Generalized Hermite Reduction, Creative Telescoping and Definite Integration of D-Finite Functions

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Based on ISSAC'18 article with A. Bostan, F. Chyzak and P. Lairez

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Examples of Creative Telescoping

$$\sum_{j,k} (-1)^{j+k} \binom{j+k}{k+l} \binom{r}{j} \binom{n}{k} \binom{s+n-j-k}{m-j} = (-1)^l \binom{n+r}{n+l} \binom{s-r}{m-n-l}$$

$$\int_0^{+\infty} x J_1(ax) I_1(ax) Y_0(x) K_0(x)\, dx = -\frac{\ln(1-a^4)}{2\pi a^2}$$

$$\frac{1}{2\pi i} \oint \frac{(1+2xy+4y^2)\exp\left(\frac{4x^2y^2}{1+4y^2}\right)}{y^{n+1}(1+4y^2)^{\frac{3}{2}}}\, dy = \frac{H_n(x)}{\lfloor n/2 \rfloor !}$$

$$\int_{-1}^{1} \frac{e^{-px} T_n(x)}{\sqrt{1-x^2}}\, dx = (-1)^n \pi I_n(p)$$

Today

$$\sum_{k=0}^{n} \frac{q^{k^2}}{(q;q)_k (q;q)_{n-k}} = \sum_{k=-n}^{n} \frac{(-1)^k q^{(5k^2-k)/2}}{(q;q)_{n-k} (q;q)_{n+k}}$$

Note: at least one free variable

Aims:
1. Prove them automatically
2. Find the rhs given the lhs

First: **find a LDE** (or LRE)

Main Example for this Talk

$$\int_{-1}^{1} \frac{e^{-px} T_n(x)}{\sqrt{1-x^2}}\, dx = (-1)^n \pi I_n(p)$$

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> f:=exp(-p*x)*ChebyshevT(n,x)/sqrt(1-x^2);
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$$f := \frac{e^{-px}\,\mathrm{ChebyshevT}(n,x)}{\sqrt{1-x^2}}$$

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> redct(Int(f,x=-1..1),[n::shift,p::diff]);
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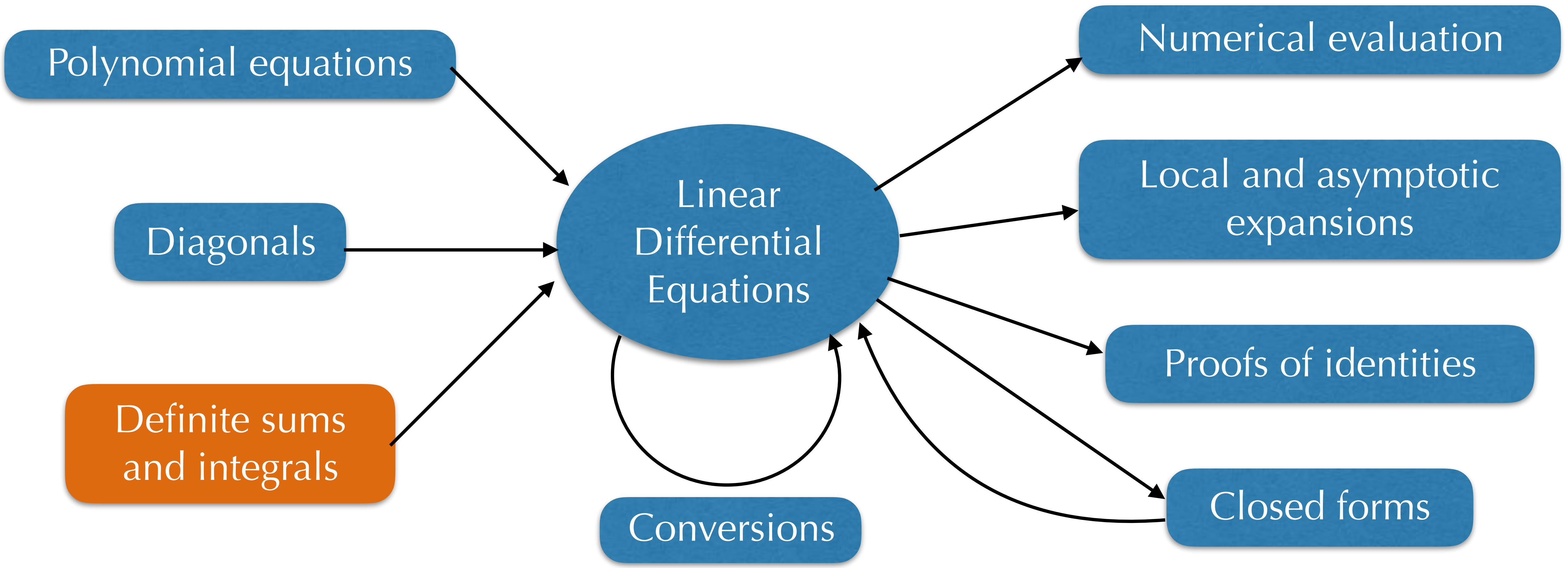
$$[pD_n + pD_p - n, pD_n^2 - 2nD_n - 2D_n - p]$$

Interpretation: the integral $F_n(p)$ satisfies

$$pF_{n+1} + pF_n' - nF_n = 0, \quad pF_{n+2} - 2(n+1)F_{n+1} - pF_n = 0$$

Context: LDEs as a Data-Structure



Solutions called differentially finite (abbrev. D-finite)

Creative Telescoping

$$I_n(x) = \int f_n(x,t)\,dt = ? \quad \text{or} \quad U_n(x) = \sum_k u_{n,k}(x) = ?$$

Input: equations (differential and/or recurrence).

Output: equations for the sum or the integral.

Example:

$$u(n,k) = \binom{n}{k} \text{ def. by } \left\{ \binom{n+1}{k} = \frac{n+1}{n+1-k}\binom{n}{k}, \binom{n}{k+1} = \frac{n-k}{k+1}\binom{n}{k} \right\}$$

$$U(n+1) = \sum_k \binom{n+1}{k} = \sum_k \left(\underbrace{\binom{n+1}{k} - \binom{n+1}{k+1}}_{\text{telesc.}} + \underbrace{\binom{n}{k+1} - \binom{n}{k}}_{\text{telesc.}} + 2\binom{n}{k} \right)$$

Pascal

IF one knows $A(n,S_n)$ and $B(n,k,S_n,S_k)$ such that

$$(A(n,S_n) + \Delta_k B(n,k,S_n,S_k)) \cdot u(n,k) = 0,$$

certificate

then the sum telescopes, leading to $A(n,S_n) \cdot U(n) = 0$.

Creative Telescoping

$$I(x) = \int f(x,t)\,dt = ?$$

IF one knows $A(x,\partial_x)$ and $B(x,t,\partial_x,\partial_t)$ such that

$$(A(x,\partial_x) + \partial_t B(x,t,\partial_x,\partial_t)) \cdot f(x,t) = 0,$$

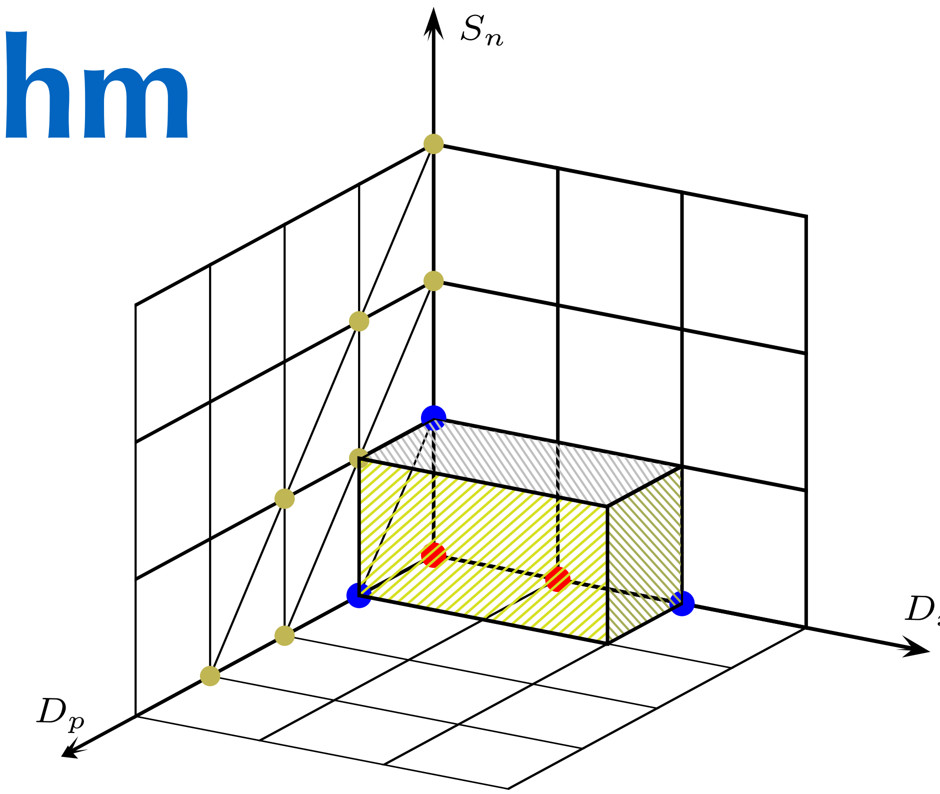
then the integral « telescopes », leading to $A(x,\partial_x) \cdot I(x) = 0$.

Then I come along and try differentiating under the integral sign, and often it worked. So I got a great reputation for doing integrals.

Richard P. Feynman 1985

Method: **integration** (summation) **by parts** and **differentiation** (difference) **under the integral** (sum) sign

Chyzak's Algorithm



$$\underbrace{\int_{-1}^{1} \frac{e^{-px}T_n(x)}{\sqrt{1-x^2}}}_{\mathsf{f}}\,\mathsf{dx} = (-1)^{\mathsf{n}}\pi \mathsf{I_n(p)}$$

Ann *f* generated by

$$D_p + x, \quad nS_n - (x^2-1)D_x - (p(1-x^2) - (n+1)x)$$

$$(1-x^2)D_x^2 - (2px^2+3x-2p)D_x - (p^2x^2+3px-n^2-p^2+1)$$

Undetermined coefficients

certificate

$$\left(\sum_{(k,m)} c_{k,m}(n,p)D_p^k S_n^m\right) + D_x\left(a_0(n,p,x) + a_1(n,p,x)D_x\right) \bmod \operatorname{Ann} f$$

Reduces to

$$\frac{\partial a_0}{\partial x} + \frac{a_1}{1-x^2}(p^2x^2+3px-n^2-p_1^2) = -\sum_{(k,m)} c_{k,m}u_{k,m}^{(0)},$$

$$\frac{\partial a_1}{\partial x} + a_0 + \frac{a_1}{1-x^2}(2px^2+3x-2p) = -\sum_{(k,m)} c_{k,m}u_{k,m}^{(1)}$$

Increase support until a soln is found

[Chyzak00]

Weakness: Certificates are Big

$$C_n := \sum_{r,s} \underbrace{(-1)^{n+r+s}\binom{n}{r}\binom{n}{s}\binom{n+s}{s}\binom{n+r}{r}\binom{2n-r-s}{n}}_{f_{n,r,s}}$$

$$(n+2)^3 C_{n+2} - 2(2n+3)(3n^2+9n+7)C_{n+1} - (4n+3)(4n+4)(4n+5)C_n$$
$$= \Delta_r(\cdots) + \Delta_s(\cdots) = \text{180 kB} \simeq \text{2 pages}$$

$$I(z) = \oint \frac{(1+t_3)^2 dt_1 dt_2 dt_3}{t_1 t_2 t_3 (1+t_3(1+t_1))(1+t_3(1+t_2)) + z(1+t_1)(1+t_2)(1+t_3)^4}$$

$$z^2(4z+1)(16z-1)I'''(z) + 3z(128z^2+18z-1)I''(z) + (444z^2+40z-1)I'(z) + 2(30z+1)I(z)$$
$$= \frac{d}{dt_1}(\cdots) + \frac{d}{dt_2}(\cdots) + \frac{d}{dt_3}(\cdots) = \text{1 080 kB} \simeq \text{12 pages}$$

and sometimes also unnecessary

Reduction-Based Creative Telescoping

$$\underbrace{\sum_{\mathbf{m}} c_{\mathbf{m}}(\mathbf{x})\partial^{\mathbf{m}}}_{\text{telescoper}} + \partial_t \underbrace{\sum_{(\mathbf{i},j)\in\mathcal{S}} a_{\mathbf{i},j}(\mathbf{x},t)\partial^{\mathbf{i}}\partial_t^j}_{\text{certificate}} \in \operatorname{Ann} \mathsf{f}$$

∂_t is a linear map in $\mathbb{Q}(\mathbf{x},t)\langle\partial_{\mathbf{x}},\partial_t\rangle/\operatorname{Ann} f$

Principle:

Reduce successive $\partial^{\mathbf{m}}$ modulo the image of ∂_t until a linear dependency is found

Previous Work

First generation of algorithms: relying on holonomy

Restrict int. by parts to $\mathbb{Q}(\boldsymbol{x})\langle \boldsymbol{\partial}_{\boldsymbol{x}}, \partial_{\mathrm{t}}\rangle$ and use elimination.

[Zeilberger90,91,Takayama90]

Second generation: faster using better certificates & algorithms

Hypergeometric summation: dim=1 + param. Gosper.

Undetermined coefficients in finite dim, Ore algebras & GB.

Idem in infinite dim.

[Wilf-Zeilberger90,92,ChyzakS.98,Chyzak00,Chyzak-Kauers-S.09]

New generation: reduction-based algorithms

Rational bivariate, hypergeometric, algebraic, mixed

Algebraic, Fuchsian, differentially finite

[Bostan et alii 10–15,Chen et alii 12–18, Hoeven 17]

I. Griffiths-Dwork Reduction for *Multiple* Integrals of *Rational* Functions

Griffiths-Dwork Reduction

$$I(t) = \oint \frac{P(t,\underline{x})}{Q^m(t,\underline{x})}\,d\underline{x}$$

Q square-free
Int. over a cycle
where Q≠0.

1. Control degrees by homogenizing $(x_1,\ldots,x_n)\mapsto(x_0,\ldots,x_n)$
2. If m=1, no reduction needed $[P/Q]:=P/Q$
3. If m>1, reduce modulo Jacobian ideal $J := \langle \partial_0 Q, \ldots, \partial_n Q\rangle$

$$P = r + v_0\partial_0 Q + \cdots + v_n \partial_n Q$$

$$\frac{P}{Q^m} = \frac{r}{Q^m} - \frac{1}{m-1}\left(\partial_0 \frac{v_0}{Q^{m-1}} + \cdots + \partial_n \frac{v_n}{Q^{m-1}}\right) + \underbrace{\frac{1}{m-1}\,\frac{\partial_0 v_0 + \cdots + \partial_n v_n}{Q^{m-1}}}_{A_{m-1}}$$

$$\left[\frac{P}{Q^m}\right] := \frac{r}{Q^m} + [A_{m-1}]$$ (recursive definition)

Thm. [Griffiths] In the regular case ($\mathbb{Q}(t)[\underline{x}]/J$ finite dim), if $R=P/Q^m$ hom of degree -n-1, $[R] = 0 \Leftrightarrow \oint R\,d\underline{x} = 0$.

→ Algo for CT: differentiate wrt *t*, reduce and iterate.

Size and Complexity

$$I(t) = \oint \underbrace{\frac{P(t,\underline{x})}{Q^m(t,\underline{x})}}_{\in \mathbb{Q}(t,\underline{x})} \, d\underline{x}$$

no regularity assumed

$$N := \deg_{\underline{x}} Q, \quad d_t := \max(\deg_t Q, \deg_t P)$$

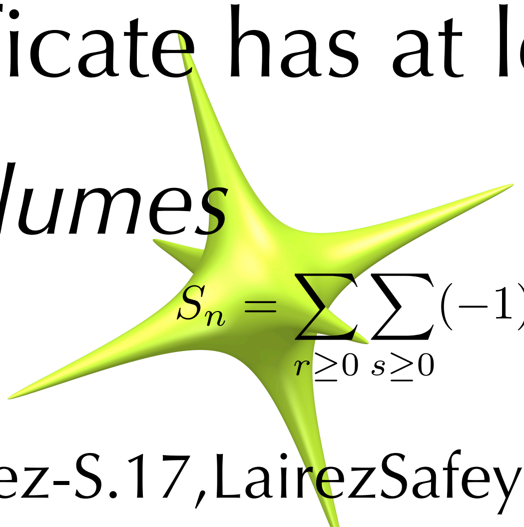
$\deg_x P$ not too big wrt N

Thm. A linear differential equation for I(t) can be computed in $O(e^{3n}N^{8n}d_t)$ operations in $\mathbb{Q}$.
It has order $\leq N^n$ and degree $O(e^n N^{3n} d_t)$.

tight

Note: generically, the certificate has at least $N^{n^2/2}$ monomials.

Applications to diagonals, volumes & multiple binomial sums



$$S_n = \sum_{r\geq 0}\sum_{s\geq 0} (-1)^{n+r+s} \binom{n}{r}\binom{n}{s}\binom{n+s}{s}\binom{n+r}{r}\binom{2n-r-s}{n}$$

[Bostan-Lairez-Salvy13,Lairez16,Bostan-Lairez-S.17,LairezSafey18]

II. Generalized Hermite Reduction

Hermite Reduction

Input: $f \in \mathbb{K}(t)$

Output: $g, h \in \mathbb{K}(t)$ s.t. $f = h + \partial_t g$

and h does not have multiple poles

classical in the integration of univariate rational functions

In other words, h is a canonical form of f modulo $\operatorname{Im} \partial_t$

Canonical: h=0 iff f is a derivative.

Lagrange's Identity

Def. Adjoint of $L = c_r D_t^r + \cdots + c_0$:

$$L^* = c_0 + \cdots + (-D_t)^r c_r.$$

Lagrange's identity: $uL(f) - L^*(u)f = D_t(P_L(f,u)).$

(proof: integration by parts and induction.)

Cor1. $g = M(f) \Rightarrow \exists q, g = M^*(1) f + D_t(q)$ rational fcn

Cor2. If $L(f) = 0$, for any u, $L^*(u)f$ is a derivative!

The computation reduces to
rational fcns x f and working modulo $\operatorname{Im} L^*$

Generalized Hermite Reduction

Input: $f \in \mathbb{K}(t)$, $M(t, \partial_t)$ a linear differential operator

Hermite: special case when $M = \partial_t$

Output: $h \in \mathbb{K}(t)$ s.t. $f = h + M(g)$ for some $g \in \mathbb{K}(t)$

and $h = 0 \Leftrightarrow f \in \operatorname{Im} M$.

Algorithm similar to rational solutions: (sketch)
local analysis at singularities, plus cleanup on polynomials.

See also [vdH17]

Example

$$\int_{-1}^{1} \underbrace{\frac{e^{-px}T_n(x)}{\sqrt{1-x^2}}}_{\mathsf{f}}\, \mathsf{dx} = (-1)^{\mathsf{n}}\pi \mathsf{I_n(p)}$$

Ann *f* generated by

$$\mathsf{D_p + x}, \quad \mathsf{nS_n - (x^2-1)D_x - (p(1-x^2) - (n+1)x)}$$

$$\mathsf{L} := \mathsf{(1-x^2)D_x^2 - (2px^2+3x-2p)D_x - (p^2x^2+3px-n^2-p^2+1)}$$

1f=(1)f reduced by L* to (1)f

D_pf=(-x)f reduced by L* to (-x)f

S_nf=((px^2+(n-1)x-p)/n)f reduced by L* to (x+n/p)f

$(D_p)^2$f=(x^2)f reduced by L* to (x/p+1+n^2/p^2)f

Conclusion: the integral $F_n(p)$ satisfies

$$F_n' + F_{n+1} = \tfrac{n}{p}F_n \qquad p^2F_n'' + pF_n' = (n^2+p^2)F_n$$

Conclusions

1. Complete algorithm for D-finite integration

2. It really works!

$$\int \frac{n^2+x+1}{n^2+1} \left(\frac{(x+1)^2}{(x-4)(x-3)^2(x^2-5)^3} \right)^n \sqrt{x^2-5}\, e^{\frac{x^3+1}{x(x-3)(x-4)^2}}\, \mathrm{d}x$$ in 1.5 sec. (HF 4 min)

$$\int (x+a)^{\gamma+\lambda-1}(a-x)^{\beta-1} C_m^{(\gamma)}(x/a) C_n^{(\lambda)}(x/a)\, \mathrm{d}x$$ in 53 sec. (HF >1h)

(HF=Koutschan's HolonomicFunctions)

3. Still an efficiency problem with apparent singularities (work in progress)

The End