Computer Algebra

Effective mathematics (what can we compute?)
Their complexity (how fast?)

Several million users. 30 years of algorithmic progress.

Thesis in this presentation:
linear differential and recurrence equations are a good data-structure.

Topics

Fast computation with high precision;
automatic proofs of identities;
«computation» of expansions,
of (multiple) integrals, of (multiple) sums.

The objects of study

Def. A power series is called differentially finite (D-finite) when it is the solution of a linear differential equation with polynomial coefficients.

Exs: sin, cos, exp, log, arcsin, arccos, arctan, arcsinh,hypergeometric series, Bessel functions,…

Def. A sequence is polynomially recursive (P-recursive) when it is the solution of a linear recurrence with polynomial coefficients.

Prop. $f = \sum_{n=0}^{\infty} f_n z^n$ D-finite $\iff$ f_n P-recursive.

Example

Coefficient of X^{20000} in $P(X)=(1+X)^{20000}(1+X+X^2)^{10000}$?

Linear differential equation of order 1
→ linear recurrence of order 2
→ unroll (cleverly).

```
> P := (1+x)^(2*N)*(1+x+x^2)^N:
> deq := gfun:-holexprtodiffeq(P,y(x)):
> rec := gfun:-diffeqtorec(%,y(x),u(k)):
> p := gfun:-rectoproc(subs(N=10000,rec),u(k)):
> p(20000);
```

23982[...10590 digits...]33952

Total time: 0.5 sec

I. Fast computation at large precision

From large integers to precise numerical values

Fast multiplication

Fast Fourier Transform (Gauss, Cooley-Tuckey, Schönhage-Strassen).
Two integers of n digits can be multiplied with
$O(n \log(n) \log\log(n))$ bit operations.

Direct consequence (by Newton iteration):

inverses, square-roots,… : same cost.

Binary Splitting for linear recurrences (70's and 80's)

- n! by divide-and-conquer:

$$n! := \underbrace{n \times \cdots \times \lfloor n/2 \rfloor}_{\text{size } O(n \log n)} \times \underbrace{(\lfloor n/2 \rfloor + 1) \times \cdots \times 1}_{\text{size } O(n \log n)}$$

Cost: $O(n \log^3 n \log\log n)$ using FFT

- linear recurrences of order 1 reduce to

$$p!(n) := (p(n) \times \cdots \times p(\lfloor n/2 \rfloor)) \times (p(\lfloor n/2 \rfloor + 1) \times \cdots \times p(1))$$

- arbitrary order: same idea, same cost (matrix factorial):

ex: $e_n := \sum_{k=0}^{n} \frac{1}{k!}$ satisfies a 2nd order rec, computed via

$$\begin{pmatrix} e_n \\ e_{n-1} \end{pmatrix} = \frac{1}{n} \underbrace{\begin{pmatrix} n+1 & -1 \\ n & 0 \end{pmatrix}}_{A(n)} \begin{pmatrix} e_{n-1} \\ e_{n-2} \end{pmatrix} = \frac{1}{n!} A!(n) \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Numerical evaluation of solutions of LDEs

Principle: $$f(x) = \underbrace{\sum_{n=0}^{N} a_n x^n}_{\text{fast evaluation}} + \underbrace{\sum_{n=N+1}^{\infty} a_n x^n}_{\text{good bounds}}$$

f solution of a LDE with coeffs in $\mathbb{Q}(x)$ (our data-structure!)

1. linear recurrence in N for the first sum (easy);
2. tight bounds on the tail (technical);
3. no numerical roundoff errors.

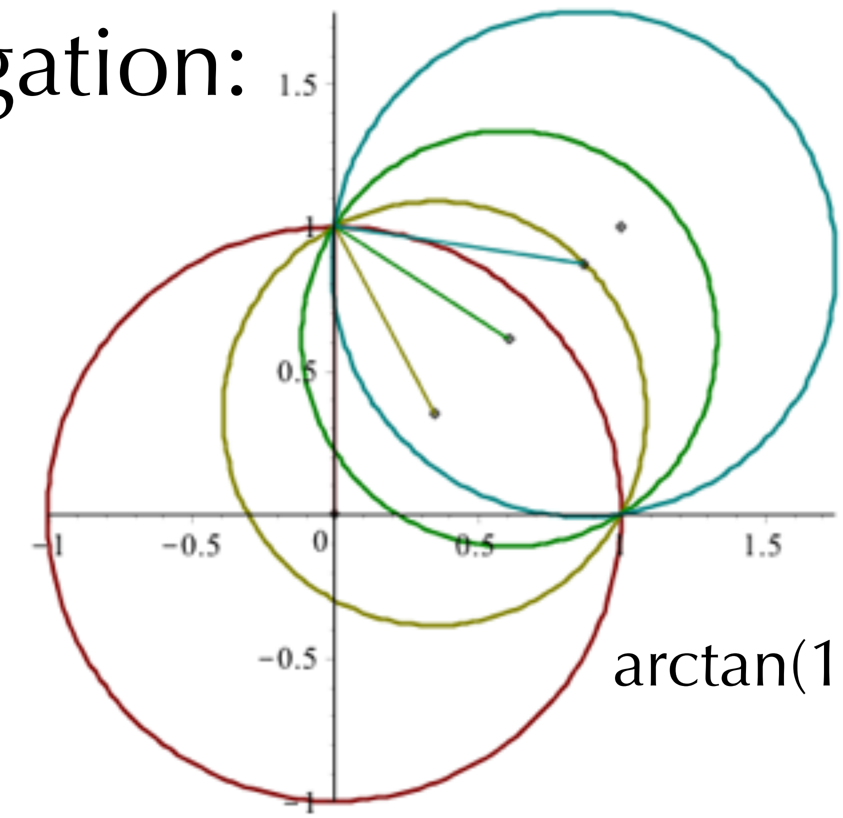
The technique used for fast evaluation of constants like

$$\frac{1}{\pi} = \frac{12}{C^{3/2}} \sum_{n=0}^{\infty} \frac{(-1)^n (6n)!(A + nB)}{(3n)!n!^3 C^{3n}}$$

with A=13591409, B=545140134, C=640320.

Analytic continuation

Compute $f(x), f'(x), \ldots, f^{(d-1)}(x)$ as new initial conditions and handle error propagation:



arctan(1+i)

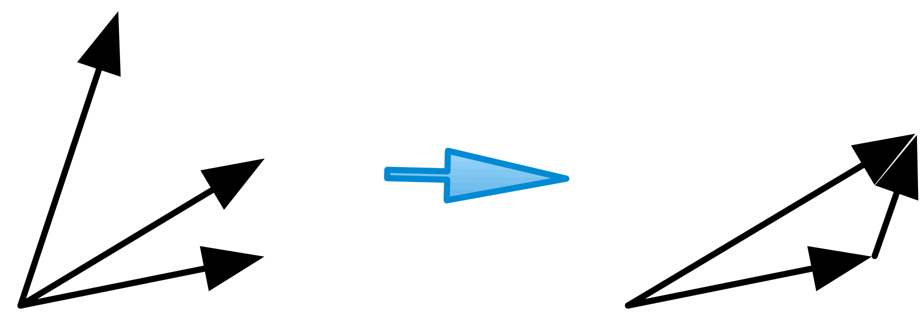
Ex: erf(π) with 15 digits:

$$0 \xrightarrow[\text{200 terms}]{} 3.1416 \xrightarrow[\text{18 terms}]{} 3.1415927 \xrightarrow[\text{6 terms}]{} 3.14159265358979$$

Again: computation on integers. No roundoff errors.

II. Proofs of Identities

Confinement



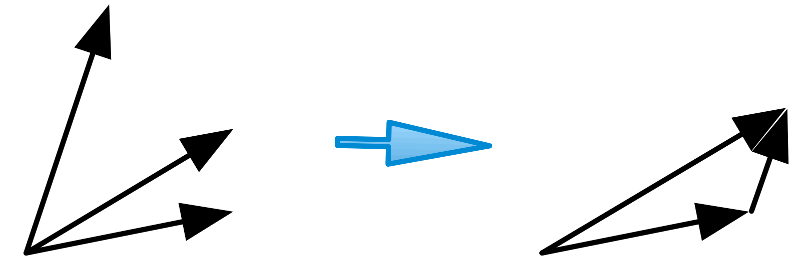
k+1 vectors in dimension k → an identity

LDE $\Longleftrightarrow$ the function and all its derivatives are confined in a finite dimensional vector space

$\Rightarrow$ the sum and product of solutions of LDEs satisfy LDEs

$\Rightarrow$ same property for P-recursive sequences

Proof technique



```
> series(sin(x)^2+cos(x)^2-1,x,4);
```

$O(x^4)$

f satisfies a LDE
$\Longleftrightarrow$
$f, f', f'', \ldots$ live in a finite-dim. vector space

Why is this a proof?

1. sin and cos satisfy a 2nd order LDE: $y''+y=0$;
2. their squares and their sum satisfy a 3rd order LDE;
3. the constant -1 satisfies $y'=0$;
4. thus $\sin^2+\cos^2-1$ satisfies a LDE of order at most 4;
5. Cauchy's theorem concludes.

Proofs of non-linear identities by linear algebra!

Example: Mehler's identity for Hermite polynomials

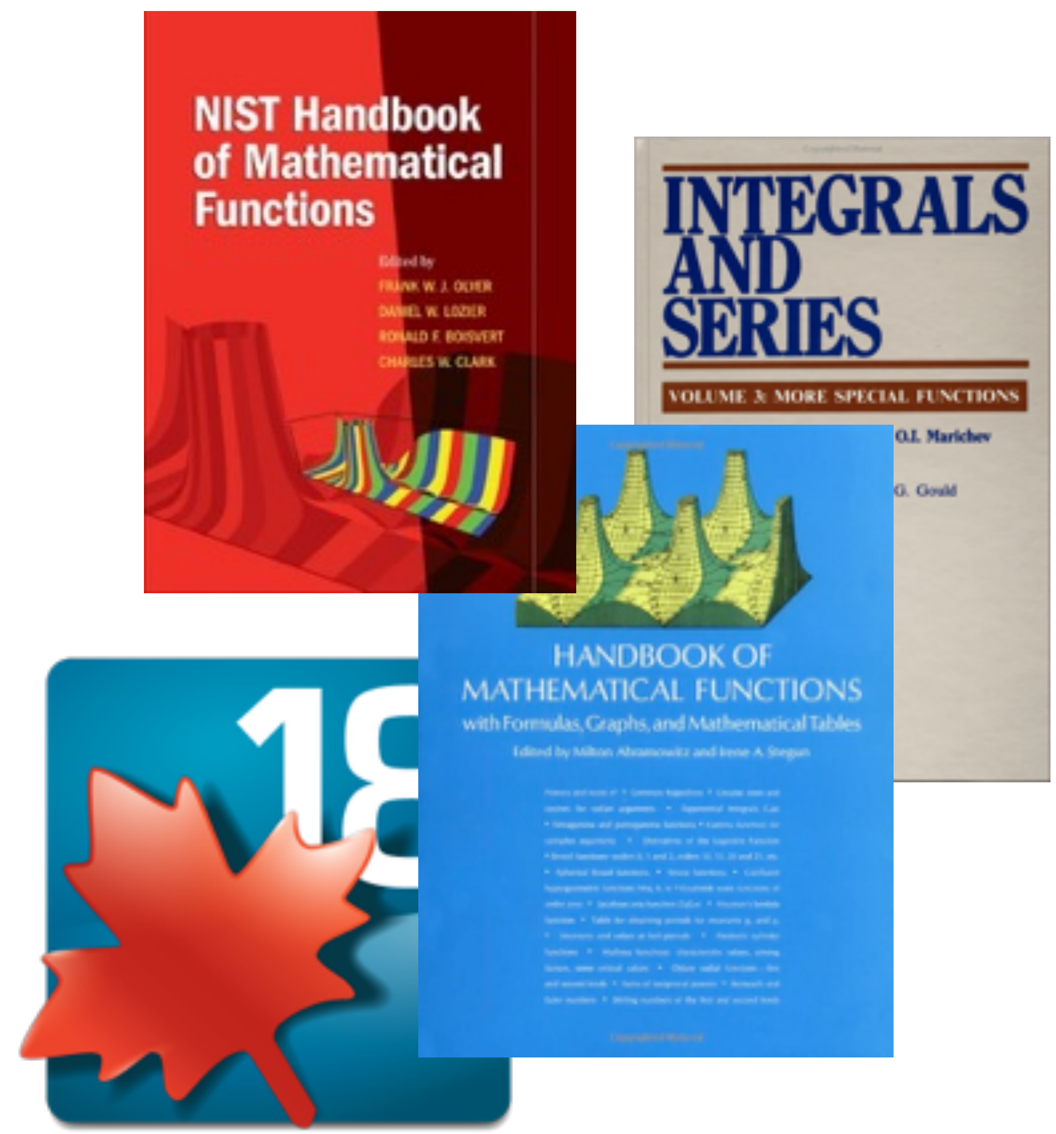
$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\frac{u^n}{n!} = \frac{\exp\left(\frac{4u(xy-u(x^2+y^2))}{1-4u^2}\right)}{\sqrt{1-4u^2}}$$

1. Definition of Hermite polynomials:
   recurrence of order 2;
2. Product by linear algebra: $H_{n+k}(x)H_{n+k}(y)/(n+k)!$, $k\in\mathbb{N}$
   generated over $\mathbb{Q}(x,n)$ by
   $$\frac{H_n(x)H_n(y)}{n!}, \frac{H_{n+1}(x)H_n(y)}{n!}, \frac{H_n(x)H_{n+1}(y)}{n!}, \frac{H_{n+1}(x)H_{n+1}(y)}{n!}$$
   → recurrence of order at most 4;
3. Translate into differential equation.

Dynamic Dictionary of Mathematical Functions

- User need
- Recent algorithmic progress
- Maths on the web

http://ddmf.msr-inria.inria.fr/



Demonstration

Dynamic Dictionary of Mathematical Functions

ddmf.msr-inria.inria.fr/1.9.1/ddmf

Home

Dynamic Dictionary of Mathematical Functions

Welcome to this interactive site on Mathematical Functions, with properties, truncated expansions, numerical evaluations, plots, and more. The functions currently presented are elementary functions and special functions of a single variable. More functions — special functions with parameters, orthogonal polynomials, sequences — will be added with the project advances.

This is release 1.9.1 of DDMF
Select a special function from the list

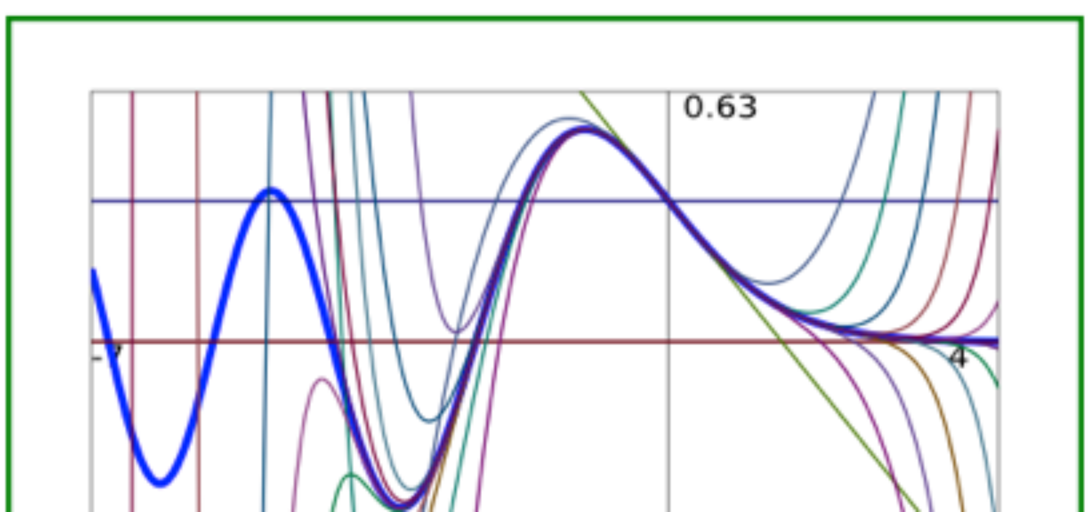
What's new? The main changes in this release 1.9.1, dated May 2013, are:

- Proofs related to Taylor polynomial approximations.

Release history.

More on the project:

- Help on selecting and configuring the mathematical rendering
- DDMF developers list
- Motivation of the project
- Article on the project at ICMS'2010
- Source code used to generate these pages
- List of related projects

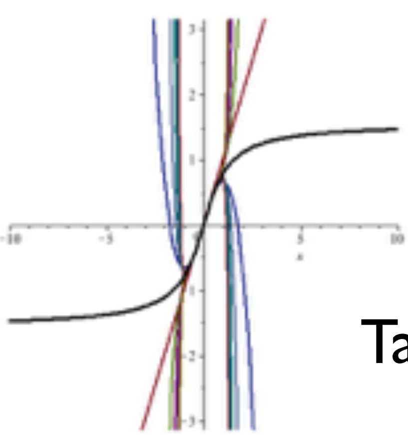


Mathematical Functions

- The Airy function of the first kind $\mathrm{Ai}(x)$
- The Airy function of the second kind $\mathrm{Bi}(x)$
- The Anger function $\mathbf{J}_n(x)$
- The inverse cosine $\arccos(x)$
- The inverse hyperbolic cosine $\operatorname{arccosh}(x)$
- The inverse cotangent $\operatorname{arccot}(x)$
- The inverse hyperbolic cotangent $\operatorname{arccoth}(x)$
- The inverse cosecant $\operatorname{arccsc}(x)$
- The inverse hyperbolic cosecant $\operatorname{arccsch}(x)$
- The inverse secant $\operatorname{arcsec}(x)$
- The inverse hyperbolic secant $\operatorname{arcsech}(x)$
- The inverse sine $\arcsin(x)$
- The inverse hyperbolic sine $\operatorname{arcsinh}(x)$
- The inverse tangent $\arctan(x)$
- The inverse hyperbolic tangent $\operatorname{arctanh}(x)$
- The modified Bessel function of the first kind $I_\nu(x)$
- The Bessel function of the first kind $J_\nu(x)$
- The modified Bessel function of the second kind $K_\nu(x)$
- The Bessel function of the second kind $Y_\nu(x)$
- The Chebyshev function of the first kind $T_n(x)$
- The Chebyshev function of the second kind $U_n(x)$
- The hyperbolic cosine integral $\mathrm{Chi}(x)$
- The cosine integral $\mathrm{Ci}(x)$
- The cosine $\cos(x)$
- The hyperbolic cosine $\cosh(x)$
- The Coulomb function $F_n(l, x)$
- The Whittaker's parabolic function $D_a(x)$
- The parabolic cylinder function $U(a, x)$
- The parabolic cylinder function $V(a, x)$

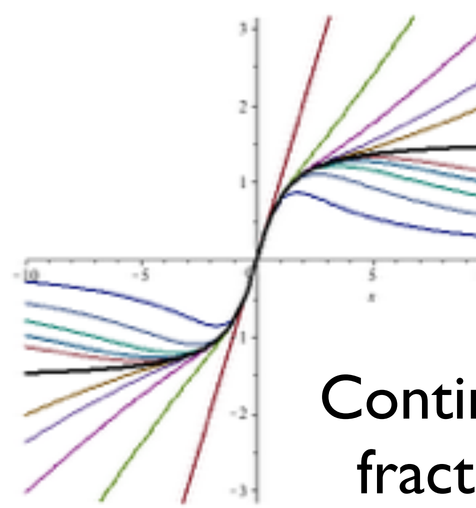
Guess & prove continued fractions

1. Differential equation produces first terms (easy):



Taylor

$$\arctan x = \cfrac{x}{1 + \cfrac{\frac{1}{3}x^2}{1 + \cfrac{\frac{4}{15}x^2}{1 + \cfrac{\frac{9}{35}x^2}{1 + \cdots}}}}$$



Continued fraction

2. Guess a formula (easy): $a_n = \dfrac{n^2}{4n^2 - 1}$

3. Prove that the CF with these a_n satisfies the differential equation.

No human intervention needed.

Automatic Proof of the guessed CF

$$\arctan x \stackrel{?}{=} \cfrac{x}{1+\cfrac{\cdots}{1+\cfrac{\frac{n^2}{4n^2-1}x^2}{1+\cdots}}}$$

- Aim: RHS satisfies $(x^2+1)y'-1=0$;
- Convergents P_n/Q_n where P_n and Q_n satisfy the LRE $u_n=u_{n-1}+a_nu_{n-2}$ (and $Q_n(0)\neq 0$);
- Define $H_n:=(Q_n)^2((x^2+1)(P_n/Q_n)'-1)$;
- H_n is a polynomial in P_n,Q_n and their derivatives;
- therefore, it satisfies a LRE that can be computed;
- from it, $H_n=O(x^n)$ visible
- from there, $(P_n/Q_n)'-1/(1+x^2)=O(x^n)$ too;
- conclude $P_n/Q_n\to$ arctan by integrating.

III. Ore Polynomials

From equations to operators

$D_x \leftrightarrow d/dx$
$x \leftrightarrow$ mult by x
product $\leftrightarrow$ composition
$D_x x = x D_x + 1$

$S_n \leftrightarrow (n \mapsto n+1)$
$n \leftrightarrow$ mult by n
product $\leftrightarrow$ composition
$S_n n = (n+1) S_n$

Taylor morphism: $D_x \mapsto (n+1)S_n$; $x \mapsto S_n^{-1}$
produces linear recurrence from LDE

Framework: Ore polynomials

$$(fg)' = f'g+fg', \quad S_n(f_n g_n) = f_{n+1} S_n(g_n), \quad \Delta_n(f_n g_n) = f_{n+1}\Delta_n(g_n)+\Delta_n(f_n)g_n,$$

and many more (e.g., q-analogues)

are captured by $\mathbb{A}\langle\partial\rangle$ ($\mathbb{A}$ integral domain) with commutation

$$\partial a = \sigma(a)\partial + \delta(a)$$

σ a ring morphism, δ a σ-derivation ($\delta(ab)=\sigma(a)\delta(b)+\delta(a)b$).

Main property: A,B in $\mathbb{A}\langle\partial\rangle$, then deg AB=deg A+deg B.

Consequence 1: (non-commutative) Euclidean division

Consequence 2: (non-commutative) Euclidean algorithm

GCRD & LCLM

greatest common right *divisor &*
least common left *multiple*

GCRD(A,B): maximal operator whose solutions are common to A and B.
LCLM(A,B): minimal operator having the solutions of A and B for solutions.

Example: closure by sum.

Computation: Euclidean algorithm or linear algebra.

Reduction of order

Input: a (large) linear recurrence equation + *ini. cond*
Output: a factor annihilating *this* solution

- Step 1: use the recurrence and its initial conditions to compute a large number of terms;

- Step 2: guess a linear recurrence equation annihilating this sequence (linear algebra);

- Step 3: take the gcrd of this operator and the initial one;

- Step 3: prove that this factor annihilates the solution by checking sufficiently many initial conditions.

Example from a continued fraction expansion

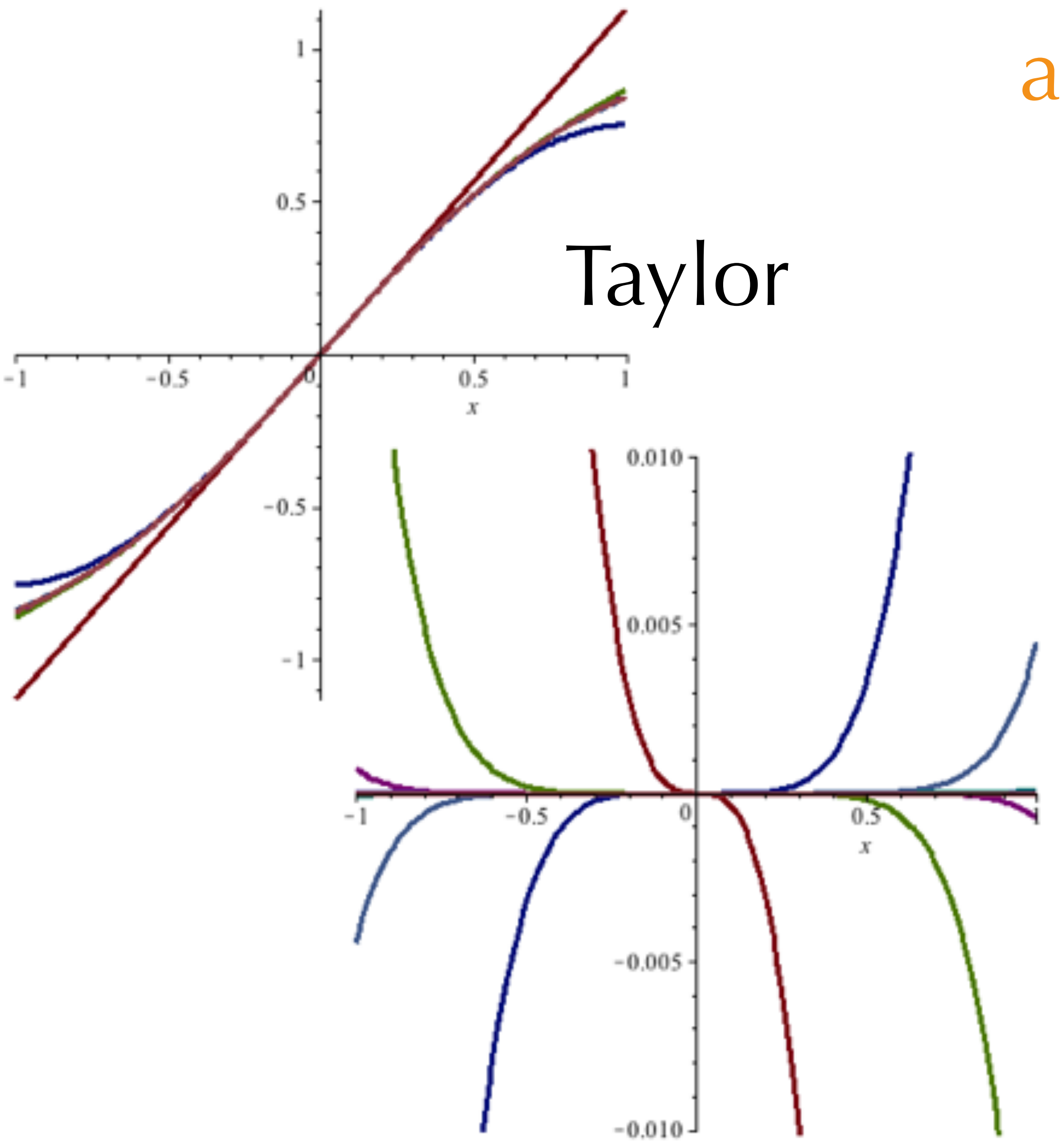
$$\mathsf{P_k = a_k x^2 P_{k-2} + P_{k-1}}, \quad \mathsf{a_k} = \begin{cases} \frac{2\mathsf{k}}{(2\mathsf{k}+1)(2\mathsf{k}+3)}, & \mathsf{k} \text{ even,} \\ \frac{-2(\mathsf{k}+2)}{(2\mathsf{k}+1)(2\mathsf{k}+3)}, & \mathsf{k} \text{ odd.} \end{cases}$$

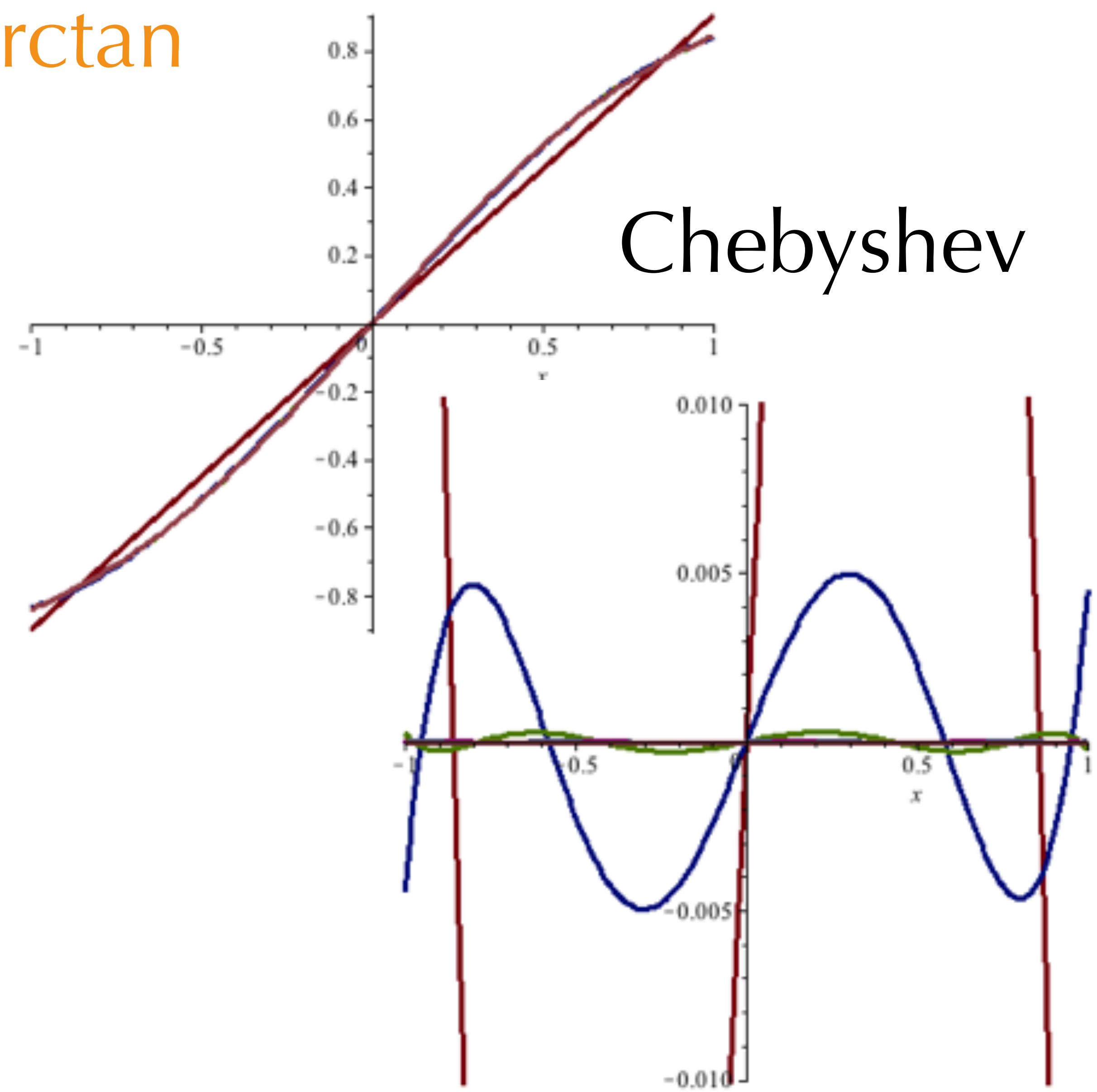
Aim: a recurrence for all k.

- Step 1: use both recurrences to find a relation between P_k,P_{k+2},P_{k+4} for even k and one for odd k;
- Step 2: compute their LCLM (order 8);
- Step 3: use the initial conditions to reduce (order 4).

Chebyshev expansions

arctan





$$z - \frac{1}{3}z^3 + \frac{1}{5}z^5 + \cdots$$

$$2(\sqrt{2}+1)\left(\frac{T_1(x)}{(2\sqrt{2}+3)} - \frac{T_3(x)}{3(2\sqrt{2}+3)^2} + \frac{T_5(x)}{5(2\sqrt{2}+3)^3} + \cdots\right)$$

Ore fractions

Generalize commutative case:

$\boxed{R=Q^{-1}P \text{ with } P \;\&\; Q \text{ operators.}}$

$B^{-1}A=D^{-1}C$ when $bA=dC$ with $bB=dD=\mathrm{LCLM}(B,D)$.

Algorithms for sum and product:

$B^{-1}A+D^{-1}C=\mathrm{LCLM}(B,D)^{-1}(bA+dC)$, with $bB=dD=\mathrm{LCLM}(B,D)$

$B^{-1}AD^{-1}C=(aB)^{-1}dC$, with $aA=dD=\mathrm{LCLM}(A,D)$.

Application: Chebyshev expansions

Extend Taylor morphism to Chebyshev expansions

Taylor

$x^{n+1}=x\cdot x^n \leftrightarrow x \mapsto X:=S^{-1}$

$(x^n)'=nx^{n-1} \leftrightarrow d/dx \mapsto D:=(n+1)S$

Chebyshev

$2xT_n(x)=T_{n+1}(x)+T_{n-1}(x)$

$\leftrightarrow x \mapsto X:=(S+S^{-1})/2$

$2(1-x^2)T_n'(x)=-nT_{n+1}(x)+nT_{n-1}(x)$

$\leftrightarrow d/dx \mapsto D:=(1-X^2)^{-1}n(S-S^{-1})/2.$

Prop. If y is a solution of L(x,d/dx), then its Chebyshev coefficients annihilate the numerator of L(X,D).

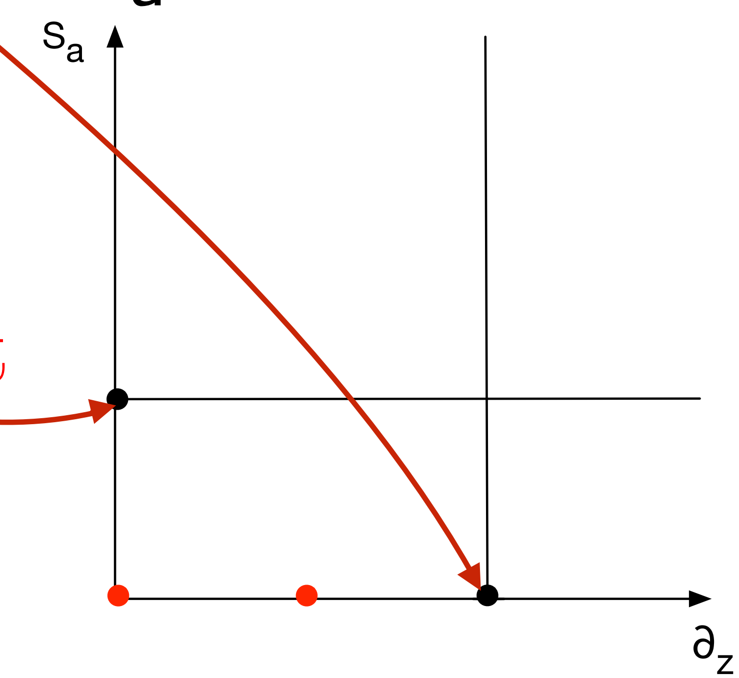
IV. Systems of equations

Example: Contiguity of Hypergeometric Series

$$F(a,b;c;z)=\sum_{n=0}^{\infty}\underbrace{\frac{(a)_n(b)_n}{(c)_n n!}}_{u_{a,n}}z^n,\qquad (x)_n:=x(x+1)\cdots(x+n-1).$$

$$\frac{u_{a,n+1}}{u_{a,n}}=\frac{(a+n)(b+n)}{(c+n)(n+1)}\to\ z(1-z)F''+(c-(a+b+1)z)F'-abF=0,$$

$$\frac{u_{a+1,n}}{u_{a,n}}=\frac{n}{a}+1\to S_aF:=F(a+1,b;c;z)=\frac{z}{a}F'+F.$$



Gauss 1812: contiguity relation.

dim=2 $\Rightarrow S_a^2F, S_aF, F$ linearly dependent

(coordinates in $\mathbb{Q}(a,b,c,z)$)

$$(a+1)(z-1)S_a^2F+((b-a-1)z+2-c+2a)S_aF+(c-a-1)F=0$$

Ore Algebras

$$\mathbb{O}:=\mathbb{K}(x_1,\ldots,x_r)\langle\partial_1,\ldots,\partial_r\rangle:=\mathbb{K}(x_1,\ldots,x_r)\langle\partial_1\rangle\cdots\langle\partial_r\rangle,$$

with commuting ∂_i's and for $i\neq j$, $\delta_i(\partial_j)=0$ and $\sigma_i(\partial_j)=\partial_j$.

Def. LM (leading monomial) on next slide.

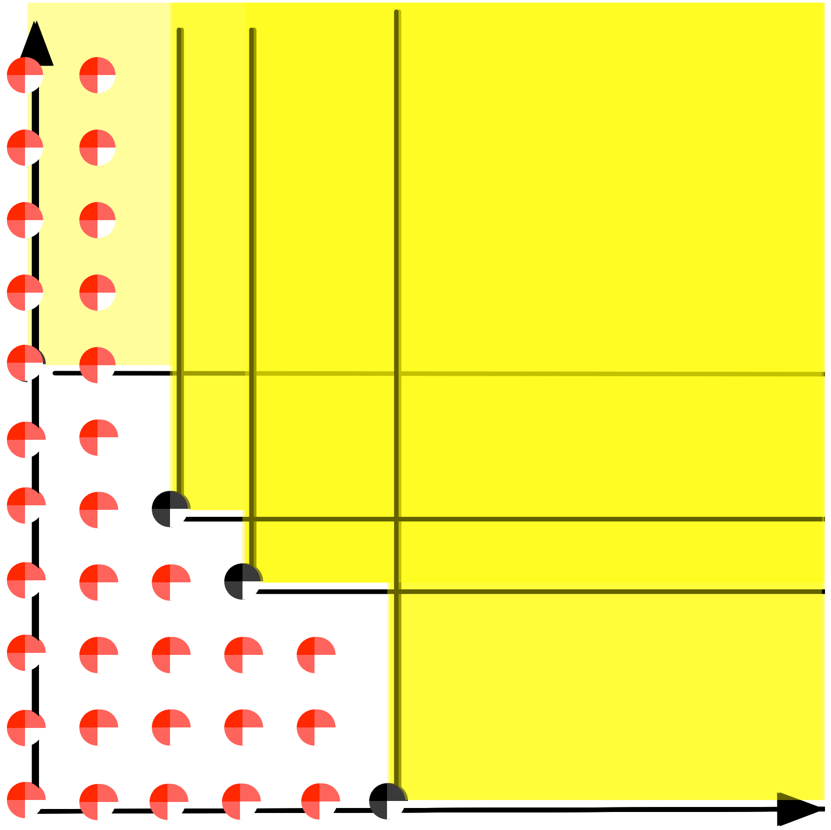
Main property: A,B in $\mathbb{O}$, then $\mathrm{LM}(AB)=\mathrm{LM}(A)\mathrm{LM}(B)$.

Consequence: (non-commutative) Gröbner bases

Gröbner bases as a data-structure to encode special functions

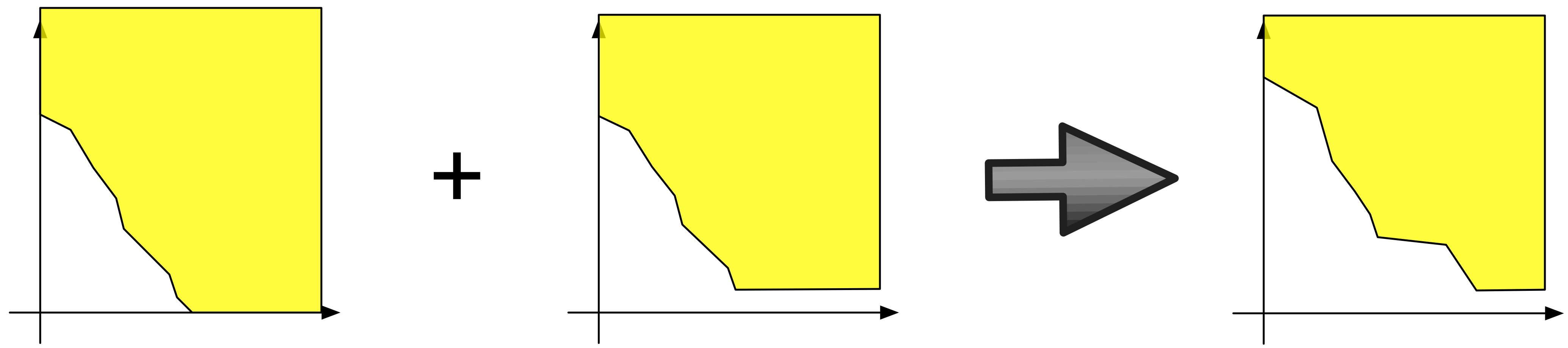
Gröbner Bases

1. Monomial ordering: order on $\mathbb{N}^k$, compatible with +, 0 minimal.
2. Leading monomial of a polynomial: the largest one.
3. Gröbner basis of a (left) ideal *I*: corners of stairs.
4. Quotient mod *I*: basis below the stairs (Vect$\{\partial^\alpha f\}$).
5. Reduction of *P*: Rewrite *P* mod *I* on this basis.
6. Dimension: « size » of the quotient.
7. D-finiteness: dimension 0.



An access to (finite-dimensional) vector spaces.

Closure Properties



Proposition.

$$\dim \operatorname{ann}(\mathsf{f}+\mathsf{g}) \le \max(\dim \operatorname{ann} \mathsf{f}, \dim \operatorname{ann} \mathsf{g}),$$
$$\dim \operatorname{ann}(\mathsf{fg}) \le \dim \operatorname{ann} \mathsf{f} + \dim \operatorname{ann} \mathsf{g},$$
$$\dim \operatorname{ann}(\partial \mathsf{f}) \le \dim \operatorname{ann} \mathsf{f}.$$

Algorithms by linear algebra

simple definitions → data-structures for more complicated functions

V. Sums and Integrals

Examples

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \sum_{j=0}^{k} \binom{k}{j}^3$$

$$\sum_{j,k} (-1)^{j+k} \binom{j+k}{k+l} \binom{r}{j} \binom{n}{k} \binom{s+n-j-k}{m-j} = (-1)^l \binom{n+r}{n+l} \binom{s-r}{m-n-l}$$

$$\int_0^{+\infty} x J_1(ax) I_1(ax) Y_0(x) K_0(x)\, dx = -\frac{\ln(1-a^4)}{2\pi a^2}$$

$$\frac{1}{2\pi i} \oint \frac{(1+2xy+4y^2)\exp\left(\frac{4x^2y^2}{1+4y^2}\right)}{y^{n+1}(1+4y^2)^{\frac{3}{2}}}\, dy = \frac{H_n(x)}{\lfloor n/2 \rfloor !}$$

$$\sum_{k=0}^{n} \frac{q^{k^2}}{(q;q)_k (q;q)_{n-k}} = \sum_{k=-n}^{n} \frac{(-1)^k q^{(5k^2-k)/2}}{(q;q)_{n-k}(q;q)_{n+k}}$$

Aims:

1. Prove them automatically
2. Find the rhs given the lhs

Note: at least one free variable

Creative telescoping

$$I(x) = \int f(x,t)\,dt = ? \quad \text{or} \quad U(n) = \sum_k u(n,k) = ?$$

Input: equations (differential for *f* or recurrence for *u*).

Output: equations for the sum or the integral.

Example:

$$u(n,k) = \binom{n}{k} \text{ def. by } \left\{ \binom{n+1}{k} = \frac{n+1}{n+1-k}\binom{n}{k}, \binom{n}{k+1} = \frac{n-k}{k+1}\binom{n}{k} \right\}$$

$$S(n+1) = \sum_k \binom{n+1}{k} = \sum_k \underbrace{\binom{n+1}{k} - \boxed{\binom{n+1}{k+1}}}_{\text{telesc.}} + \underbrace{\boxed{\binom{n}{k+1}} - \binom{n}{k}}_{\text{telesc.}} + 2\boxed{\binom{n}{k}} = 2S(n).$$

Pascal

IF one knows $A(n,S_n)$ and $B(n,k,S_n,S_k)$ such that

$$(A(n,S_n) + \Delta_k B(n,k,S_n,S_k)) \cdot u(n,k) = 0,$$

certificate

then the sum telescopes, leading to $A(n,S_n) \cdot U(n) = 0$.

Creative Telescoping

$$I(x) = \int f(x,t)\,dt = ?$$

IF one knows $A(x,\partial_x)$ and $B(x,t,\partial_x,\partial_t)$ such that

$$(A(x,\partial_x) + \partial_t B(x,t,\partial_x,\partial_t)) \cdot f(x,t) = 0,$$

then the integral « telescopes », leading to $A(x,\partial_x) \cdot I(x) = 0$.

Then I come along and try differentiating under the integral sign, and often it worked. So I got a great reputation for doing integrals.

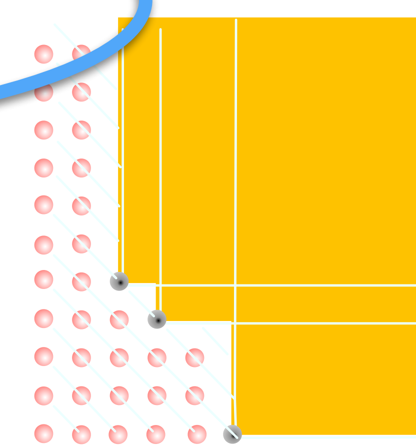
Richard P. Feynman 1985

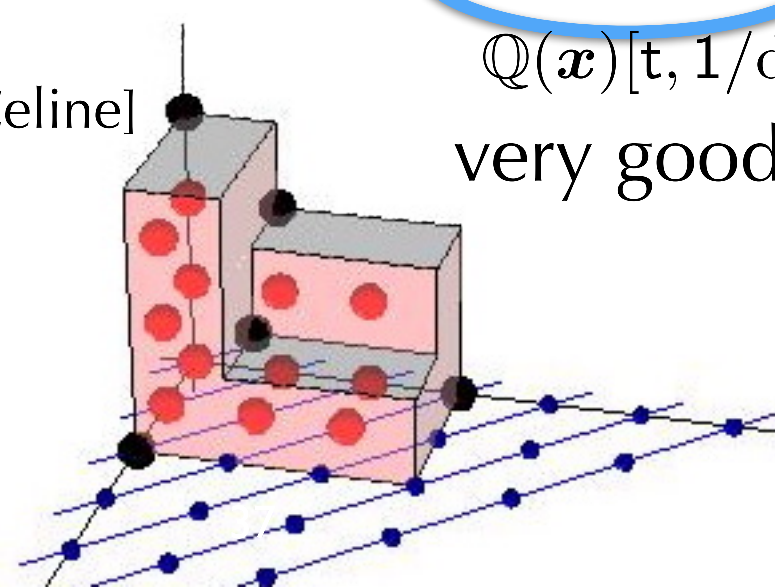
Method: integration (summation) by parts and differentiation (difference) under the integral (sum) sign

Telescoping Ideal

$$\mathrm{T_t(f)} := \Big(\mathrm{Ann}\,\mathrm{f} + \underbrace{\partial_t \mathbb{Q}(\boldsymbol{x},t)\langle \boldsymbol{\partial}_{\boldsymbol{x}}, \partial_t\rangle}_{\text{int. by parts}}\Big) \cap \underbrace{\mathbb{Q}(\boldsymbol{x})\langle\boldsymbol{\partial}_{\boldsymbol{x}}\rangle}_{\text{diff. under } \int} .$$

- hypergeometric summation: dim=1 + param. Gosper. [Zeilberger]
- holonomy: restrict int. by parts to $\mathbb{Q}(\boldsymbol{x})\langle\boldsymbol{\partial}_{\boldsymbol{x}}, \partial_{\mathrm{t}}\rangle$ and Gröbner bases. [Wilf-Zeilberger, also Sister Celine]
- finite dim, Ore algebras & GB [Chyzak]
- infinite dim & GB
- rational *f* and restrict to $\mathbb{Q}(\boldsymbol{x})[\mathrm{t}, 1/\mathrm{den\,f}]\langle\boldsymbol{\partial}_{\boldsymbol{x}}, \partial_{\mathrm{t}}\rangle$ in very good complexity.





Chyzak's Algorithm

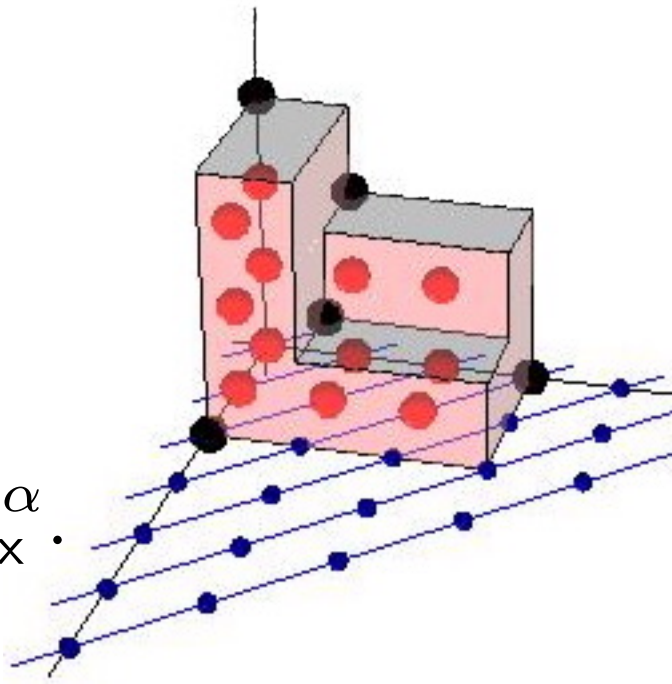
$$T_t(f) := \Big(\mathrm{Ann}\, f + \underbrace{\partial_t \mathbb{Q}(\boldsymbol{x}, t)\langle \boldsymbol{\partial_x}, \partial_t\rangle}_{\text{int. by parts}}\Big) \cap \underbrace{\mathbb{Q}(\boldsymbol{x})\langle \boldsymbol{\partial_x}\rangle}_{\text{diff. under } \int} .$$

Input: a Gröbner basis G for Ann f in $\mathbb{A}$=$\mathbb{Q}(x,t)\langle\partial_x,\partial_t\rangle$
Output: P in $\mathbb{Q}(x)\langle\partial_x\rangle$ and Q in $\mathbb{A}$,
reduced wrt G and such that $(P+\partial_t Q)f=0$.

For r=1,2,3,...

1. use indeterminate coefficients to define

$$Q = \sum_{(i,j) \text{ below stairs}} q_{i,j}(x,t)\partial_x^i\partial_t^j, \quad P = \sum_{|\alpha|\le r} p_\alpha(x)\partial_x^\alpha .$$



2. reduce $P+\partial_t Q$ using G, leading to a 1st order system for $q_{i,j}(x,t)$ and $p_\alpha(x)$;
3. stop if a rational solution is found.

Examples of applications

- **Hypergeometric**: binomial sums, hypergeometric series;

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3 = (-1)^n \frac{(3n)!}{n!^3}$$

- **Higher dimension**: classical orthogonal polynomials, special functions like Bessel, Airy, Struve, Weber, Anger, hypergeometric and generalized hypergeometric,...

$$J_0(z) = \frac{2}{\pi} \int_0^1 \frac{\cos(zt)}{\sqrt{1-t^2}}\, dt$$

- **Infinite dimension**: Bernoulli, Stirling or Eulerian numbers, incomplete Gamma function,...

$$\int_0^\infty \exp(-xy)\Gamma(n,x)\, dx = \frac{\Gamma(n)}{y}\left(1 - \frac{1}{(y+1)^n}\right)$$

VI. Faster Creative Telescoping

Certificates are big

$$C_n := \sum_{r,s} \underbrace{(-1)^{n+r+s}\binom{n}{r}\binom{n}{s}\binom{n+s}{s}\binom{n+r}{r}\binom{2n-r-s}{n}}_{f_{n,r,s}}$$

$$(n+2)^3C_{n+2} - 2(2n+3)(3n^2+9n+7)C_{n+1} - (4n+3)(4n+4)(4n+5)C_n = 180\text{ kB} \simeq 2\text{ pages}$$

$$I(z) = \oint \frac{(1+t_3)^2 dt_1 dt_2 dt_3}{t_1t_2t_3(1+t_3(1+t_1))(1+t_3(1+t_2)) + z(1+t_1)(1+t_2)(1+t_3)^4}$$

$$z^2(4z+1)(16z-1)I'''(z) + 3z(128z^2+18z-1)I''(z) + (444z^2+40z-1)I'(z) + 2(30z+1)I(z) = 1\,080\text{ kB}$$

$\simeq$ 12 pages

Next, in $T_t(f) := \Big(\mathrm{Ann}\, f + \underbrace{\partial_t \mathbb{Q}(\boldsymbol{x}, t)\langle \boldsymbol{\partial_x}, \partial_t\rangle}_{\text{int. by parts}}\Big) \cap \underbrace{\mathbb{Q}(\boldsymbol{x})\langle \boldsymbol{\partial_x}\rangle}_{\text{diff. under } \int}$.

we restrict to rational f and $\partial_t \mathbb{Q}(\boldsymbol{x})[t, 1/\operatorname{den} f]\langle \boldsymbol{\partial_x}, \partial_t\rangle$

Bivariate integrals by Hermite reduction

$$I(t) = \oint \frac{P(t,x)}{Q^m(t,x)}\,dx$$

Q square-free

Int. over a cycle where Q≠0.

If m=1, Euclidean division: P=aQ+r, $\deg_x r<\deg_x Q$

$$\frac{P}{Q} = \frac{r}{Q} + \partial_x \int a$$

Def. Reduced form: $\left[\frac{P}{Q}\right] := \frac{r}{Q}$

If m>1, Bézout identity and integration by parts

$$P = uQ + v\partial_x Q \quad \rightarrow \quad \frac{P}{Q^m} = \underbrace{\frac{u + \frac{\partial_x v}{m-1}}{Q^{m-1}}}_{A_{m-1}} + \partial_x \frac{v/(1-m)}{Q^{m-1}}$$

$$\left[\frac{P}{Q^m}\right] := [A_{m-1}]$$

Algorithm: $R_0:=[P/Q^m]$

for i=1,2,… do $R_i:=[\partial_t R_{i-1}]$

when there is a relation $c_0(t)R_0+\ldots+c_i(t)R_i=0$

return $c_0+\ldots+c_i\partial_t^i$

More variables: Griffiths-Dwork reduction

$$I(t) = \oint \frac{P(t,\underline{x})}{Q^m(t,\underline{x})}\, d\underline{x}$$

Q square-free
Int. over a cycle where Q≠0.

1. Control degrees by homogenizing $(x_1,\dots,x_n)\mapsto(x_0,\dots,x_n)$
2. If $m=1$, $[P/Q]:=P/Q$
3. If $m>1$, reduce modulo Jacobian ideal $J := \langle \partial_0 Q, \dots, \partial_n Q\rangle$

$$P = r + v_0\partial_0 Q + \cdots + v_n \partial_n Q$$

$$\frac{P}{Q^m} = \frac{r}{Q^m} - \frac{1}{m-1}\left(\partial_0 \frac{v_0}{Q^{m-1}} + \cdots + \partial_n \frac{v_n}{Q^{m-1}}\right) + \underbrace{\frac{1}{m-1}\,\frac{\partial_0 v_0 + \cdots + \partial_n v_n}{Q^{m-1}}}_{A_{m-1}}$$

$$\left[\frac{P}{Q^m}\right] := \frac{r}{Q^m} + [A_{m-1}]$$

Thm. [Griffiths] In the regular case ($\mathbb{Q}(t)[\underline{x}]/J$ finite dim), if $R=P/Q^m$ hom of degree -n-1, $[R]=0 \Leftrightarrow \oint R\,d\underline{x} = 0$.

→ SAME ALGORITHM.

Size and complexity

$$I(t) = \oint \underbrace{\frac{P(t,\underline{x})}{Q^m(t,\underline{x})}}_{\in\mathbb{Q}(t,\underline{x})} d\underline{x}$$

no regularity assumed

$$N := \deg_{\underline{x}} Q, \quad d_t := \max(\deg_t Q, \deg_t P)$$

$\deg_x P$ not too big

Thm. A linear differential equation for I(t) can be computed in $O(e^{3n}N^{8n}d_t)$ operations in $\mathbb{Q}$.
It has order $\leq N^n$ and degree $O(e^n N^{3n} d_t)$.

tight

Note: generically, the certificate has at least $N^{n^2/2}$ monomials.

This has consequences for multiple binomial sums.

Conclusion

- Linear differential equations and recurrences are a great data-structure;
- Numerous algorithms have been developed in computer algebra;
- Efficient code is available;
- More is to be found (certificate-free algorithms, diagonals,…)