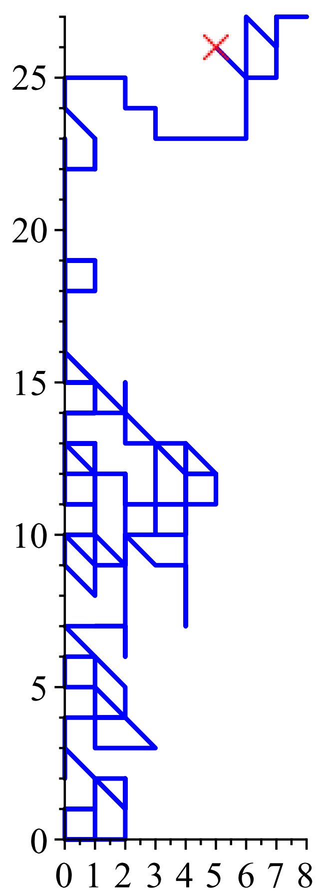
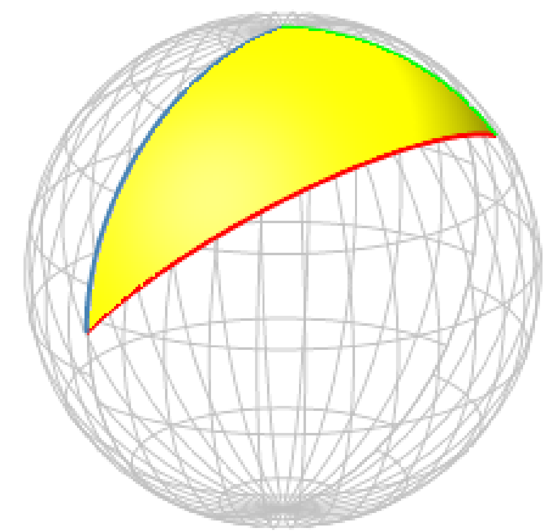
Lattice Walks and Reliable Dominant Eigenvalues of the Laplacian on Spherical Triangles



Bruno Salvy

AriC, Inria, ENS de Lyon



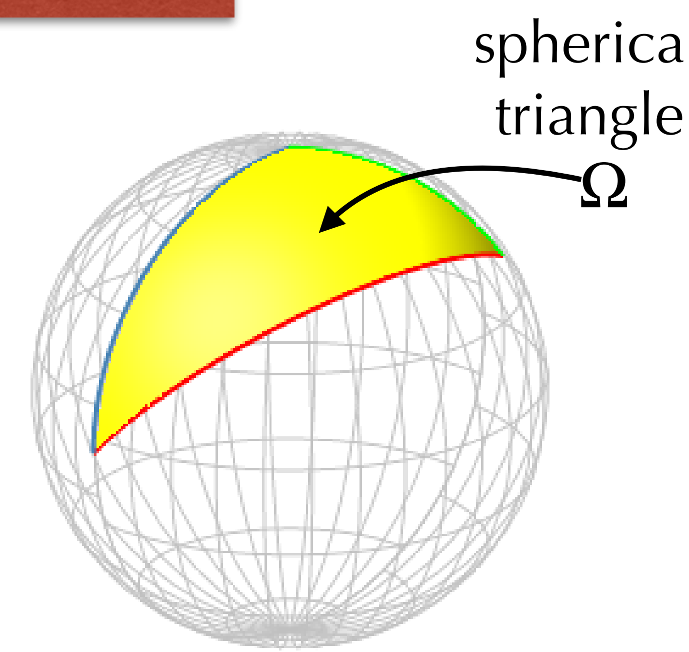
Joint work with Joel Dahne

Dominant Eigenvalue of the Laplace-Beltrami Operator on the Unit Sphere

Laplace operator in spherical coordinates in $\mathbb{R}^d$

Laplace-Beltrami

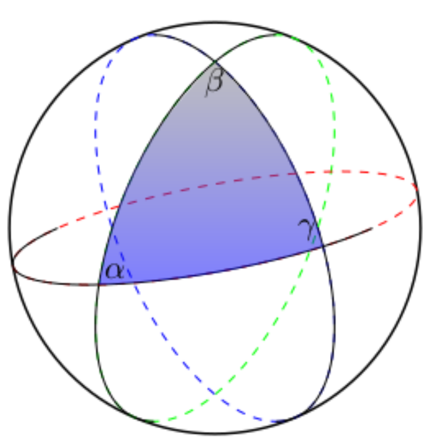
$$\Delta f = r^{1-d}\frac{\partial}{\partial r}\left(r^{d-1}\frac{\partial f}{\partial r}\right) + r^{-2}\Delta_{\mathbb{S}^{d-1}} f$$



$$\boxed{\begin{array}{l}\text{Eigenvalue problem for } \Omega \subset \mathbb{S}^{d-1}: \\ \quad \Delta_{\mathbb{S}^{d-1}} f + \lambda f = 0 \text{ in } \Omega, \quad f|_{\partial\Omega} = 0.\end{array}}$$

Classical fact: $0 < \lambda_1 < \lambda_2 \leq \cdots, \quad \lambda_n \to \infty$

dominant eigenvalue



Goal: $(\alpha, \beta, \gamma) \mapsto \lambda_1$ with high precision

(dimension d=3)

Results

Angles	BPRT	new
$(3\pi/4, \pi/3, \pi/2)$	12.400051	$12.400051652843377905\ldots \pm 10^{-26}$
$(2\pi/3, \pi/3, \pi/2)$	13.74435	$13.744355213213231835\ldots \pm 10^{-94}$
$(2\pi/3, \pi/4, \pi/2)$	20.5719**64**	$20.571973537984730557\ldots \pm 10^{-28}$
$(2\pi/3, \pi/3, \pi/3)$	21.309407	$21.309407630190445260\ldots \pm 10^{-159}$
$(3\pi/4, \pi/4, \pi/3)$	24.45691**0**	$24.456913796299111694\ldots \pm 10^{-40}$
$(2\pi/3, \pi/4, \pi/4)$	49.10994**2**	$49.109945263284609920\ldots \pm 10^{-129}$
$(2\pi/3, 3\pi/4, 3\pi/4)$	4.261735	$4.3 \pm 5\ 10^{-2}$
$(2\pi/3, 2\pi/3, 2\pi/3)$	5.159146	$5.16 \pm 5\ 10^{-3}$
$(\pi/2, 2\pi/3, 3\pi/4)$	6.241748	$6.2 \pm 5\ 10^{-2}$

finite elements &
convergence acceleration

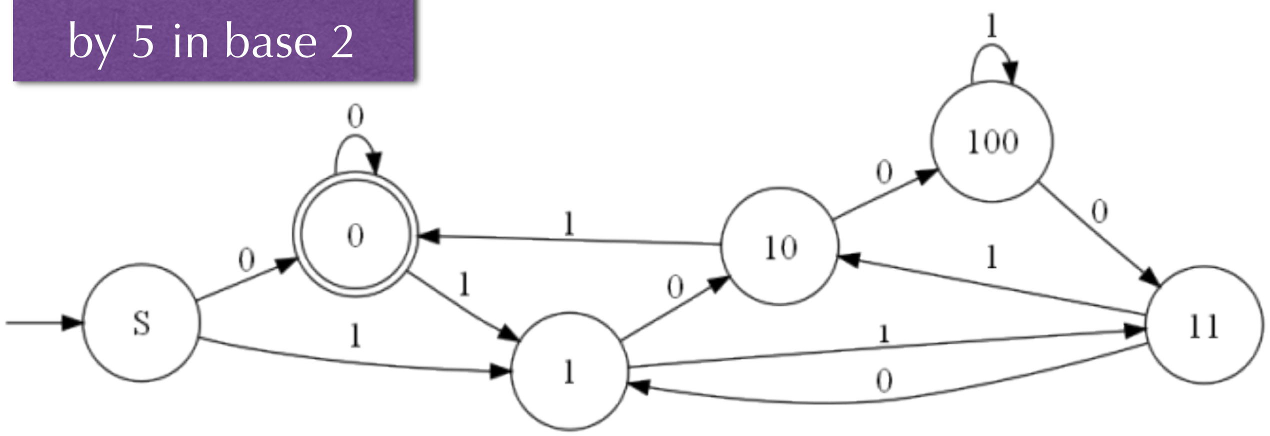
I. Why do we care?
II. How do we do it?
III. What is going on?

[BogoselPerrollazRaschelTrotignon18]

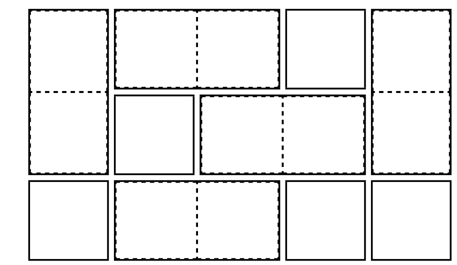
I. Linear Recurrences with Constant Coefficients

Linear Recurrent Sequences

Numbers divisible by 5 in base 2



Tilings of rectangles of bounded height by dominos and monominos



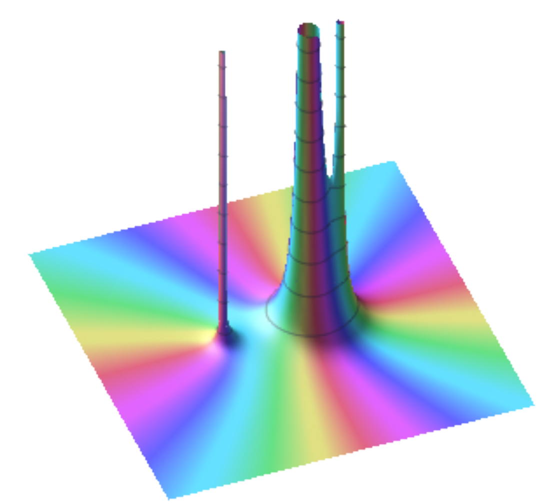
$$u_{n+k} = a_0 u_n + \cdots + a_{k-1} u_{n+k-1} \quad \text{with initial conditions} \quad u_0, \ldots, u_{k-1}$$

very well understood

(u_n) is a LRS $\iff$ its generating series $U(z) := \sum_{n=0}^{\infty} u_n z^n$ is rational

Ex. Fibonacci: $F_{n+2} = F_{n+1} + F_n, \quad F_0 = F_1 = 1$

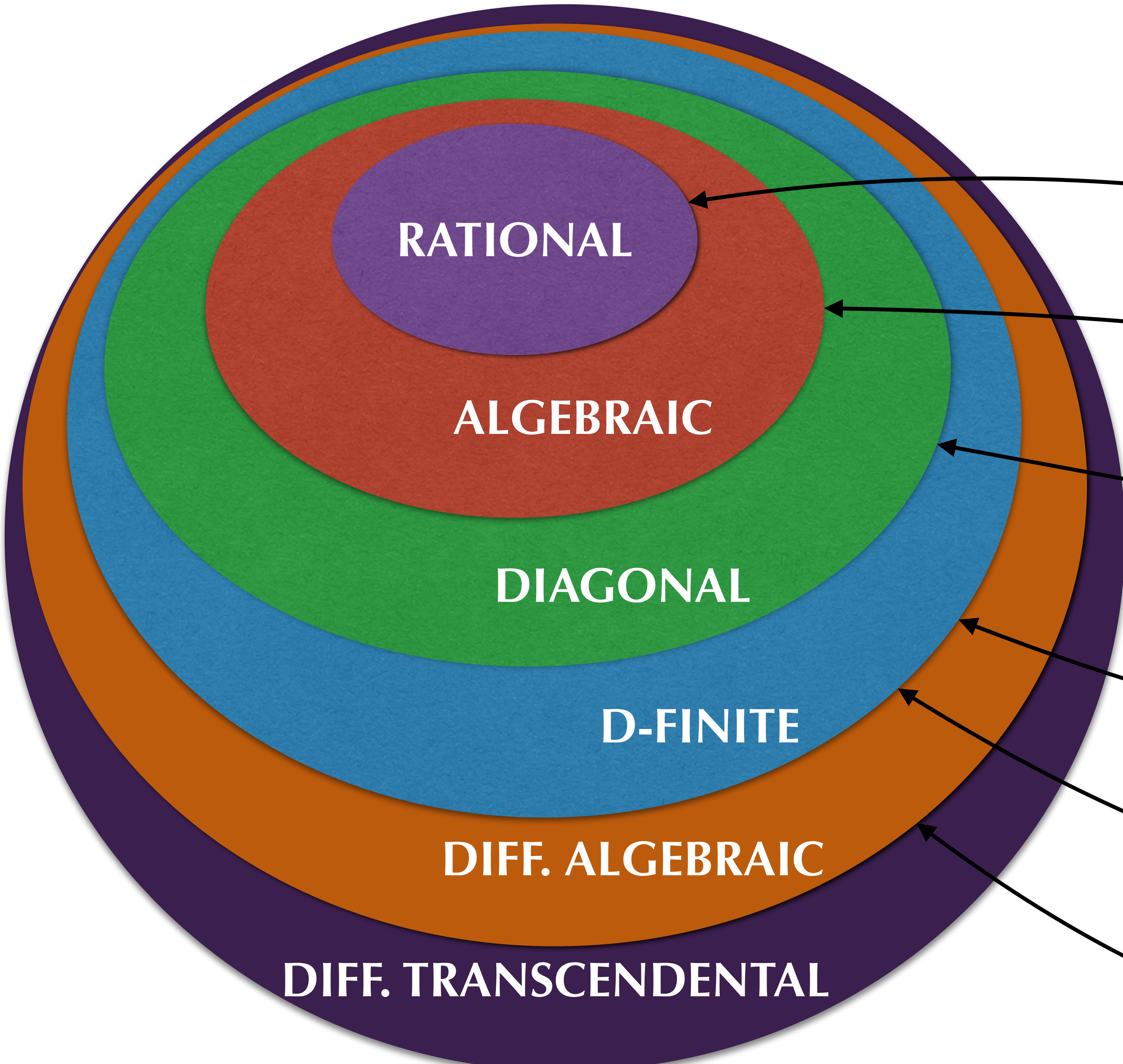
$$F(z) = \frac{z}{1 - z - z^2} = \frac{(2\phi - 1)/5}{1 - z\phi} - \frac{(2\phi - 1)/5}{1 + z/\phi}$$



$$F_n = \frac{1}{2\pi i} \oint \frac{F(z)}{z^{n+1}} \, dz$$

Classes of Univariate Power Series

$$U(z) := \sum_{n=0}^{\infty} u_n z^n$$



linear recurrent sequences

$P(z, U(z)) = 0$

words in context-free languages

Christol's conjecture: All D-finite power series with integer coefficients and radius of convergence in $(0,\infty)$ are diagonals.

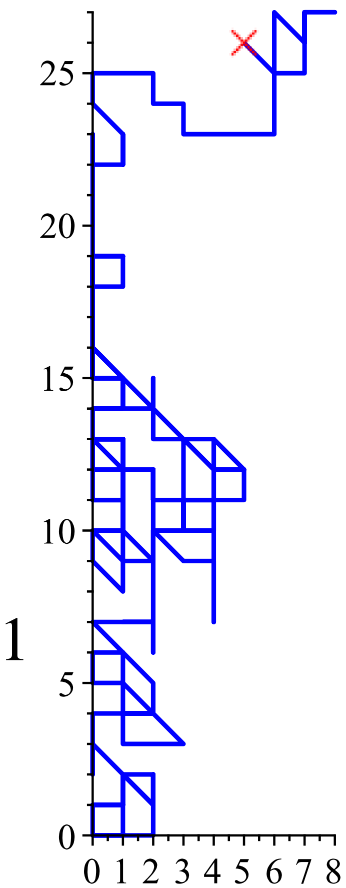
(u_n) satisfies a linear recurrence with polynomial coefficients

$U(z)$ satisfies a linear diff. eqn. with polynomial coefficients

$P(z, U(z), U'(z), \ldots) = 0$

Knowing where $U(z)$ fits helps deduce properties of (u_n) .

Lattice Walks: a Mine of Linear Recurrences Waiting for Tools



Num. walks from (0,0) to $(i,j) \in \mathbb{Z}^2$ using n steps in $\mathcal{S}$

Ex.: $\mathcal{S} = \{\uparrow, \downarrow, \rightarrow, \leftarrow, \nwarrow\}$

$$u_{i,j,n} = u_{i-1,j,n-1} + u_{i,j-1,n-1} + u_{i+1,j,n-1} + u_{i,j+1,n-1} + u_{i+1,j+1,n-1}$$

$$U(x,y,z) := \sum_{i,j,n} u_{i,j,n} x^i y^j z^n, \quad U(0,0,z) = \sum_{n\geq 0} e_n z^n, \quad U(1,1,z) = \sum_{n \geq 0} u_n z^n.$$

Applications: queuing theory, statistical physics, combinatorics,...

Question: $\mathcal{S}$, boundary conditions $\rightarrow$ nature of these series?

Boundary conditions:

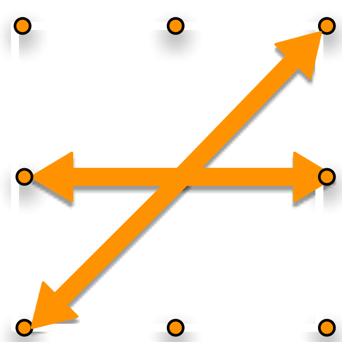
no constraint : U rational;

$u_{i,j,n} = 0$ for $i < 0$: U algebraic;

$u_{i,j,n} = 0$ for $i < 0$ and $j < 0$: depends on $\mathcal{S}$.

[BanderierFlajolet02]

Walks in $\mathbb{N}^2$: Recent Progress



79 quadrant models
with small steps

Motivation for this work

Guess & Prove (4)
algebraic
(1965,1985,2010)

Creative telescoping (19)
D-finite
(2010)

Probabilities & number theory (56)
non-D-finite
(2014)

Tutte invariants (9)
Diff. algebraic
(2017)

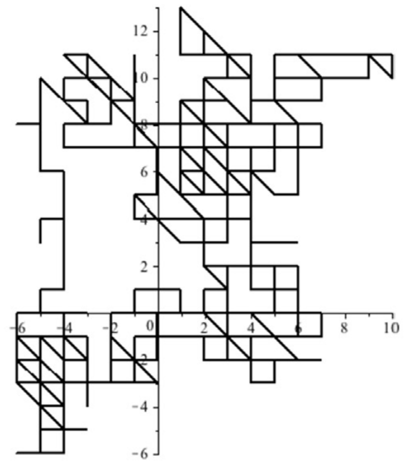
Differential Galois theory (47)
Diff. transcendental
(2018)

[Bernardi,Bostan,Bousquet-Mélou,Gessel,Gouyou-Beauchamps,Hardouin,Kauers, Kreweras,Melczer,Mishna,Raschel,Rechnitzer,Roques,Salvy,Singer]

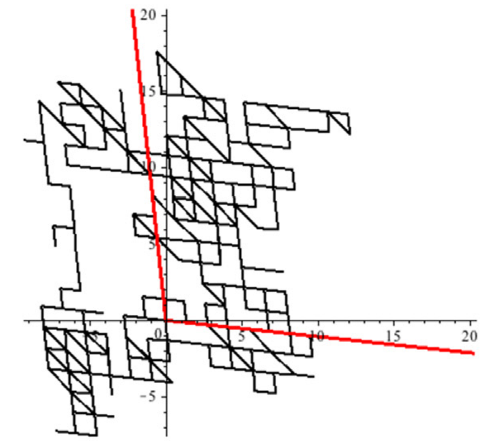
Probabilities & Number Theory for Walks in $\mathbb{N}^d$

$$\mathcal{S} = \{\uparrow, \downarrow, \rightarrow, \leftarrow, \searrow\}$$

Idea: Normalize so that the asymptotic behaviour is a Brownian motion



a. fix probabilities for each step that remove drift



b. linear transform to remove correlation

$\mathbb{N}^d$ becomes a cone K

Probabilistic ingredient:

$$e_n \sim K\rho^n n^{-p/2} \text{ with } p = \sqrt{\lambda_1 + (d/2-1)^2} - (d/2-1) > 0,$$

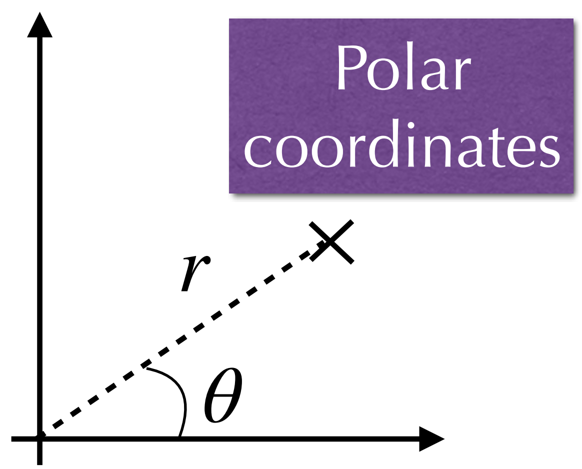
λ_1 dominant eigenvalue of $\Delta_{\mathbb{S}^{d-1}}$ on $K \cap \mathbb{S}_{d-1}$.

Arithmetic ingredient:

$U(z)$ D-finite, convergent, with integer coefficients $\Rightarrow p \in \mathbb{Q}$.

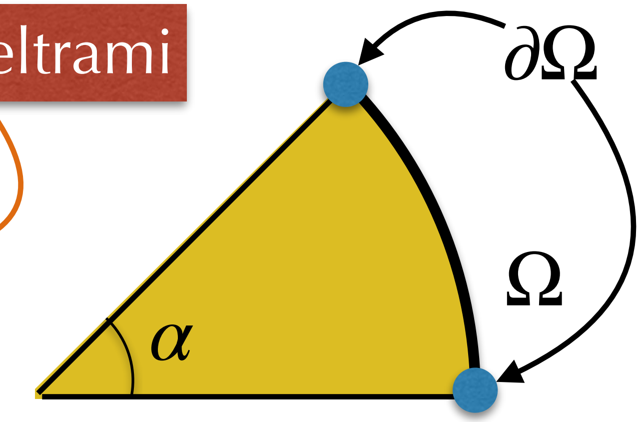
[DenisovWachtel15;Chudnovsky85;André89;Katz70]

Planar Case



Laplace-Beltrami

$$\Delta f(r,\theta) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}$$



Eigenfunctions: $\sin(\sqrt{\lambda}\theta + c)$

Boundary: $\begin{cases} \theta = 0 \to c = 0 \\ \theta = \alpha \to \lambda = \lambda_k := \left(\dfrac{k\pi}{\alpha}\right)^2, \; k \in \mathbb{N}^\star . \end{cases}$

Ex. $\mathcal{S} = \{\uparrow, \downarrow, \rightarrow, \leftarrow, \searrow\}$

$$-\frac{p}{2} = 1 - \frac{\pi}{\arccos(u)},\ 8u^3 - 8u^2 + 6u - 1 = 0,\ u > 0 \Rightarrow p \notin \mathbb{Q} \Rightarrow U \text{ not D-finite.}$$

Automatic proof of 51 of the 56 non-Dfinite cases.

[BostanRaschelSalvy14]

II. 3D Walks: Laplacian on Spherical Triangles

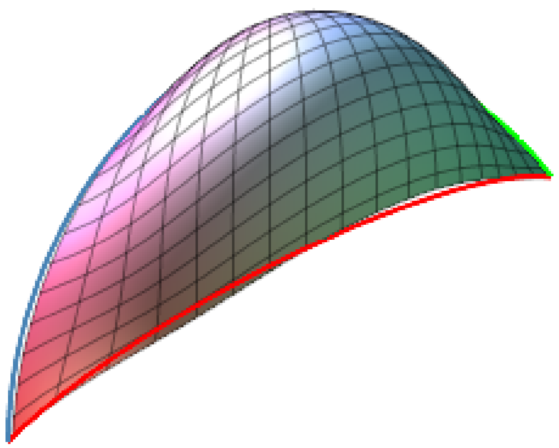
No more closed-forms? Turn to numerical approximation.

Method of Particular Solutions

Eigenvalue problem for Ω :

$$\Delta f + \lambda f = 0 \text{ in } \Omega, \quad f|_{\partial\Omega} = 0.$$

replace by f small



A 3-step method:

1. Find a basis of solutions of $\Delta f + \lambda f = 0$ in Ω;
2. Use it to find $(f^\star, \lambda^\star)$ s.t. $f^\star$ is small on $\partial\Omega$;
3. A close-by (eigenvalue,eigenfunction) satisfies

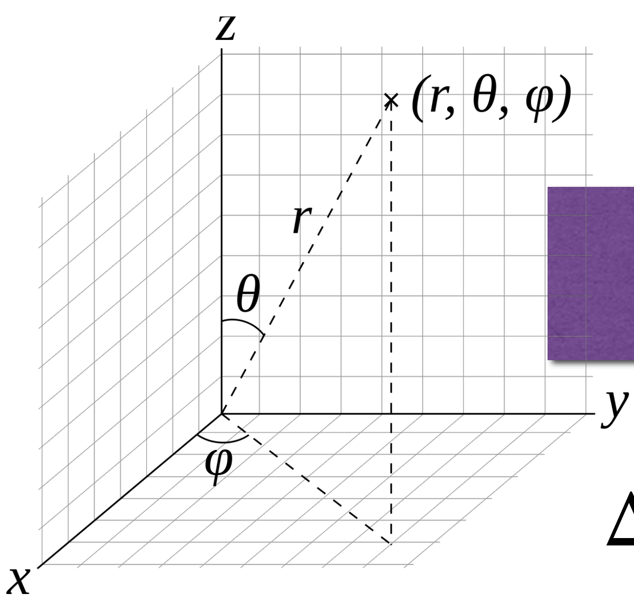
High precision needed

$$\frac{|\lambda - \lambda^\star|}{\lambda^\star} \le \frac{\sup_{x\in\partial\Omega} |f^\star(x)|}{\|f^\star\|} \mathrm{Vol}(\Omega)\,.$$

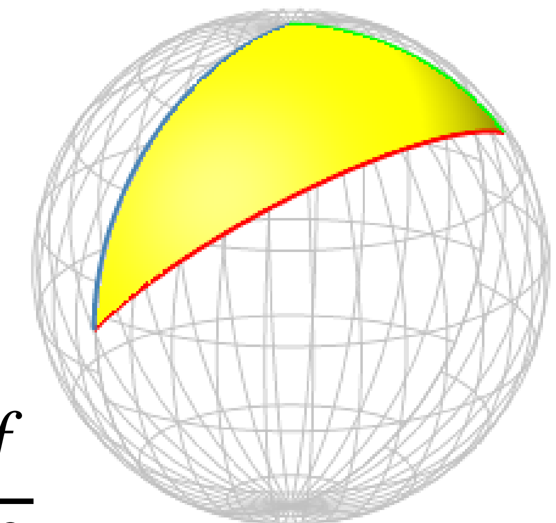
Reliable estimate needed

[FoxHenriciMoler67]

Step 1. EigenFunctions



Spherical coordinates



$$\Delta_{\mathbb{S}^2} f(\theta,\phi) = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

Separation of variables: $f = F(\phi)G(\theta)$ gives

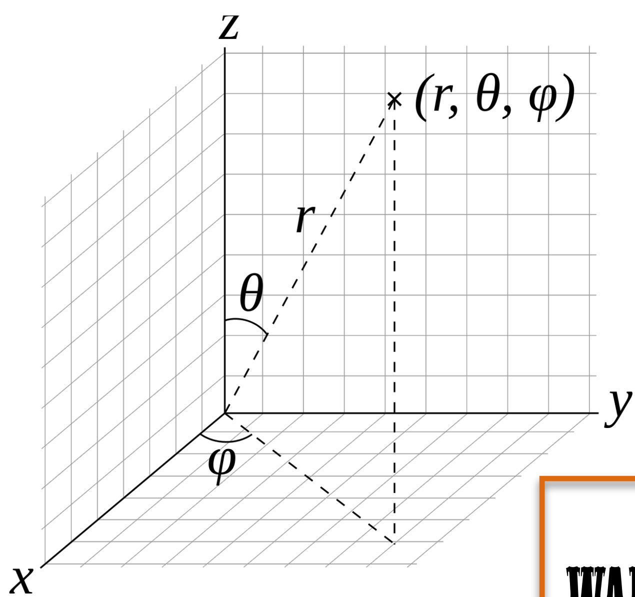
$$F(\phi) = \sin(\mu\phi + c), \quad G(\theta) = \mathrm{P}_\nu^\mu(\cos\theta) \quad (\mu \le 0)$$

with $\lambda = \nu(\nu+1)$.

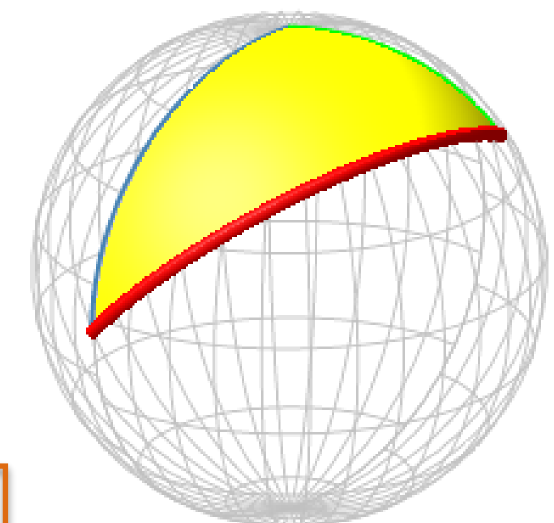
Ferrers function of the 1st kind
(cousins of the Legendre functions; spherical harmonic when μ,ν integers; known to Arb)

First 2 boundaries:
$$\begin{cases} \phi = 0 \to c = 0 \\ \phi = \phi_{\max} \to \mu = \mu_k := -\dfrac{k\pi}{\phi_{\max}},\ k \in \mathbb{N}. \end{cases}$$

Step 2. The Final Boundary

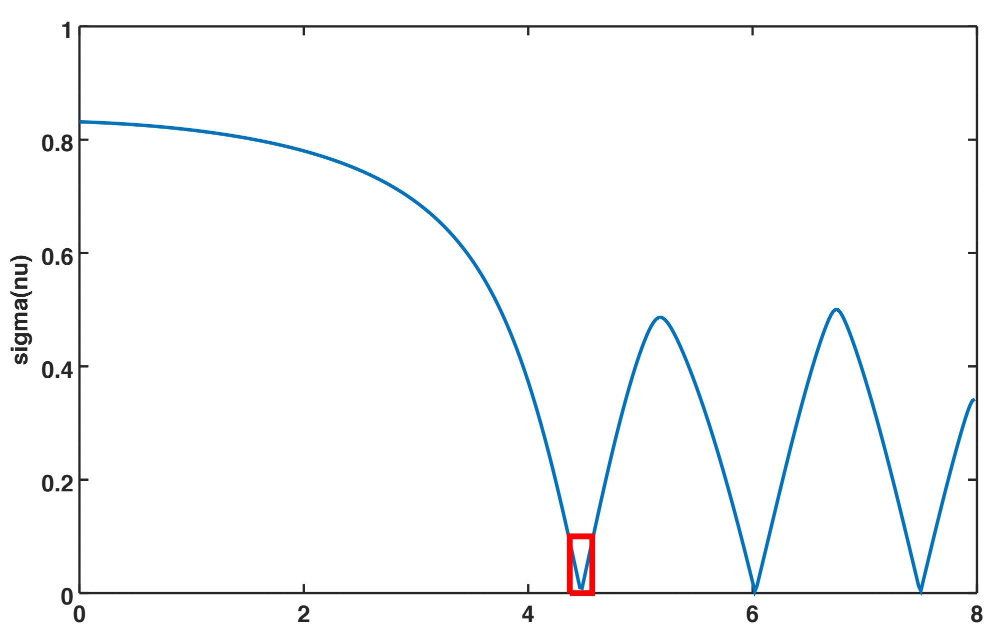


$$B_\nu^N(\phi) := \sum_{k=0}^{N} c_k \sin(\mu_k \phi) P_\nu^{\mu_k}(\cos\theta(\phi))$$

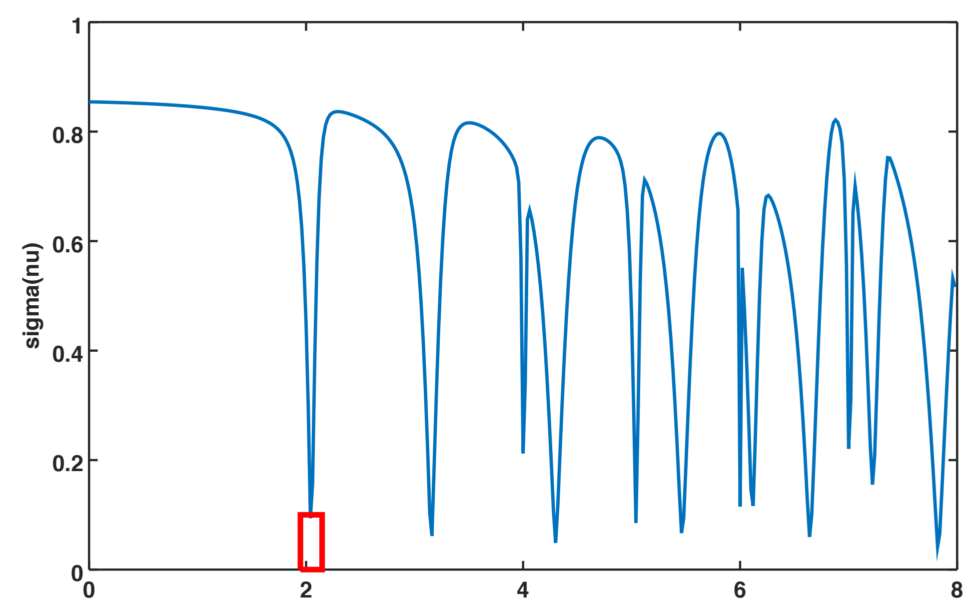


WANTED: ν s.t. $B_\nu^\infty|_{[0,\phi_{\max}]} = 0, \; \|B_\nu\|_\Omega \approx 1.$

For each ν, use [BetckeTrefethen05]

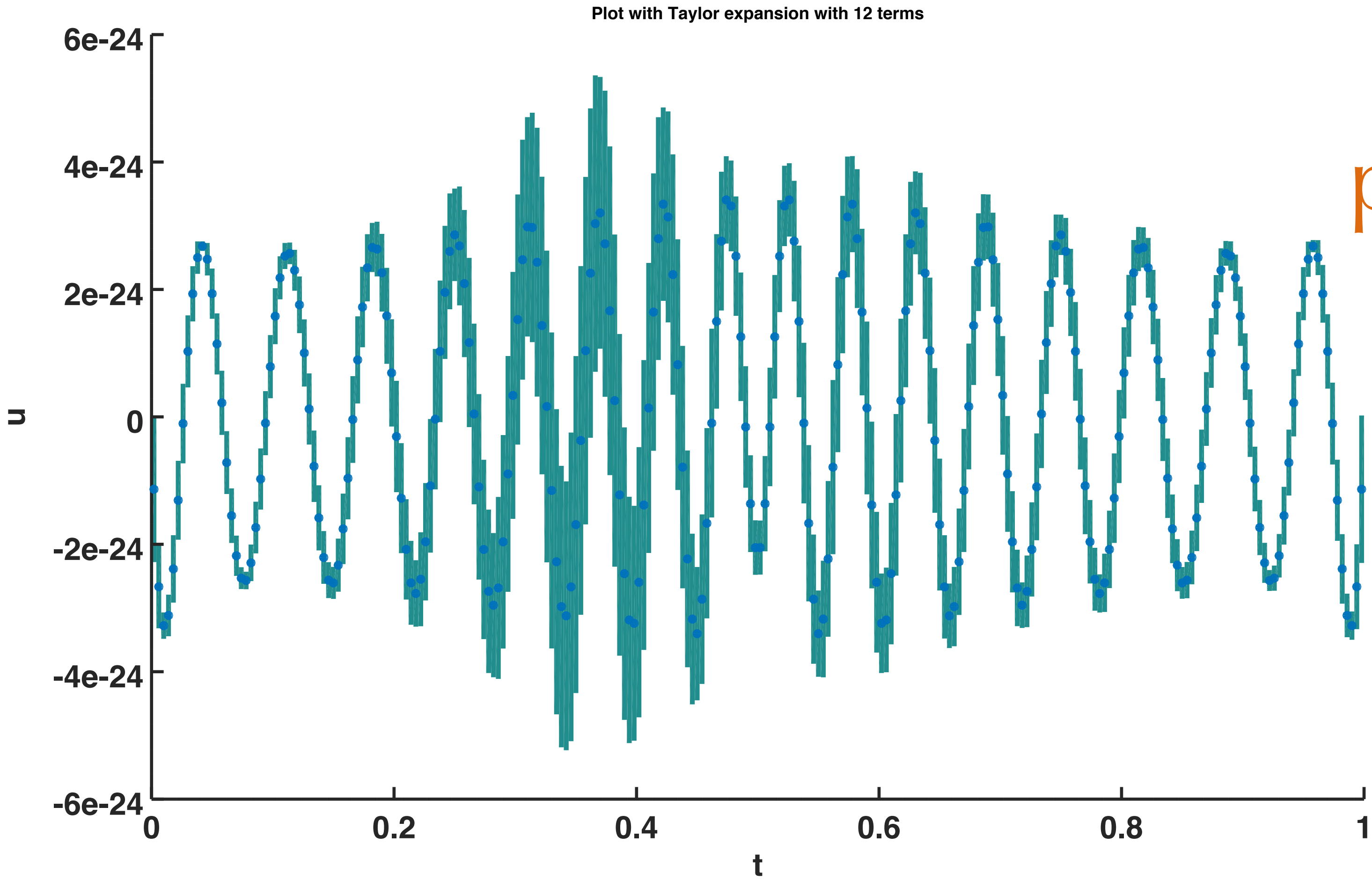


good case → many digits



not so good case → fewer digits

Step 3. Rigorous Bound on the Boundary



Use a rigorous polynomial approximation

Taylor coefficients from gfun

interval evaluation in Arb.

Summary & Conclusion

Linear recurrences with constant coefficients remain mysterious;

lattice walks provide a simple source of examples;

more and more tools are available;

high-precision is useful in experimental mathematics;

work still in progress (improve speed, work on the bad cases).

Thank you.