Algorithms Proving Inequalities Between Sequences

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Pascaline, Inria at ENS de Lyon

Around Alaa Ibrahim's PhD thesis and its context

Slides available at

Luminy, March 2026

I. (Long) Introduction

I.1. Examples

Examples

$$s_n = \sum_{k=0}^{n} (-27)^{n-k} 2^{2k-n} \frac{(3k)!}{k!^3} \binom{k}{n-k} > 0$$

$$2(n+2)^2 s_{n+2} = (81n^2 + 243n + 186)s_{n+1} - 81(3n+2)(3n+4)s_n.$$

Family of tests

$$u_n^{(k)} = \sum_{j=0}^{n} (-1)^j \frac{(kn - (k-1)j)! \, k!^j}{(n-j)!^k j!} \geq 0 \text{ for } k \geq 4$$

linear rec of order k with coeffs of degree $k(k-1)/2$

Was conjectured by Gillis-Reznick-Zeilberger (1983)

More Examples

For $c_1 > \cdots > c_k > 0$ and $\mu_i > 0$, $\mu_1 + \cdots + \mu_k \leq 1$,

Family of tests

$$\prod_{j=1}^{k} (1 - c_j x)^{\mu_j} = 1 - \sum_{n \geq 1} d_n x^n \text{ with } d_n > 0 \qquad \text{[Aharonov 2017]}$$

linear rec of order k with coeffs of degree 1

The power series $s(x)$ solution to

$$xs^4 - 2xs^3 + 2s - 1 = 0 \qquad \text{[Bostan 2020]}$$
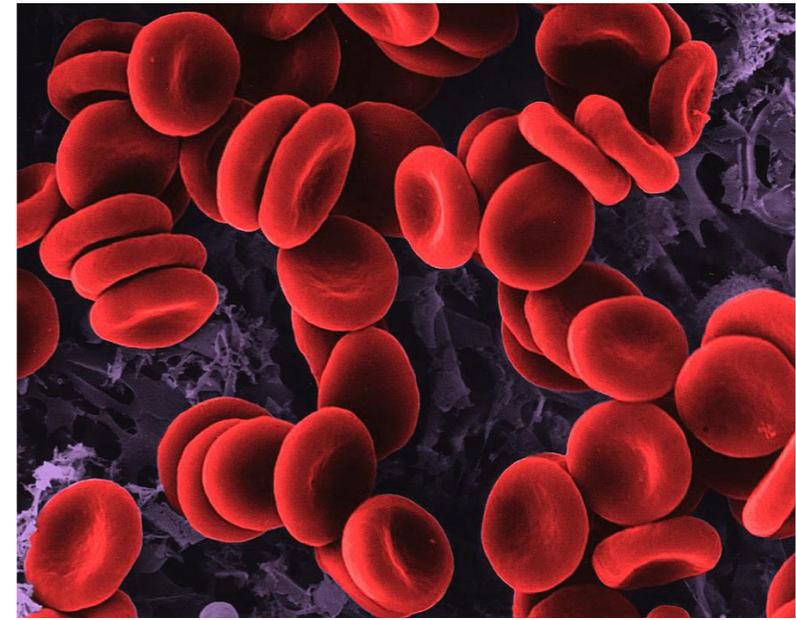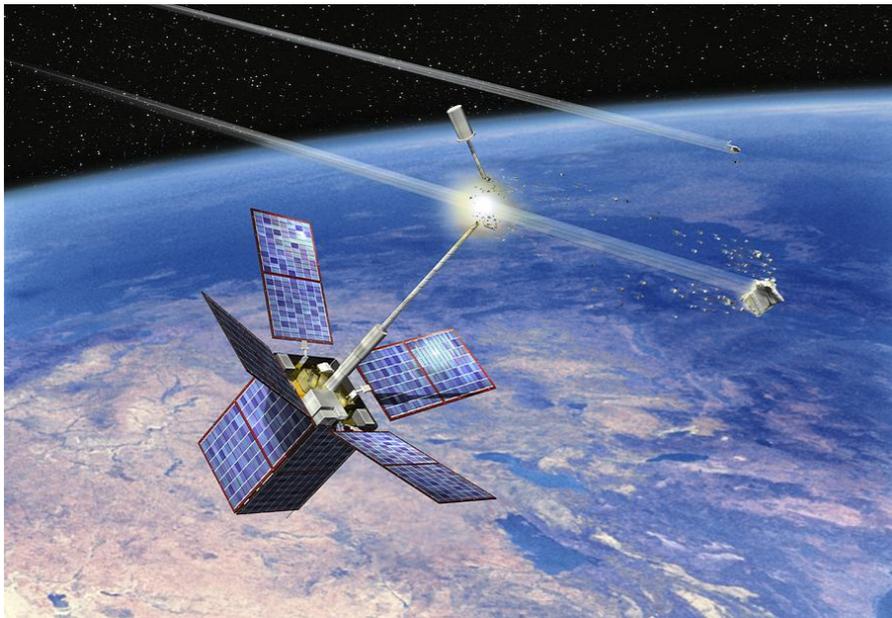(2 proofs)

has positive coefficients (order 2, deg 3).

The power series $y(x)$ solution to

$$y = x - y^8 + y^{15} \qquad \text{[Dotsenko, Markl, Remm 2020]}$$

is of the form $xh(x^7)$, h with positive coefficients (order 2, deg 3).

Other Examples

Probability of collision: $\sum_{n \geq 0} u_n$.
Numerical stability ensured by the positivity of (u_n), linear recurrence of order 4, degree 1, with 2 parameters.

[Serra-Arzelier-Joldes-Lasserre-Rondepierre-S. 2016]

Uniqueness of the Canham model for biomembranes. Reduced to the positivity of a solution of a linear recurrence of order 6, coeffs of degree 11.

[Melczer-Mezzarobba 2022]
[Bostan-Yurkevich 2022]

Subject of the course:

Which of these can be proved by computer algebra?

P-finite Sequences

Def. A sequence (u_n) is P-finite when it is defined by

$$p_d(n)u_{n+d} = p_{d-1}(n)u_{n+d-1} + \cdots + p_0(n)u_n, \quad n \geq 0$$

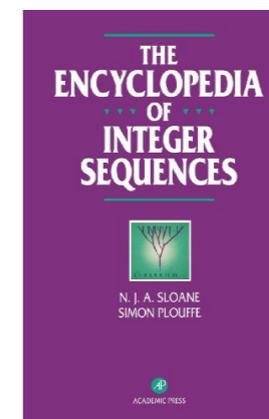and $u_0, \ldots, u_{d-1}$ with $p_0, \ldots, p_d$ in $\mathbb{K}[n]$, $u_0, \ldots, u_{d-1}$ in $\mathbb{K}$.

Our data-structure

In this talk, $p_0 p_d \neq 0$ and

Exs. F_n (Fibonacci), $n!$, $1/n!$, $\binom{2n}{n}$, C_n (Catalan)...

THE ENCYCLOPEDIA OF INTEGER SEQUENCES

N. J. A. SLOANE
SIMON PLOUFFE

ACADEMIC PRESS

$\simeq$25% of the litterature

Quiz

Which of the following is P-finite?

$$\frac{1}{\dbinom{2n}{n}}$$ ✔

$$2^{2^n}$$ ✘

$$\cos n$$ ✔ ✘

$$1 + \frac{1}{2} + \cdots + \frac{1}{n}$$ ✔

$$\lfloor \log n \rfloor$$ ✘

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$ ✔

See the course by
C. Koutschan (JNCF2024)

Sum & Product of P-finite Sequences

Thm. If $(a_n), (b_n)$ in $\mathbb{K}^{\mathbb{N}}$ are P-finite, and $\alpha \in \mathbb{K}$, then $(\alpha a_n), (a_n + b_n)$ and $(a_n b_n)$ are P-finite.

Proof next page

Ex. Cassini's identity for the Fibonacci numbers

$$u_n = F_{n+2}F_n - F_{n+1}^2 = (-1)^{n+1}$$

Blackboard proof

Proof

$$p_d(n)a_{n+d} + \cdots + p_0(n)a_n = 0, \quad n \geq 0,$$

$$q_\delta(n)b_{n+\delta} + \cdots + q_0(n)b_n = 0, \quad n \geq 0,$$

There exists $K \in \mathbb{N}$ s.t. $p_d(n)q_\delta(n) \neq 0$ for $n \geq K$.

Then for $k \in \mathbb{N}$,

$$a_{n+k} = r_{d-1}^{[k]}(n)a_{n+d-1} + \cdots + r_0^{[k]}(n)a_n, \quad n \geq K$$

$$b_{n+k} = s_{\delta-1}^{[k]}(n)b_{n+\delta-1} + \cdots + s_0^{[k]}(n)b_n, \quad n \geq K$$

with $r_i^{[k]}, s_i^{[k]}$ rational functions whose denominators
do not vanish in $\mathbb{N}$.

> Algorithm in both cases:
> construct a matrix in
> $\mathbb{K}(n)^{D \times (D+1)}$ and find its kernel

$$\implies (a_n + b_n) \text{ and } (a_n b_n) \text{ satisfy linear recurrences for } n \geq K$$

$$\implies \text{ they are P-finite}$$

From Positivity to Inequalities

If $(u_n), (v_n)$ are P-finite, deciding

monotonicity	$u_{n+1} \geq u_n$	all reduce to
convexity	$u_{n+1} + u_{n-1} \geq 2u_n$	positivity,
log-convexity	$u_{n+1}u_{n-1} \geq u_n^2$	using closure
inequality	$u_n \geq v_n$	properties.

LDE/LREs as Data-Structures

Polynomial equations

Diagonals

Definite sums and integrals

Linear Differential or Recurrence Equations

Conversions

Numerical evaluation

Local and asymptotic expansions

Proofs of identities

Closed forms

Inequalities

This course
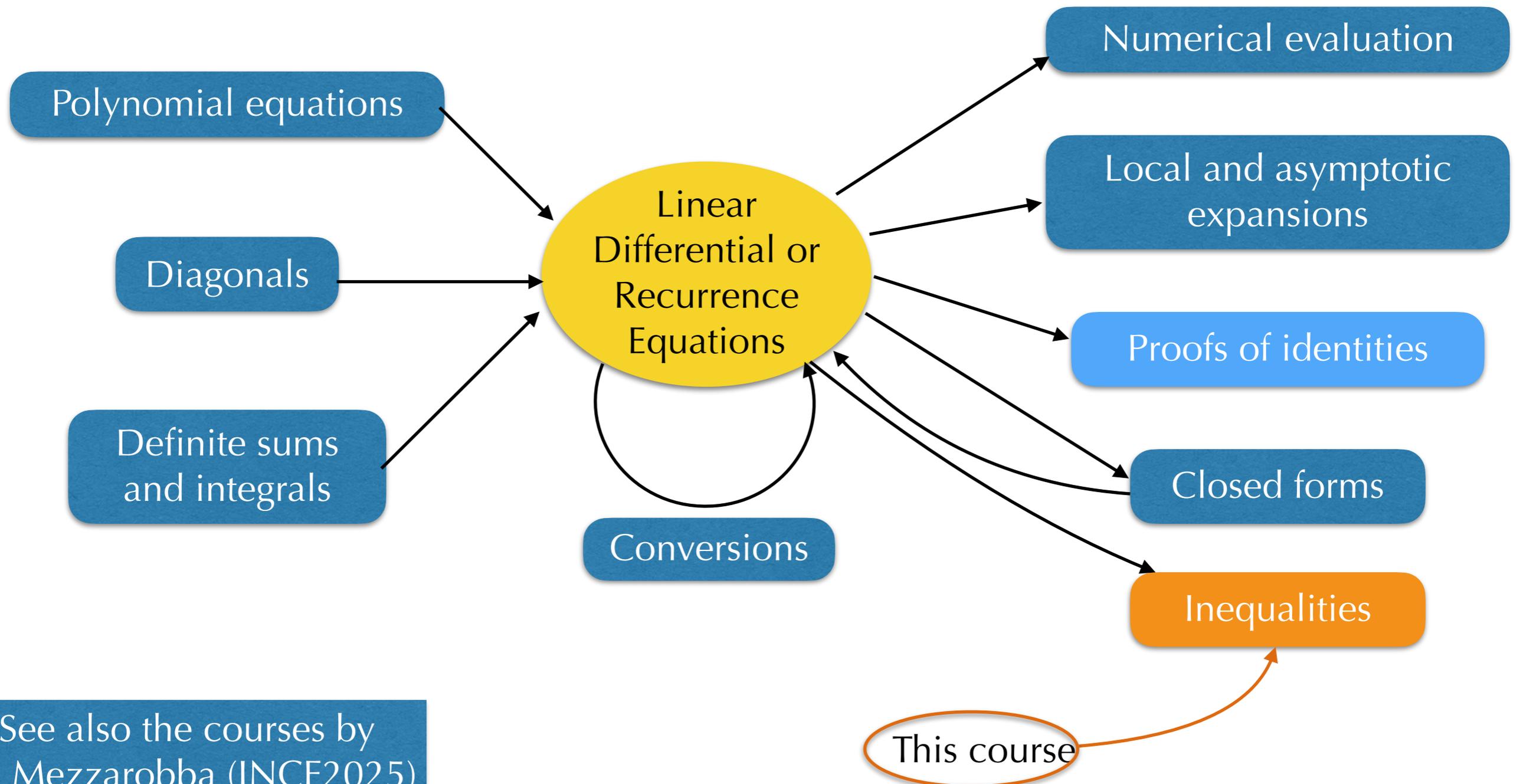
See also the courses by
M. Mezzarobba (JNCF2025)
C. Koutschan (JNCF2024)

I.2. Constant Coefficients are not Easy

Help from Closed Forms

$$u_{n+d} = c_{d-1}u_{n+d-1} + \cdots + c_0 u_n, \qquad c_i \in \mathbb{Q}$$

Characteristic polynomial: $x^d - \sum_{i=0}^{d-1} c_i x^i = \prod_{i=1}^{k} (x - \lambda_i)^{m_i}$

Closed form: $u_n = C_1(n)\lambda_1^n + \cdots + C_k(n)\lambda_k^n$

$C_i \in \overline{\mathbb{Q}}[n]$
computable
from $u_0, \ldots, u_{d-1}$

Easy situation: $\boxed{|\lambda_1| > |\lambda_i|, i \neq 1}$ and $C_1 \neq 0.$

Aim: extension to polynomial coefficients

property of the recurrence

property of the initial conditions

Example from the AMM (Apr 2010)

11501 [2010, 834]. *Proposed by Finbarr Holland, University College Cork, Cork, Ireland.* (Corrected) Let

$$g(z) = 1 - \frac{3}{\frac{1}{1-az} + \frac{1}{1-iz} + \frac{1}{1+iz}}.$$

Show that the coefficients in the Taylor series expansion of g about 0 are all nonnegative if and only if $a \geq \sqrt{3}$.

Skolem's Problem Reduces to Positivity

Input: $u_{n+d} = c_{d-1}u_{n+d-1} + \cdots + c_0 u_n,$ $c_0, \ldots, c_{d-1}, u_0, \ldots, u_{d-1} \in \mathbb{Q}$

Problem: is $u_n = 0$ for some n?

1. Construct a recurrence for $v_n = m^n u_n$, where m is the lcm of the denominators of $c_0, \ldots, c_{d-1}, u_0, \ldots, u_{d-1}$. Then $v_n \in \mathbb{Z}$ for all n.

2. Construct a recurrence for $w_n = v_n^2 - 1$. Then $w_n \geq 0 \Leftrightarrow u_n \neq 0$.

Only known to be decidable for $d \leq 4$

Decidability Results

$$u_{n+d} = c_{d-1}u_{n+d-1} + \cdots + c_0 u_n$$

$d \leq 5$ in general

$d \leq 9$ if the characteristic polynomial is square-free

$d \leq 11$ for integer, reversible sequences

$d \leq 17$ if both properties hold

and arbitrary d, $|\lambda_1| > |\lambda_i|, i \neq 1$ and $C_1 \neq 0$

[Ouaknine, Worrell et al. 2014–today]

A Difficult Example

$$u_n = \left((-4 + 7i)^n + (-4 - 7i)^n + 2(8 - i)^n + 2(8 + i)^n\right)^2 - 2^n$$

is a sequence of integers that satisfies

$$u_{n+10} + u_{n+9} + 378u_{n+8} - 749576u_{n+7} + 2333386u_{n+6} - 55996590u_{n+5}$$
$$+ 205750047100u_{n+4} - 856834394000u_{n+3} - 13815580471875u_{n+2}$$
$$- 20682499470546875u_{n+1} + 41423825675781250u_n = 0$$

$u_n/65^n$

Is $u_n \geq 0$ for all n?
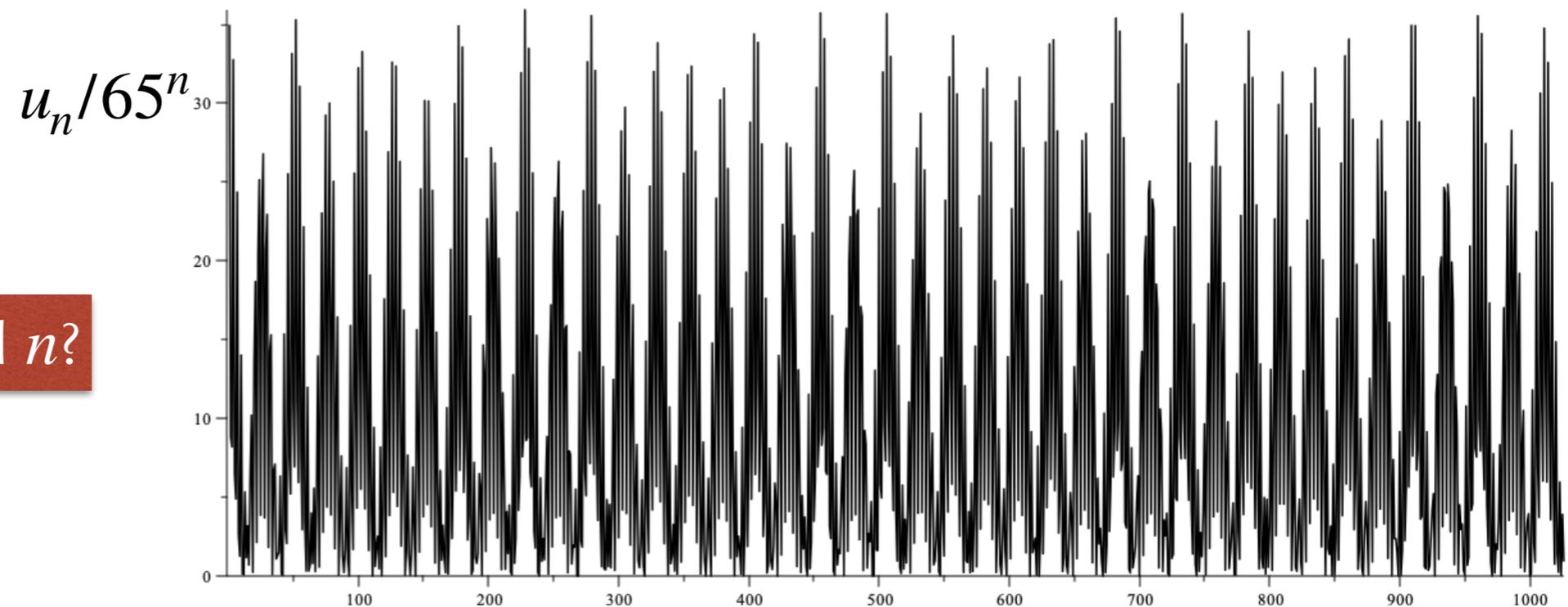
I.3. Algorithms for Polynomial Coefficients

Gerhold-Kauers Method by Quantifier Elimination

Use quantifier elimination to look for m s.t.

$\forall n \geq 0, \forall u_n \geq 0, \forall u_{n+1} \geq 0, \ldots, \forall u_{n+d-1} \geq 0,$

$\quad u_{n+d} \geq 0 \wedge \cdots \wedge u_{n+m} \geq 0 \Rightarrow u_{n+m+1} \geq 0.$

Gives a proof when it terminates

Not restricted to linear recurrences

Ex.: Turán's inequality for the Legendre polynomials

$$P_n(x)^2 - P_{n-1}(x)P_{n+1}(x) > 0, \quad -1 < x < 1$$

Ex.: it does not terminate on $u_n \geq 0$ for

$$u_{n+2} = 3u_{n+1} - 2u_n, \ u_0 = 1, u_1 = 3$$

[Gerhold-Kauers 2005]
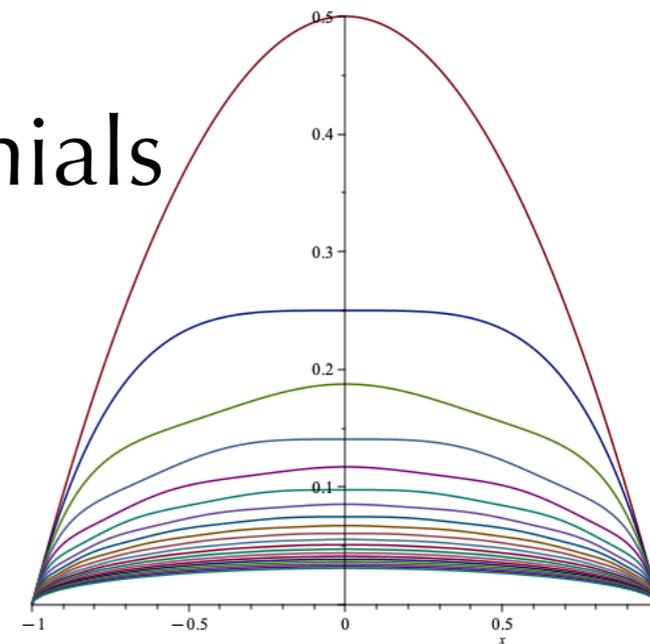
Kauers–Pillwein Cone
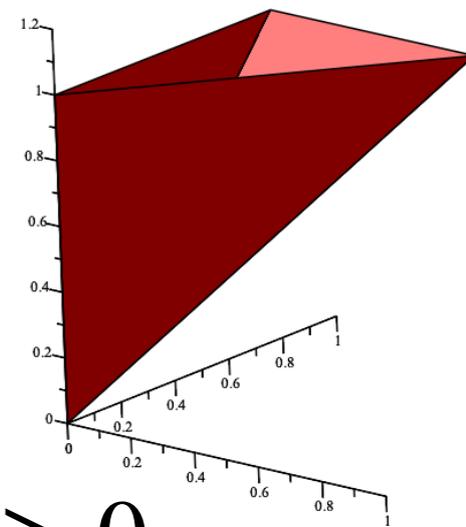
Idea: replace $u_{i+1} \geq 0$ by $u_{i+1} \geq \mu u_i$ for some $\mu \geq 0$

1. Construct a formula

$$\forall(x_0, \ldots, x_{d-1}, n), n \geq n_0 \wedge x_{d-1} \geq \mu x_{d-2} \wedge \cdots \wedge x_1 \geq \mu x_0 \wedge x_0 \geq 0 \Rightarrow x_d \geq \mu x_{d-1}$$

2. Quantifier eliminate $\rightarrow \Phi(n_0, \mu)$

3. For $n_0 = 0, 1, \ldots$ check whether

$$u_{n_0+d-1} \geq \mu u_{n_0+d-1} \geq \cdots \geq \mu^{d-1} u_{n_0} \geq 0 \wedge \Phi(\mu, n_0)$$

is satisfiable for some $\mu \geq 0$.

Termination (under genericity hypothesis)
for $d = 2$ and cases of $d = 3$.

[Pillwein et al. 2010–2022]

Proof by Induction on an Example

$$s_0 = 1, \; s_1 = 12,$$

$$2(n+2)^2 s_{n+2} = (81n^2 + 243n + 186)s_{n+1} - 81(3n+2)(3n+4)s_n.$$

$$s_n \geq 0, \ldots, s_{n+m-1} \geq 0 \text{ do not imply } s_{n+m} \geq 0.$$

Add $s_{n+1} \geq 18 s_n$ to the induction hypothesis, then

$$2(n+2)^2 \left(s_{n+2} - 18 s_{n+1} \right) =$$

$$\left(45n^2 + 99n + 42 \right) \underbrace{\left(s_{n+1} - 18 s_n \right)}_{\geq 0} + \left(81n^2 + 324n + 108 \right) \underbrace{s_n}_{\geq 0}$$

Method:
synthesize extra
hypotheses

$$\text{implies } s_{n+2} \geq 18 s_{n+1} \geq 0.$$

First terms: $s_0 = 1, s_1 = 12, s_2 = 198, s_3 = 3720 > 18 \times s_2$.

Other Algorithms

Order 2:

Tools

- for initial conditions outside of a line;   continued fractions
- for coefficients of degree 1 and a characteristic polynomial with two distinct rational roots.

linear relations
of 1-periods

[Ouaknine, Worrell et al.
2021–today]

Arbitrary order: This course

$|\lambda_1| > |\lambda_i|, i \neq 1$ and initial
conditions outside of a hyperplane

Perron-Frobenius
for cones

I.5 Cone-based Approach

Vector Version of the Recurrence

$$p_d(n)u_{n+d} = p_{d-1}(n)u_{n+d-1} + \cdots + p_0(n)u_n \Leftrightarrow$$

$$\begin{pmatrix} u_{n+1} \\ \vdots \\ u_{n+d} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ \frac{p_0(n)}{p_d(n)} & \cdots & \cdots & \cdots & \frac{p_{d-1}(n)}{p_d(n)} \end{pmatrix} \begin{pmatrix} u_n \\ \vdots \\ u_{n+d-1} \end{pmatrix}$$

U_n

$A(n)$

$$U_{n+1} = A(n)U_n$$

Matrix factorial

$$= A(n)A(n-1)\cdots A(0)U_0$$

Companion matrix

$$u_n \geq 0 \text{ for all } n \Leftrightarrow U_n \in \mathbb{R}_+^d \text{ for all } n$$

Simple Cone-Based Proof

$$u_{n+2} = \frac{2a}{3}u_{n+1} - \frac{1}{3}u_n \ (n \geq 1) \text{ with } a \geq \sqrt{3}$$

AMM11501

Cone: $y \geq ax/3 \geq 0$ stable by the recurrence:

$$\frac{2a}{3}y - \frac{1}{3}x \geq \frac{ay}{3} + \frac{ay - x}{3} \geq \frac{ay}{3} + \underbrace{\frac{a^2/3 - 1}{3}}_{\geq 0}x.$$
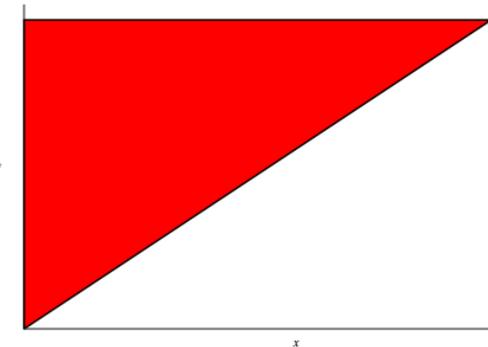
First values:

$$u_1 = \frac{a}{3}, \ u_2 = 2\frac{a^2 - 3}{9}, \ u_3 = \frac{4a(a^2 + 3)}{27} \geq \frac{u_2}{3}.$$

Running Example

$$\left(16n + 1\right) u_{n+3} - (32n - 2) u_{n+2} + (20n - 4) u_{n+1} - \left(5n - 3\right) u_n = 0$$

$$u_0 = 5, \ u_1 = 5, \ u_2 = 1$$
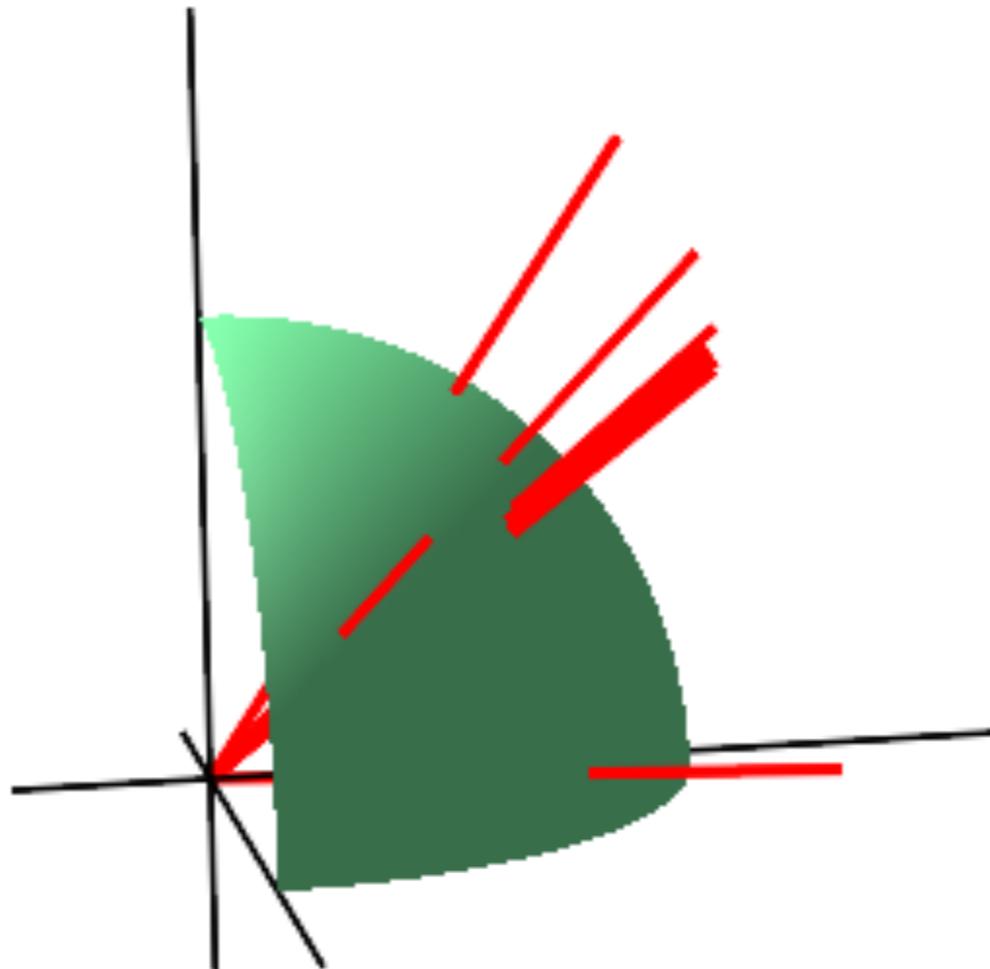
Recurrences of Poincaré Type

$$U_{n+1} = A(n)U_n$$

Def. Poincaré type: $A := \lim_{n \to \infty} A(n)$ finite.

Polynomial coefficients as a perturbation

Ex. $2(n+2)^2 s_{n+2} = (81n^2 + 243n + 186)s_{n+1} - 81(3n+2)(3n+4)s_n$. ✔

$$A(n) = \begin{pmatrix} 0 & 1 \\ -\dfrac{81(3n+2)(3n+4)}{2(n+2)^2} & \dfrac{3(27n^2 + 81n + 62)}{2(n+2)^2} \end{pmatrix} \to A = \begin{pmatrix} 0 & 1 \\ -\dfrac{729}{2} & \dfrac{81}{2} \end{pmatrix}$$

Ex. $u_{n+3} + u_{n+2} + nu_{n+1} + (n+1)u_n = 0$ ✖

Positivity-preserving reduction

$$v_n := \psi_n u_n$$

$$(n+2)\psi_{n+2} = \psi_n \quad \xrightarrow{\hspace{2cm}}$$

$$\psi_0 = \psi_1 = 1 \quad \sim 1/\sqrt{n!}$$

$$(n+6)(n+4)v_{n+6} + 2(n+4)(n+1)v_{n+4}$$
$$+ (n^2 - n - 5)v_{n+2} - (n+1)v_n = 0 \quad ✔$$

[Mezzarobba-S. 2010]

Cone-Based Proofs

0. Check $\lambda_1 > |\lambda_i|$, $i \neq 1$;

Depends on $A(n)$ only

1. Construct a cone $K \subset \mathbb{R}_+^d$ s.t. $A(K \setminus \{0\}) \subset K^\circ$;

2. Compute m s.t. for all $n \geq m$, $A(n)K \subset K$;

3. For $i = 1, 2, \dots$ do

Depends also on U_0

    1. $U_i = A(i)U_{i-1}$;

    2. If $U_i \notin \mathbb{R}_+^d$ return false

    3. If $i \geq m$ and $U_i \in K$ return true
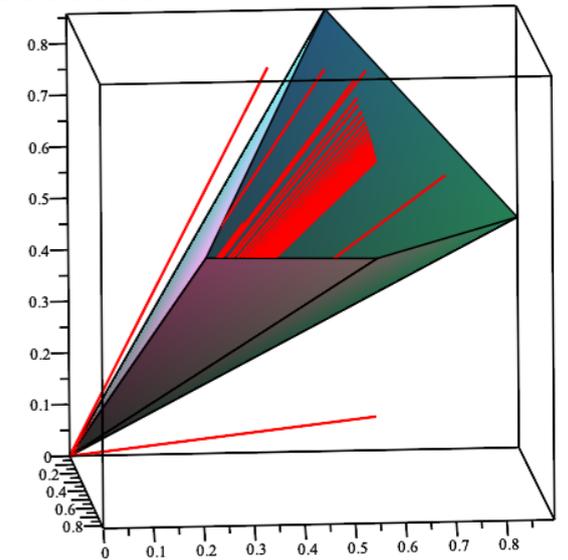
Constructs a formula $\forall n \geq m, U_n \in K \subset \mathbb{R}_+^d$ and proves it by induction

Thm. If $\lambda_1 > |\lambda_i|, i \neq 1$, λ_1 simple, then for arbitrary order d, the algorithm terminates for U_0 outside of a hyperplane.

Next: algorithms for the computation of K and m.

It works in practice!

```
> rec:=(16*n+1)*u(n+3)-(32*n-2)*u(n+2)+(20*n-4)*u(n+1)-(5*n-3)*u(n):
> ini:=u(0)=5,u(1)=5,u(2)=1:
> Positivity({rec,ini},u,n);
```

$$true, \begin{bmatrix} \frac{44}{79} & 2 & 0 \\ \frac{41}{64} & \frac{213}{250} & \frac{301}{500} \\ \frac{81}{110} & \frac{909}{5000} & \frac{513}{1000} \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix}, 5$$

Certificate that can be checked separately

The sequence is positive

M, V s.t. $M \cdot V$ are the extremal vectors of a contracted cone in $\mathbb{R}^3_{>0}$

Starting at $n = 5$, U_n is in that cone

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Plan of the Course

We are here

I. Introduction
II. Matrices which leave a cone invariant

III. Properties of the generalized power method
IV. Choices of cones

II. Matrices Which Leave a Cone Invariant

Cones in $\mathbb{R}^d$

Def. $K \subset \mathbb{R}^d$ is a cone when

. $\mathbb{R}_+ K \subset K$;

. $K + K = K$ (it is convex).

Def. The cone K is proper when

. $K \cap (-K) = \{0\}$;

. K contains a basis of $\mathbb{R}^d$ (it is solid);

. K is closed.

Def. For a matrix A, the cone K is

. invariant by A if $AK \subset K$;

. contracted by A if $A(K \backslash \{0\}) \subset K^\circ$.
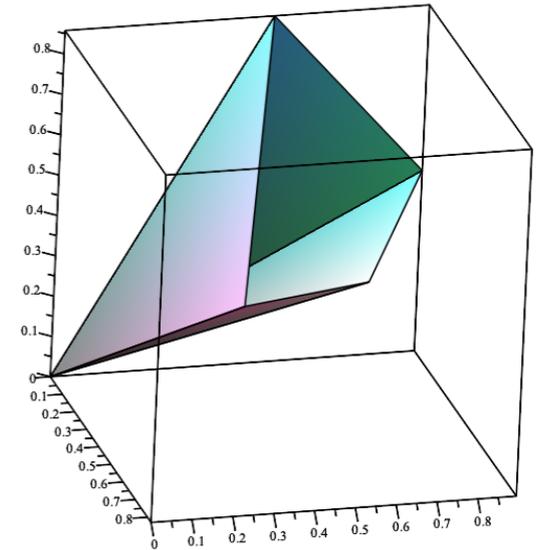
polyhedral:
finite number of
extremal vectors
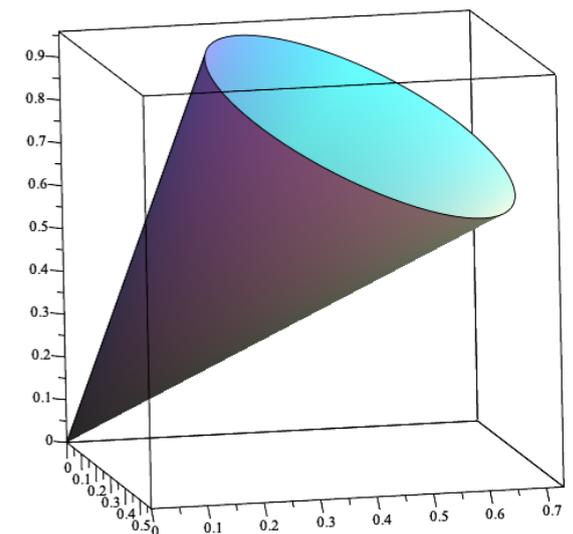
Spectral Characterization

$$A \in \mathbb{R}^{d \times d}, \quad \text{spectrum } \sigma(A) := \{\lambda_1, \ldots, \lambda_r\},$$

$$\rho(A) := |\lambda_1| = \cdots = |\lambda_\nu| > |\lambda_j|, \quad j > \nu.$$

Jordan form in $\mathbb{C}^d$: there exists a basis V_{jr} where

Take $V_{jr} = \overline{V}_{jr}$ if $\lambda_j = \overline{\lambda_k} \notin \mathbb{R}$

$$A = \mathrm{diag}(J_1, \ldots, J_r), \qquad J_j = \begin{pmatrix} \lambda_j & 1 & & 0 \\ & \lambda_j & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_j \end{pmatrix}, \qquad m(\lambda_j) := \dim J_j \text{ (index of } \lambda_j)$$

Thm. 1. There exists a proper cone K invariant by A iff

$$\lambda_1 = \rho(A) \text{ and } m(\lambda_j) \leq m(\lambda_1) \text{ for } j \leq \nu.$$

Then K contains an eigenvector V_1 for λ_1.

2. There exists a proper cone K contracted by A iff

$$\lambda_1 > |\lambda_j|, j \neq 1, \lambda_1 \text{ simple}.$$

up to relabelling the λ_i

Perron-Frobenius for cones

[Birkhoff 1967; Vandergraft 1968]

Examples

simple eig.

multiple eig.
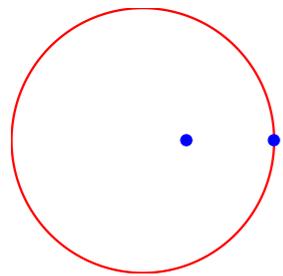
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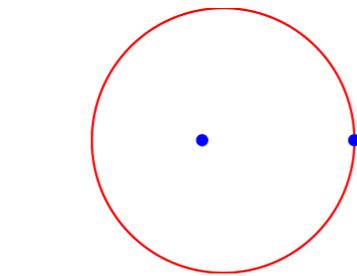
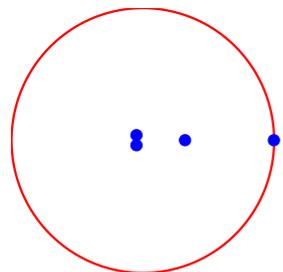$$\sum_{k=0}^{n} (-27)^{n-k} 2^{2k-n} \frac{(3k)!}{k!^3} \binom{k}{n-k}$$
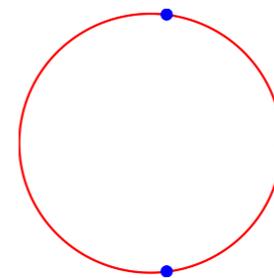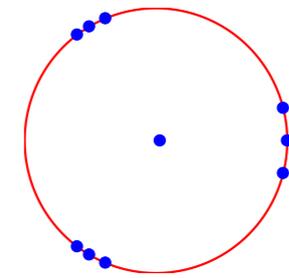
Gillis-Reznick-Zeilberger $k = 4$
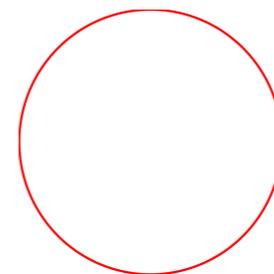
Gillis-Reznick-Zeilberger $k = 7$

Serra et al.

Running example

Turán

Ouaknine 2022
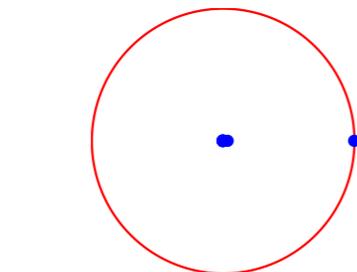
Bostan 2020

Melczer-Mezzarobba

Contracted cones

Invariant, but not contracted, cones

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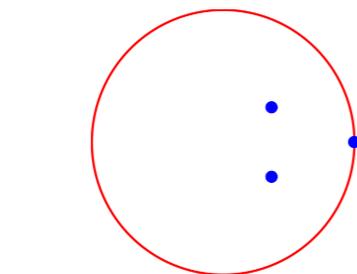
Application: Power Method

1. Pick a random U_0
2. For $n = 0, 1, \dots$
$$U_{n+1} := AU_n / \|U_n\|$$

Eigenvalues of A : $\lambda_1, \dots, \lambda_k$

Hyp.: $\boxed{|\lambda_1| > |\lambda_i|, i \neq 1}$ and $\boxed{\lambda_1 \text{ simple.}}$

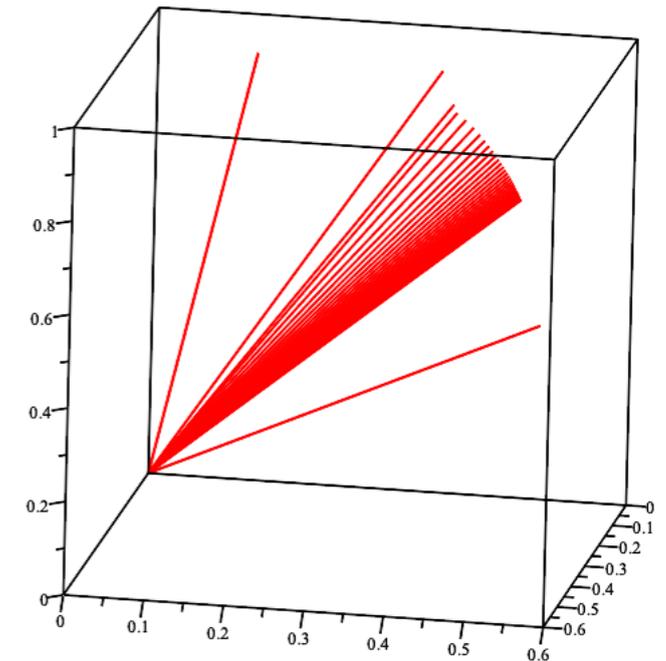> Polynomial coefficients will be seen as a perturbation

Then $A^n / \lambda_1^n \to VW^\top$ $(AV = \lambda_1 V, A^\top W = \lambda_1 W)$

> Convergence to a rank 1 matrix

Generic $U_0 : W^\top U_0 \neq 0$

> Convergence to a dominant eigenvector

$$\Rightarrow U_n / \|U_n\| \to \pm V / \|V\|$$

Steps of the Proof Next Slides

$$\implies$$

1. $\lambda_1 > 0$ and of maximal index and $V_1 \in K$

2. K contracted $\Rightarrow \lambda_1$ simple

3. K contracted $\Rightarrow \lambda_1 > |\lambda_i|,\ i \neq 1$

$$\impliedby$$

4. A with these properties has an invariant/contracted cone K

Effective

$\rho(A)$ **eigenvalue of maximal index**

Case $\rho(A) = 0$: look at the sequence $A^k x$ for $x \in K$

1. Take $x = \sum c_{jr} V_{jr} \in K$ with all $c_{ij} \neq 0$ K is solid

2. Asymptotic growth:

$$A^k x = \rho^{k-M+1} \frac{k^{M-1}}{(M-1)!} \left(\sum_{j \in J} c_{j,M} e^{i(k-M+1)\theta_j} V_{jM} + o(1) \right),$$

$$J_i^k = \begin{pmatrix} \lambda_i^k & \binom{k}{1}\lambda_i^{k-1} & \binom{k}{2}\lambda_i^{k-2} & \cdots & \binom{k}{m-1}\lambda_i^{k-(m-1)} \\ 0 & \lambda_i^k & \binom{k}{1}\lambda_i^{k-1} & \ddots & \vdots \\ \vdots & 0 & \lambda_i^k & \ddots & \binom{k}{2}\lambda_i^{k-2} \\ \vdots & & \ddots & \ddots & \binom{k}{1}\lambda_i^{k-1} \\ 0 & \cdots & \cdots & 0 & \lambda_i^k \end{pmatrix}$$

$$M := \max\{m(\lambda_j) \mid j \leq \nu\}, \quad J := \{j \leq \nu \mid m(\lambda_j) = M\}, \quad \lambda_j = \rho e^{i\theta_j}$$

3. $A^k x / \|A^k x\|$ has a convergent subsequence

4. Limit direction: $x' := \sum_{j \in J} d_j V_{jM} \in K \backslash \{0\}$

   > **Lemma**: if $\lambda_j \neq \rho$, there exists r and $w_i > 0$ s.t. $\sum_{i=0}^{r} w_i \lambda_j^i = 0$.

5. Eliminate non-positive eigenvalues

   if $d_j \neq 0$ for $\lambda_j \neq \rho$, then $x'' := \sum_{i=0}^{r} w_i A^i x' \in K \backslash \{0\}$ has $d'_j = 0$.

[Birkhoff 1967] $\rightarrow$ the eigenvector corresponding to $\rho(A)$ is in K

K contracted $\Rightarrow \lambda_1$ simple

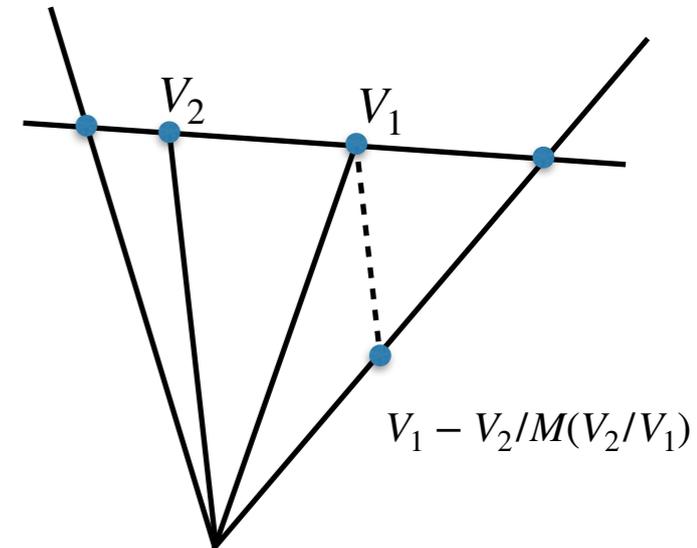$$\lambda_1 := \rho(A); \quad V_1 \in K \text{ an eigenvector for } \lambda_1$$

1. $V_1 \in K°: AV_1 = \lambda_1 V_1 \in K°$ Contraction

2. There is no other eigenvector for λ_1:

   if $V_2 \notin \mathbb{R}V_1$ is an eigenvector,

   $$M(V_2/V_1) := \min\{\beta \in \mathbb{R} \mid \beta V_1 - V_2 \in K\} > 0$$

   $$V_3 = M(V_2/V_1)V_1 - V_2 \in \partial K \text{ but } AV_3 = \lambda_1 V_3 \in K°. \quad \text{Contradition}$$

3. λ_1 is simple: if $AV_2 = \lambda_1 V_2 + V_1$,

   $$. - V_2 \notin K, \; AV_3/\lambda_1 = \beta V_1 - V_2 \in K, \beta < M(V_2/V_1) \quad \text{Contradition}$$

   $$. - V_2 \in K, \text{then } A^k(-V_2)/\lambda_1^{k-1} = -\lambda_1 V_2 - kV_1 \in K \quad \text{Contradition}$$

[Vandergraft 1968]

K **contracted** $\Rightarrow \lambda_1 > |\lambda_i|$, $i \neq 1$

Assume $\lambda_2 = \lambda_1 e^{i\theta}$, $AV_2 = \lambda_2 V_2$, $A\overline{V}_2 = \overline{\lambda}_2 V_2$

1. $x_\phi := \text{Re}(e^{i\phi}V_2) \in K$ for some ϕ: if $x_\phi \notin K$,

   let $t_\phi := \min\{t > 0 \mid tV_1 + x_\phi \in K\}$; $y_\phi = t_\phi V_1 + x_\phi \in \partial K$

   $Ay_\phi / \lambda_1 = t_\phi V_1 + \text{Re}(e^{i(\theta+\phi)}V_2) \in K° \Rightarrow t_{\theta+\phi} < t_\phi \Rightarrow \min t_\phi = 0.$

2. $x_\phi = 0$. Take $w_i > 0$ s.t. $\displaystyle\sum_{i=0}^{r} w_i \lambda_2^i = 0.$

    $\displaystyle\sum w_i A^i x_\phi = 0 \Rightarrow A^i x_\phi = 0$ for all i.

$W_2 := e^{i\phi + i\pi/2}V_2 \in \mathbb{R}^d$; $AW_2 = \lambda_2 W_2 \Rightarrow \lambda_2 = -\lambda_1$

$\Rightarrow W_2 \notin K$.   Let $t_0 := \min\{t > 0 \mid tV_1 + W_2 \in K\}$

$\qquad A^2(t_0 V_1 + W_2)/\lambda_1^2 = t_0 V_1 + W_2 \in K°.$ Contradition
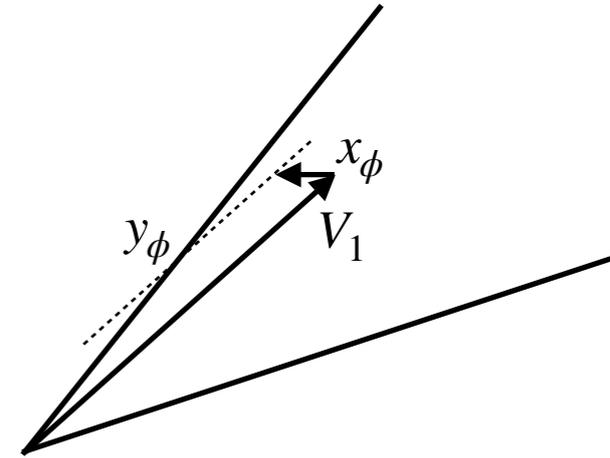
[Vandergraft 1968]

Reciprocal on our Example

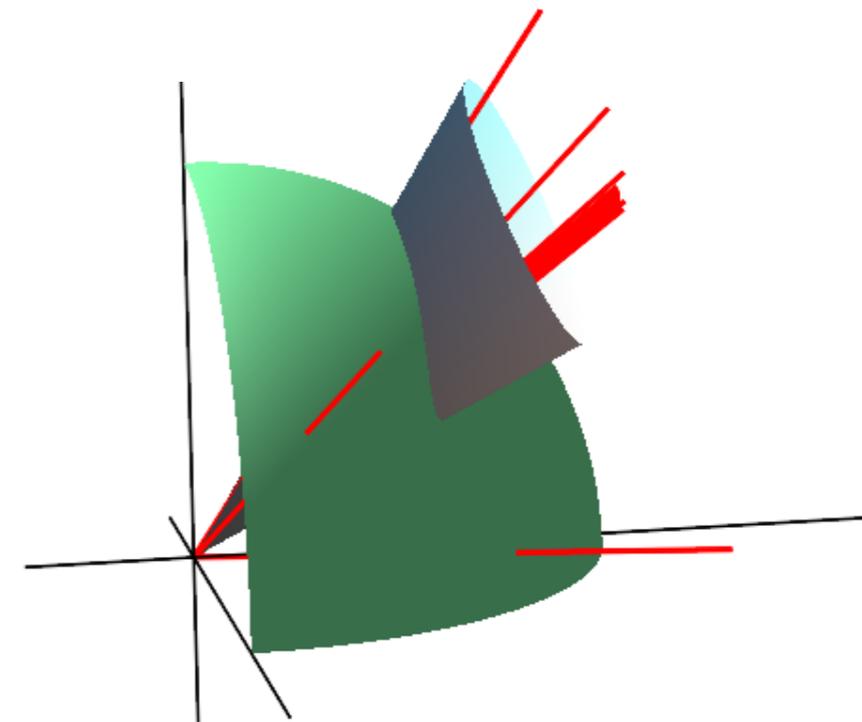$$\left(16n + 1\right) u_{n+3} = \left(5n - 3\right) u_n + (20n - 4) u_{n+1} + (32n - 2) u_{n+2}$$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \dfrac{5}{16} & -\dfrac{5}{4} & 2 \end{pmatrix}$$
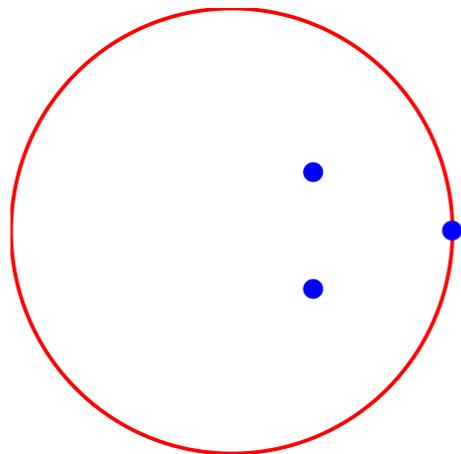
Eigenvectors: $V_1, V_2, \overline{V_2}$

$V_i = (1, \lambda_i, \lambda_i^2)^\mathsf{T}$

$\lambda_1 \approx 1.15 > |\lambda_2|,$

$\lambda_2 \approx 0.43 + 0.30i\,.$

$$K_\mu := \{a\mu V_1 + b V_2 + \overline{b}\,\overline{V_2} \mid |b| \le a\} \subset \mathbb{R}^3$$

$\forall \mu > 0,\ A K_\mu \subset \mathring{K}_\mu$

$\exists \mu > 0,\ K_\mu \subset \mathbb{R}^d_{>0}$

Vandergraft's Cone (contracted case)

Take a basis $V_1, \ldots, V_d$ where A has the form

$$A = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & J_2 & \ddots & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & J_k \end{pmatrix} \quad J_i = \begin{pmatrix} \lambda_i & \varepsilon & 0 & \cdots & 0 \\ 0 & \lambda_i & \varepsilon & \cdots & 0 \\ 0 & 0 & \lambda_i & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \varepsilon \\ 0 & 0 & 0 & \cdots & \lambda_i \end{pmatrix} \quad \begin{array}{c} \text{with} \\ 0 < \varepsilon < \lambda_1 - |\lambda_i| \\ \text{and } V_j = \overline{V_i} \\ \text{when } \lambda_j = \overline{\lambda_i}. \end{array}$$

$$K := \{ a_1 V_1 + \cdots + a_d V_d \mid |a_i| \le a_1 \text{ and } a_j = \overline{a_i} \text{ when } V_j = \overline{V_i} \}$$

satisfies $AK \subset \mathring{K}$.

Extends to the general case

Back to the Algorithm

Depends on $A(n)$ only

0.   Check $\lambda_1 > |\lambda_i|$, $i \neq 1$;

1.   Construct a cone $K \subset \mathbb{R}_+^d$ s.t. $A(K\backslash\{0\}) \subset K°$;

2.   Compute m s.t. for all $n \geq m$, $A(n)K \subset K$;

3.   For $i = 1,2,\ldots$ do

       1. $U_i = A(i)U_{i-1}$;

Depends also on U_0

       2.   If $U_i \notin \mathbb{R}_+^d$ return false

       3.   If $i \geq m$ and $U_i \in K$ return true

Constructs a formula $\forall n \geq m, U_n \in K \subset \mathbb{R}_+^d$ and proves it by induction

Thm. If $\lambda_1 > |\lambda_i|$, $i \neq 1$, λ_1 simple, then for arbitrary order d, the algorithm terminates for U_0 outside of a hyperplane.

Step 1 given by Vandergraft's construction;
Existence of m in Step 2 by continuity.

Next: generic U_0

Algorithms Proving Inequalities Between Sequences

Part II

Slides available at
https://perso.ens-lyon.fr/bruno.salvy/talks/JNCF2026-PartII.pdf

Luminy, March 2026

I. (Short) Introduction

Summary of Part I (1/2)

. P-finite sequence $(u_n) \to$ sequence of vectors

$$U_n = (u_n, \ldots, u_{n+d-1})^\top \text{ with}$$

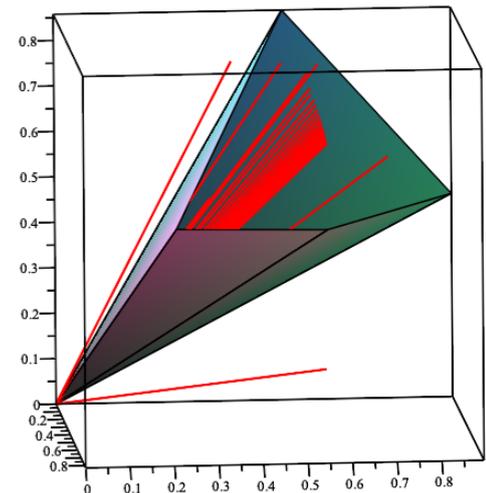$$U_{n+1} = A(n)U_n = A(n)A(n-1)\cdots A(0)U_0;$$

. $A = \lim A(n)$ with eigenvalues

$$|\lambda_1| = \cdots = |\lambda_\nu| > |\lambda_i| \ (i > \nu)$$

contracts a cone iff $\nu = 1$ and $\lambda_1 > 0$;

. Such a cone can be obtained from a Jordan form of A;

. This leads to a simple algorithm in that situation:

Summary of Part I (2/2)

$A := \lim\limits_{n \to \infty} A(n)$
λ_i eigenvalues of A

Depends on $A(n)$ only

0. Check $\lambda_1 > |\lambda_i|$, $i \neq 1$;
1. Construct a cone $K \subset \mathbb{R}_+^d$ s.t. $A(K \setminus \{0\}) \subset K^\circ$;
2. Compute m s.t. for all $n \geq m$, $A(n)K \subset K$;
3. For $i = 1, 2, \dots$ do
    1. $U_i = A(i)U_{i-1}$;
    2. If $U_i \notin \mathbb{R}_+^d$ return false
    3. If $i \geq m$ and $U_i \in K$ return true

Depends also on U_0
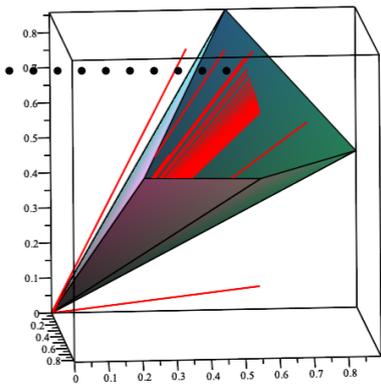
Constructs a formula $\forall n \geq m, U_n \in K \subset \mathbb{R}_+^d$ and proves it by induction

. When $A(n)$ is constant,

$$A^n / \lambda_1^n \to VW^\top \quad (AV = \lambda_1 V, A^\top W = \lambda_1 W)$$

and the algorithm terminates for any U_0 s.t.
$$W^\top U_0 \neq 0.$$

Generalized Power Method

Thm. $A(n) \in \mathrm{GL}_d(\mathbb{R})$ with limit A s.t. $\lambda_1 > |\lambda_i|, i \neq 1,$ λ_1 simple, then

$$\frac{A(n)A(n-1)\cdots A(0)}{\|A(n)A(n-1)\cdots A(0)\|} \to VW^\top,$$

> Recall that
> $U_{n+1} = A(n)\cdots A(0)U_0$

with $AV = \lambda_1 V$. Initial conditions generic when $W^\top U_0 \neq 0$.

Ex. Apéry recurrence

$$(n+2)^3 u_{n+2} = (2n+3)(17n^2 + 51n + 39)u_{n+1} - (n+1)^3 u_n$$

$$V = \begin{pmatrix} 1 \\ (3 + 2\sqrt{2})^2 \end{pmatrix}, \quad W = \begin{pmatrix} 1 \\ (6/\zeta(3) - 5)^{-1} \end{pmatrix}$$

algebraic

difficult to control in general

Today's Menu

II. Positivity proofs with various choices of cones

III. Sequences of cones

IV. Genericity issues

I.   Proof of Friedland's theorem

III. Variants of Vandergraft's Cone

Example from yesterday (1/2)

$$(16n + 1)\, u_{n+3} = (5n - 3)\, u_n - (20n - 4)\, u_{n+1} + (32n - 2)\, u_{n+2}$$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{5}{16} & -\frac{5}{4} & 2 \end{pmatrix}$$

Eigenvectors: $V_1, V_2, \overline{V_2}$

$$V_i = (1, \lambda_i, \lambda_i^2)^\mathsf{T}$$

$$\lambda_1 \approx 1.15 > |\lambda_2|,$$

$$\lambda_2 \approx 0.43 + 0.30i.$$
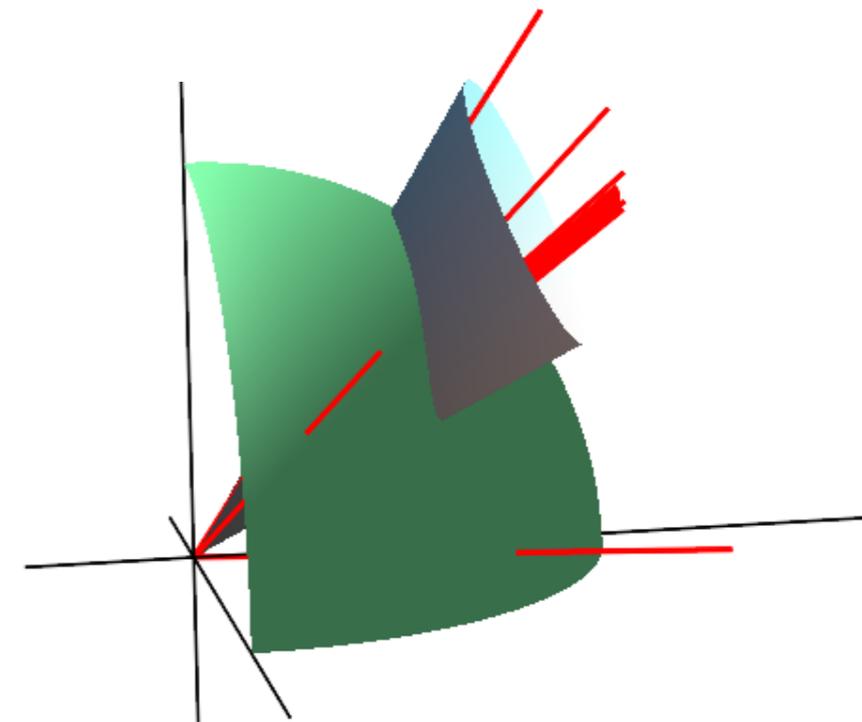
$$K_\mu := \{a\mu V_1 + bV_2 + \overline{b}\,\overline{V_2} \mid |b| \leq a\} \text{ (real)}$$

$$\forall \mu > 0,\ AK_\mu \subset \mathring{K}_\mu$$

$$\exists \mu > 0,\ K_\mu \subset \mathbb{R}^d_{>0}$$
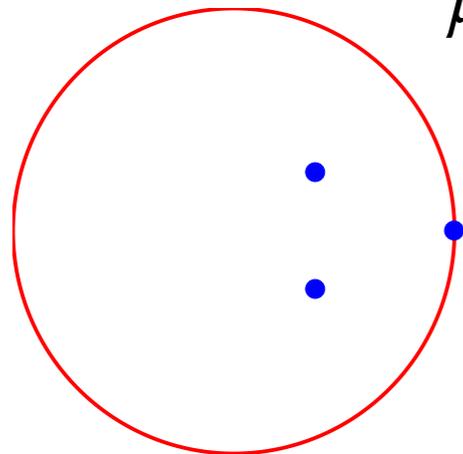
Example (2/2)

$$K = \{aV_1 + bV_2 + \overline{b}\,\overline{V_2} \mid |b| \leq a\}$$

Extremal vectors: $V_1 + e^{it}V_2 + e^{-it}\overline{V_2}$

Image of extremal vectors

$$\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} = V^{-1}A(n)V \begin{pmatrix} 1 \\ c + is \\ c - is \end{pmatrix} \qquad c^2 + s^2 = 1$$
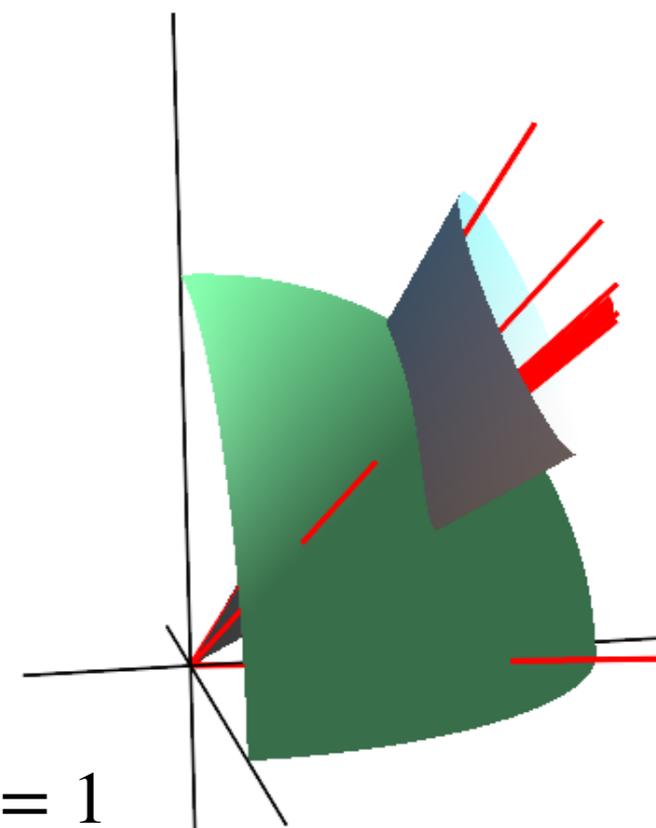
Wanted: m s.t. $a_n \geq 0$ and $\underbrace{a_n^2 - b_n c_n}_{} \geq 0$ for all c, s and $n \geq m$.

$$\in \mathbb{Q}(n, \lambda_1, \lambda_2, \overline{\lambda_2})[c, s]/(c^2 + s^2 - 1)$$

m computable by quantifier elimination (in theory)

Approximate Cones

1. Contracted cone

$$\tilde{K} := \{a\tilde{V}_1 + b\tilde{V}_2 + \overline{b}\,\overline{\tilde{V}_2} \mid |b| \leq a\},$$

$\tilde{V}_1, \tilde{V}_2$ rational approximations of V_1, V_2

$$\tilde{\lambda}_1 = \frac{7}{6}, \qquad \tilde{\lambda}_2 = \frac{2}{5} + \frac{i}{3}$$

Check that $A\tilde{K} \subset \tilde{K}^\circ \subset \mathbb{R}^d_+$

otherwise, increase precision

2. Contraction index

Image of extremal vectors

$$\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} = \tilde{V}^{-1} A(n) \tilde{V} \begin{pmatrix} 1 \\ c + is \\ c - is \end{pmatrix}$$
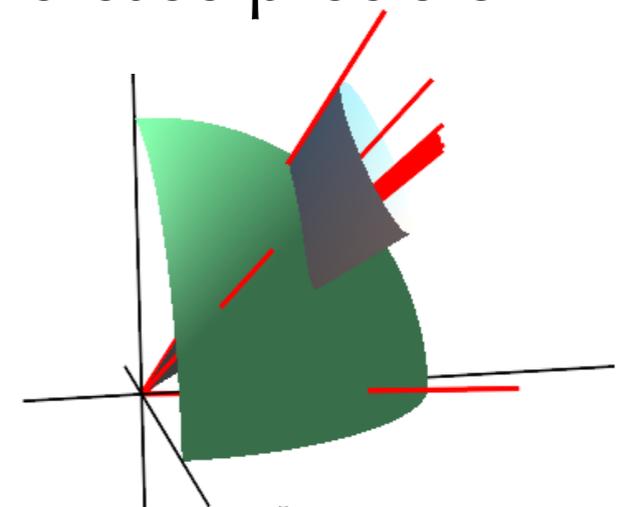
Wanted: m s.t. $a_n \geq 0$ and $a_n^2 - b_n c_n \geq 0$ for all c, s and $n \geq m$.

$\underbrace{p_2(s,c)n^2 + p_1(s,c)n + p_0(s,c)}$ now in $\mathbb{Q}[n,s,c]/(s^2 + c^2 - 1)$

instead of $\mathbb{Q}(n, \lambda_1, \lambda_2, \overline{\lambda_2})[c,s]/(c^2 + s^2 - 1)$

Polyhedral Contracted Cones

$$|\mathfrak{R}(b)| + |\mathfrak{I}(b)|$$

$$\hat{K} := \{aV_1 + bV_2 + \overline{b}\,\overline{V_2} \mid \|b\|_1 \leq a\},$$

$$\|uv\|_1 \leq \|u\|_1\|v\|_1, \quad \|\lambda_2\|_1 < \lambda_1 \Rightarrow A\hat{K} \subset (\hat{K})^\circ$$

+ approximate version as before
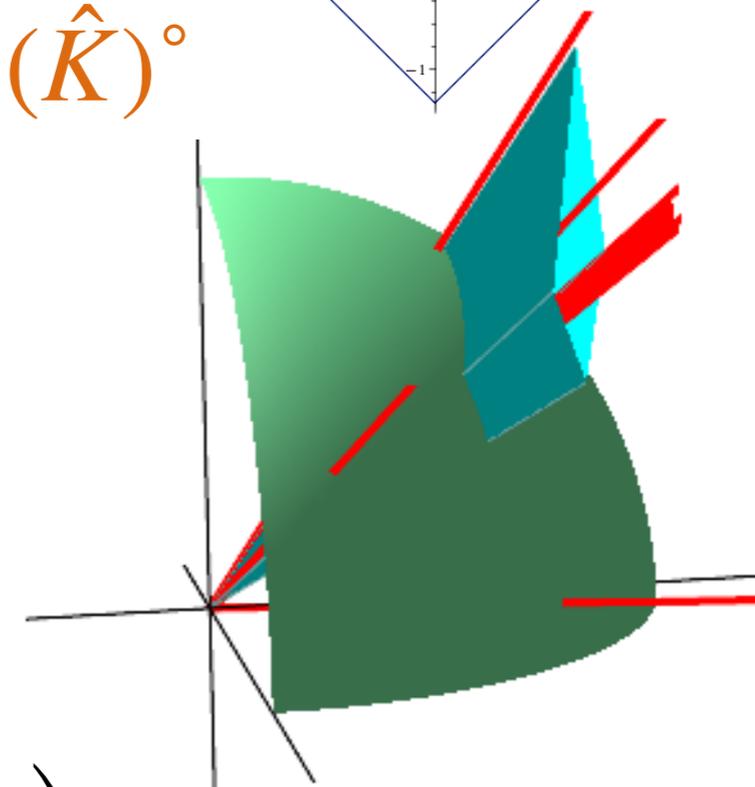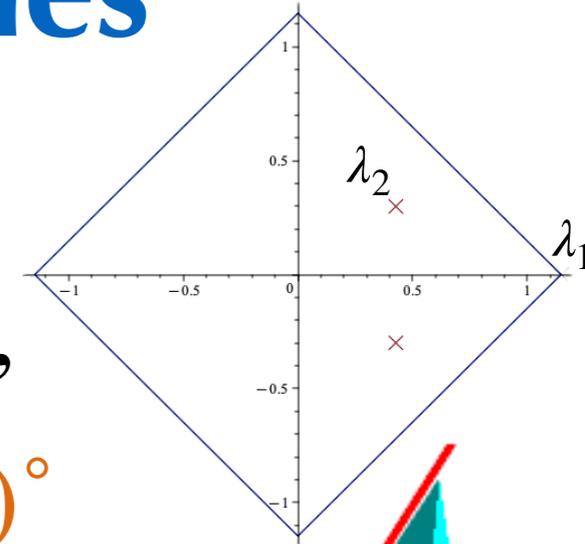
2. Contraction index

Image of 4
extremal vectors

$$\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} = \tilde{V}^{-1} A(n) \tilde{V} \begin{pmatrix} 1 \\ c + is \\ c - is \end{pmatrix}$$

Wanted: m s.t. $a_n \geq \|b_n\|_1$ for $\begin{pmatrix} c \\ s \end{pmatrix} \in \{\begin{pmatrix} \pm 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \pm 1 \end{pmatrix}\}$ and $n \geq m$.

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \frac{18811}{3774}n - \frac{452275}{30192} > 0 \text{ for } n \geq 4, \qquad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \frac{138361}{18870}n + \frac{576563}{150960} > 0 \text{ for } n \geq 5,$$

$$\begin{pmatrix} 0 \\ -1 \end{pmatrix} \to \frac{139657}{18870}n + \frac{2215391}{150960} > 0 \text{ for } n \geq 3, \qquad \begin{pmatrix} -1 \\ 0 \end{pmatrix} \to \frac{681019}{94350}n - \frac{3927583}{754800} > 0 \text{ for } n \geq 1$$
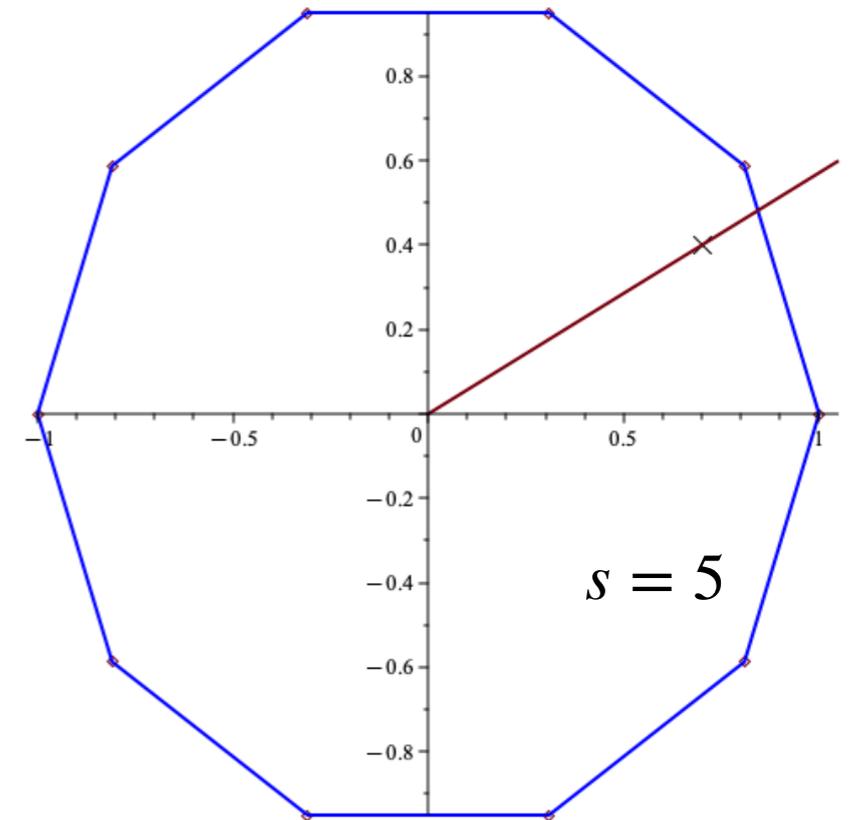
$\tilde{\hat{K}}$ contracted
by A_n for $n \geq 5$

simple computations in $\mathbb{Q}[n]$

General Case: Minkowski Functional

For $\mathscr{P} \subset \mathbb{C}$,

$$\|z\|_{\mathscr{P}} = \inf\{r > 0 \,|\, z \in r\mathscr{P}\}$$

$\mathscr{P}_s :=$ polygon with vertices at $e^{ik\pi/s}$

$\|\cdot\|_{\mathscr{P}_s}$ is a sub-multiplicative norm

$s = 5$

$$K := \{a_1 V_1 + \cdots + a_d V_d \,|\, \|a_i\|_{\mathscr{P}_{s_i}} \leq a_1 \text{ and } a_j = \overline{a}_i \text{ when } V_j = \overline{V}_i\}$$
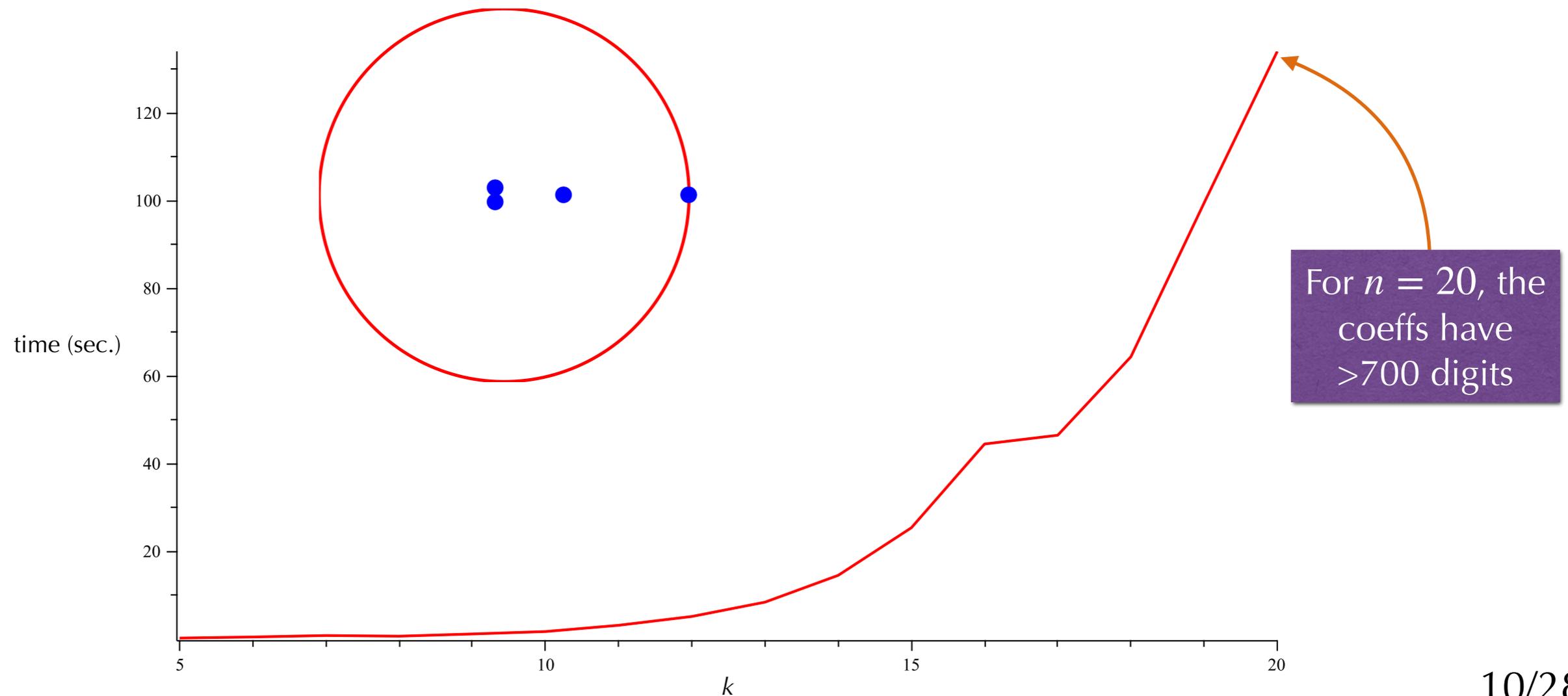
with s_i minimal s.t. $\lambda_i \in \mathscr{P}_{s_i}$

is a polyhedral cone that satisfies $AK \subset \mathring{K}$.

Timings

$$u_n^{(k)} = \sum_{j=0}^{n} (-1)^j \frac{(kn - (k-1)j)!\, k!^j}{(n-j)!^k\, j!} \geq 0 \text{ for } k \geq 4$$

Gillis-
Reznick-
Zeilberger

linear rec of order k with coeffs of degree $k(k-1)/2$

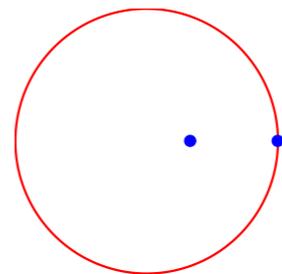For $n = 20$, the coeffs have >700 digits

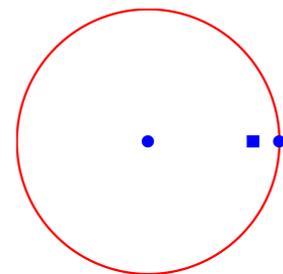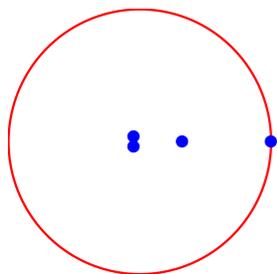# Back to the Examples with Contracted Cones
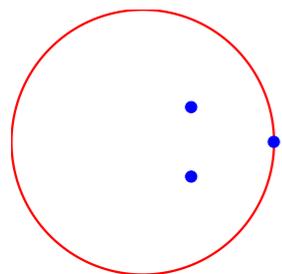
Running example
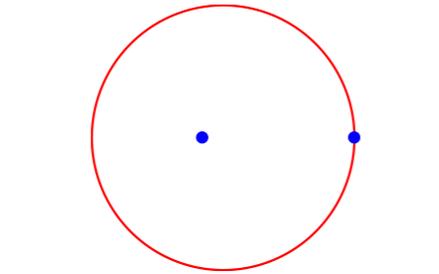
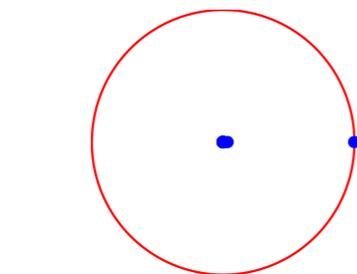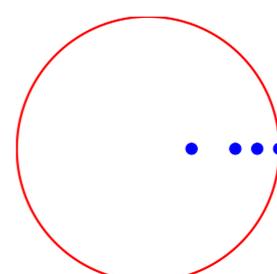$$\sum_{k=0}^{n} (-27)^{n-k} 2^{2k-n} \frac{(3k)!}{k!^3} \binom{k}{n-k}$$

AMM 11501

Serra et al.

Gillis-Reznick-Zeilberger $k = 4$

Gillis-Reznick-Zeilberger $k = 7$

Aharonov 2017

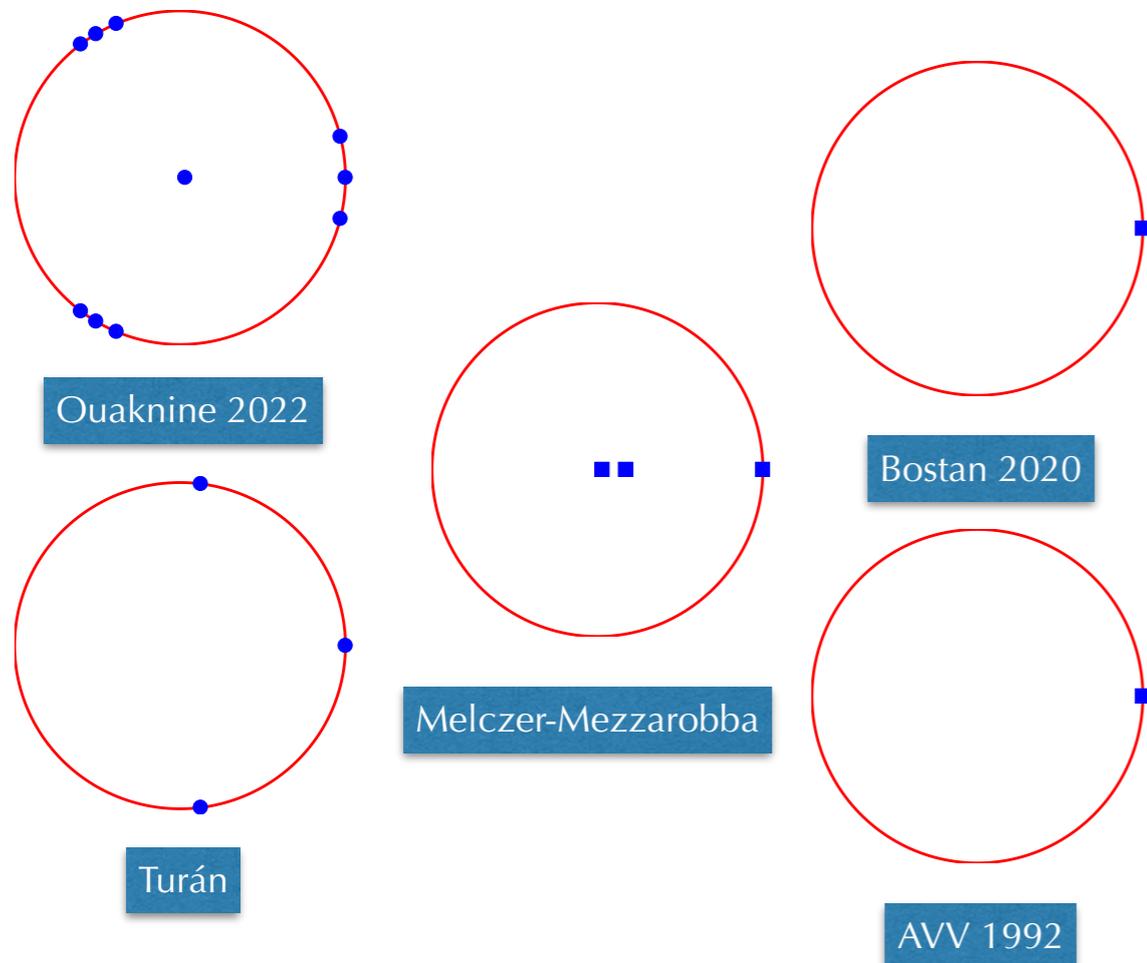All those can be handled by the algorithm

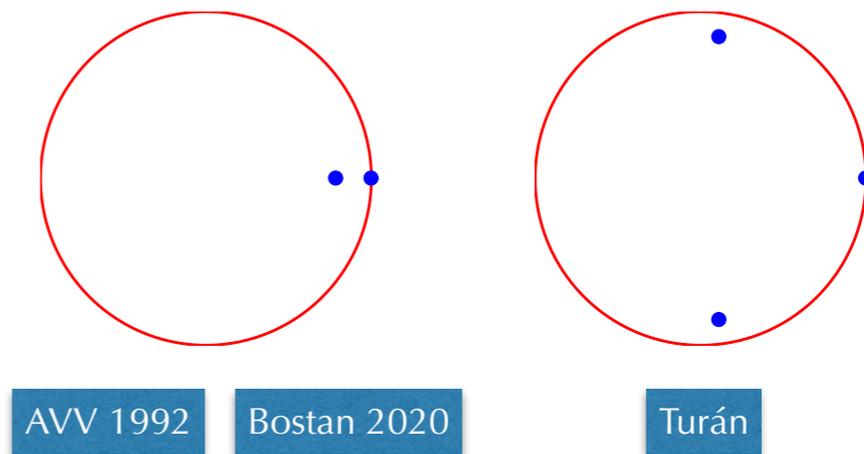In parameterized cases, generally only for given values

IV. Sequences of Cones
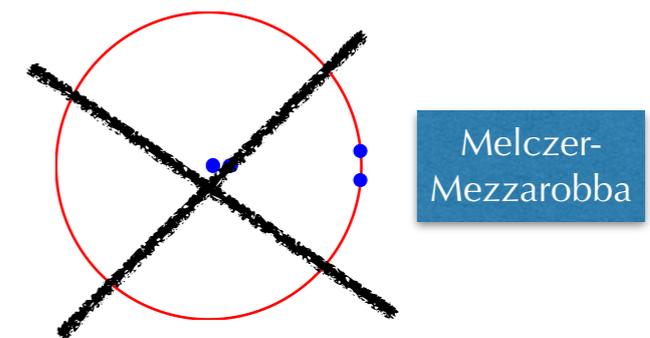
See also Alaa's slides (JNCF2025)

Examples with Invariant, but not Contracted, Cones

View for finite n

Ouaknine 2022

Melczer-Mezzarobba

Bostan 2020

Turán

AVV 1992

AVV 1992   Bostan 2020   Turán

Contracted cones!
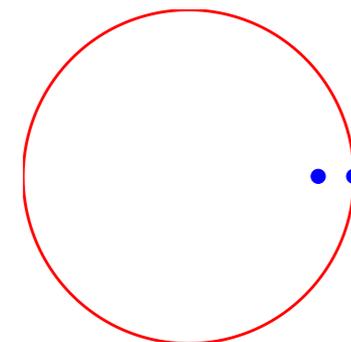
Melczer-Mezzarobba

● simple eig.

■ multiple eig.

The power series $s(x)$ solution to

$$xs^4 - 2xs^3 + 2s - 1 = 0$$

has positive coefficients.

AVV: $(1 - x)^{1/4}{}_2F_1([1/2, 1/2], [1]; x)$

strictly ↓ on (0,1)

[Anderson, Vamanamurthy, Vuorinen 92]

Proof by Sequence of Cones

The coefficients of $s(x)$ satisfy

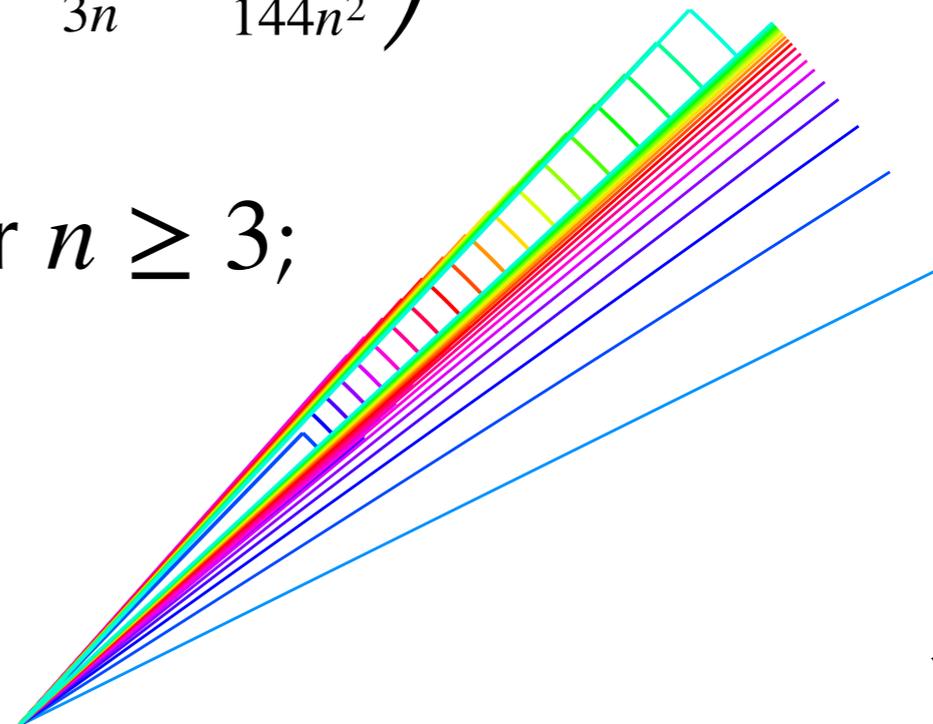$$2n\left(3n+2\right)\left(3n+1\right)u_n - \left(36n^3 + 108n^2 + 106n + 35\right)u_{n+1} + 2\left(3n+5\right)\left(3n+7\right)\left(n+1\right)u_{n+2} = 0$$

Eigenvalues of $A(n)$: $\quad 1 - \dfrac{2}{3n} + \dfrac{131}{144n^2} + O\!\left(\dfrac{1}{n^3}\right), \; 1 - \dfrac{10}{3n} + \dfrac{1325}{144n^2} + O\!\left(\dfrac{1}{n^3}\right)$
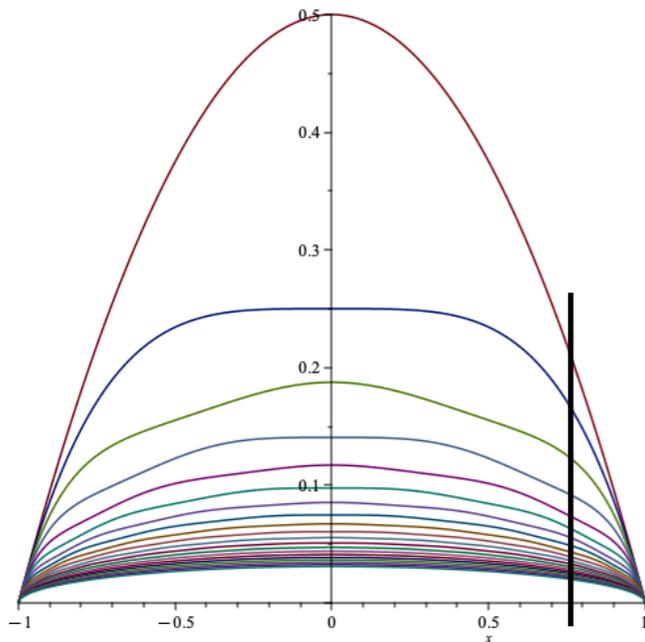
The cone $\quad K_n = c_1 \begin{pmatrix} 2 \\ 2 - \frac{4}{3n} + \frac{131}{72n^2} \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 - \frac{10}{3n} + \frac{1325}{144n^2} \end{pmatrix} \quad |c_2| \le c_1$

satisfies $K_n \subset \mathbb{R}^2_{>0}$; $\quad A(n)K_n \subset K_{n+1}$ for $n \ge 3$; $\quad (u_8, u_9)^\top \in K_8$.

For AVV92, $y \ge x(1 - 5/(4n) + 2/n^2) \ge 0$
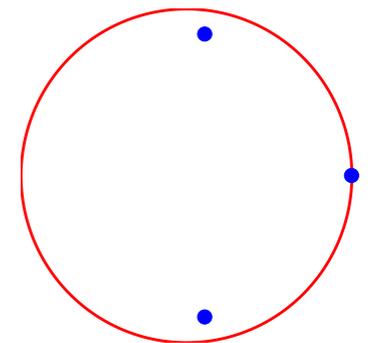works for n $\ge$ 3.

Turán's Inequality at $x = 3/4$

$$-16\,(2n+7)\,(n+1)\,(n+2)^2\,u_n$$

$$+\left(2n+5\right)(n+2)\left(20n^2+108n+137\right)u_{n+1}$$

$$-(2n+7)\,(n+4)\left(20n^2+132n+209\right)u_{n+2}$$

$$+16\left(n+5\right)\left(2n+5\right)(n+4)^2\,u_{n+3}=0$$

Eigenvalues

$$1 - \frac{1}{n} + O(\frac{1}{n^2})$$

$$\frac{1 \pm 3\sqrt{7}i}{8} - \frac{3 \pm 9\sqrt{7}i}{8n} + O(\frac{1}{n^2}) \qquad |\cdot| = 1 - \frac{3}{n} + O(\frac{1}{n^2})$$

$\rightarrow$ a sequence of cones K_n such that

$$K_n \subset \mathbb{R}^2_{>0};\ A(n)K_n \subset K_{n+1} \text{ for } n \geq 20;$$
$$(u_{20}, u_{21}, u_{22})^\top \in K_{20}.$$

Sufficient Conditions

Prop. If $A(n) \to A$ with A having only simple dominant eigenvalues $\lambda_1, \ldots, \lambda_\nu$, one of which is positive and

$$\max |\lambda_{i,n} - \lambda_{i+1,n}| = o(\lambda_{1,n} - \max_{j \neq 1} |\lambda_{j,n}|),$$

then there exists a sequence of proper positive cones K_n s.t. $K_n \subset A(n)K_{n+1}$ for n sufficiently large.

Typically, $\lambda_{i,n} - \lambda_{i,n+1} = O(\frac{1}{n^2})$, $\lambda_{1,n} - \max |\lambda_{j,n}| = O(\frac{1}{n})$.

V. Positive Sequences with Nongeneric Initial Conditions

Initial conditions generic when $W^\top U_0 \neq 0$.

From Friedland's Theorem
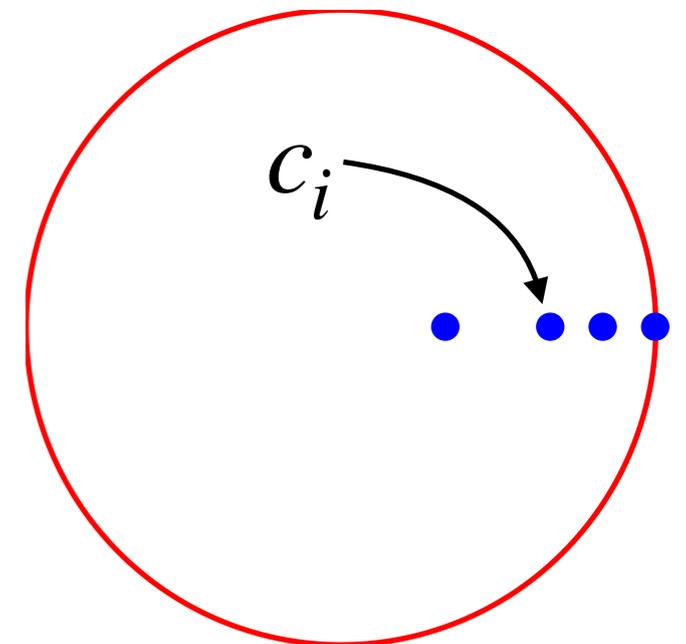
Genericity from the Problem

Ex.: For $c_1 > \cdots > c_k > 0$ and $\mu_i > 0$, $\mu_1 + \cdots + \mu_k \leq 1$,

$$\prod_{j=1}^{k} (1 - c_j x)^{\mu_j} = 1 - \sum_{n \geq 1} d_n x^n \text{ with } d_n > 0. \quad \text{[Aharonov 2017]}$$

Recurrence of order k, eigenvalues $c_1, \ldots, c_n$, singularity at $1/c_1$ $\rightarrow$ generic initial conditions.

$\rightarrow$ automatic proof for any $(c_1, \ldots, c_k, \mu_1, \ldots, \mu_k)$.

Non-Genericity from the Problem

Ex.: The power series $y(x)$ solution to

[Dotsenko, Markl, Remm 2020]

$$y = x - y^8 + y^{15}$$

is of the form $xh(x^7)$, h having positive coefficients.

Method of proof:

1. Analysis (radius of convergence $\frac{21^7}{5^{15}}$);

   From information outside of the recurrence

2. the sequence v_n of coefficients satisfies a linear recurrence of order 2 with eigenvalues $\frac{5^{15}}{21^7} < \frac{3^{15}}{7^7}$;

   algebraic $\Rightarrow$ D-finite

3. construct another solution w_n of the recurrence;

4. prove its positivity;

   Ok, generically

5. construct a recurrence for $u_n := w_{n+1}v_n - v_{n+1}w_n$ and show its positivity;

   use the algo

6. conclusion: v_n/w_n decreases to $0 \Rightarrow$ positive!

II. Generalized Power Method

Why the algorithm works for generic initial conditions

Generalized Power Method

Thm. $A(n) \in \mathrm{GL}_d(\mathbb{R})$ with limit A s.t. $\lambda_1 > |\lambda_i|, i \neq 1$, λ_1 simple, then
$$\frac{A(n)A(n-1)\cdots A(0)}{\|A(n)A(n-1)\cdots A(0)\|} \to VW^\top,$$

Recall that
$U_{n+1} = A(n)\cdots A(0)U_0$

with $AV = \lambda_1 V$. Initial conditions generic when $W^\top U_0 \neq 0$.

Ex. Apéry recurrence
$$(n+2)^3 u_{n+2} = (2n+3)(17n^2 + 51n + 39)u_{n+1} - (n+1)^3 u_n$$

$$V = \begin{pmatrix} 1 \\ (3+2\sqrt{2})^2 \end{pmatrix}, \quad W = \begin{pmatrix} 1 \\ 6/\zeta(3) - 5 \end{pmatrix}.$$

algebraic

difficult to control in general

[Friedland 2006]

Ingredients for the Proof Next 7 Slides

1. There exists a cone K contracted by A

2. For $n \geq n_0$, $A(n)$ contracts K

3. Let d_K be Hilbert's pseudo-metric in K

4. The diameters of AK and $A(n)K$, $n \geq n_0$ for d_K are bounded by a finite Δ

5. The matrices $A(n)$ all have Lipschitz constant upper bounded by $L = \tanh(\Delta/4) < 1$

6. The transpose matrices $A(n)^\top$ contract the dual cone $K^\star$ with same diameter for $d_{K^\star}$.
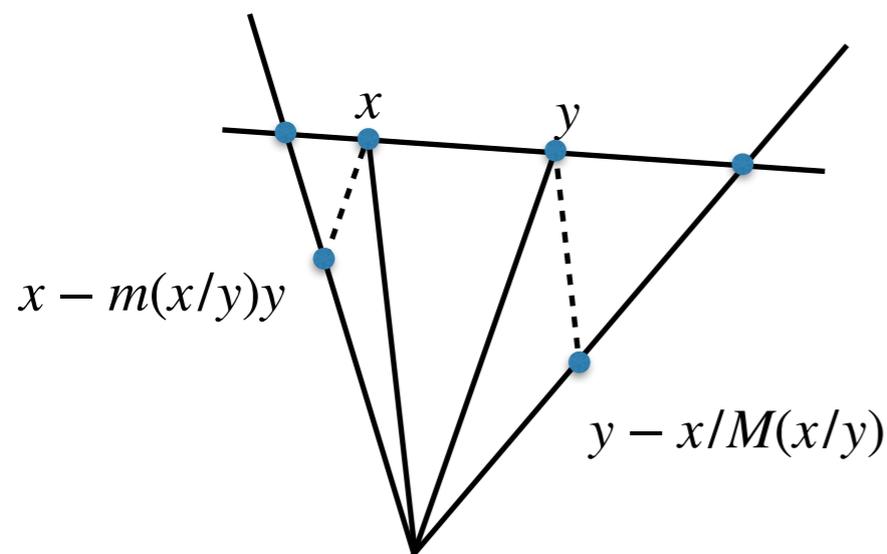
Birkhoff (1957)
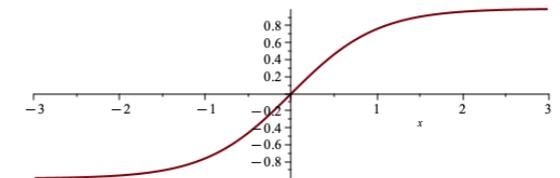
Hilbert's Pseudo-Metric

K a proper cone

Notation: $x \leq_K y$ when $y - x \in K$

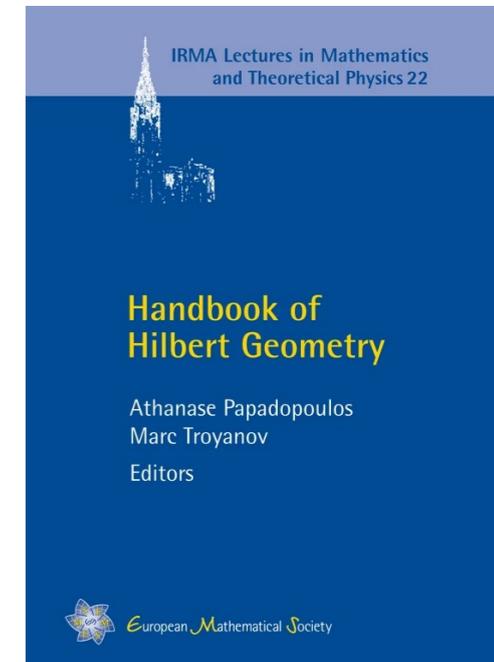$$M(x/y) = \inf\{\beta > 0 \mid x \leq_K \beta y\}, \quad m(x/y) = \sup\{\alpha > 0 \mid \alpha y \leq_K x\}$$

Ex.: $K = \mathbb{R}^d_+$, $M(x/y) = \max x_i/y_i$
$$m(x/y) = \min x_i/y_i$$

$x - m(x/y)y$

$y - x/M(x/y)$

$$d_K(x, y) := \log \frac{M(x/y)}{m(x/y)}$$

Prop.
1. $d(x, y) \geq 0$;
2. $d(x, y) = d(y, x)$;
3. $d(x, z) \leq d(x, y) + d(y, z)$;
4. $d(x, y) = d(\lambda x, \mu y)$, $\lambda, \mu > 0$;
5. $d(x, y) = 0 \Leftrightarrow y = \lambda x$, $\lambda > 0$.

[Birkhoff 1957]

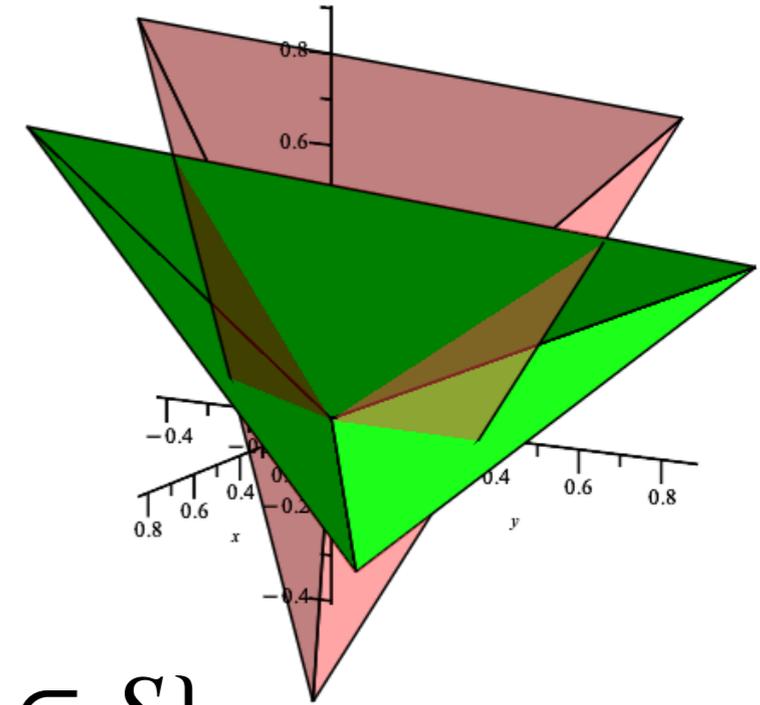# Proof with the Ingredients (1/2)

$$Q(n) := A(n)A(n-1)\cdots A(0) \qquad AV = \lambda_1 V$$

1. The columns of Q converge in direction to V

. Since $A(n) \to A$, $\varepsilon_n := d_K(A(n)V, V) \to 0$

. Let $y_n = A(n)y_{n-1}$ with $y_0 = x \in K\backslash\{0\}$. Then

$$\begin{aligned} d_K(y_n, V) &= d_K(A(n)y_{n-1}, V) \\ &\leq d_K(A(n)y_{n-1}, A(n)v) + d_K(A(n)V, V) \\ &\leq Ld_K(y_{n-1}, V) + \varepsilon_n \end{aligned}$$

. Thus $d_K(Q(n)x, V) \to 0$ for all x.

Duality

Def. If $S \subset \mathbb{R}^d$, its dual is the set

$$S^{\star} := \{y \in \mathbb{R}^d \mid (y, z) \geq 0 \text{ for all } z \in S\}.$$

Prop. If K is a proper cone, then
. $x \notin K \Rightarrow \exists y \in K^{\star}, (y, x) < 0$;
. $K^{\star\star} = K$;
. $K^{\star}$ is a proper cone;
. $(K^{\star})^{\circ} = \{y \in \mathbb{R}^d \mid (y, z) > 0 \text{ for all } z \in K \backslash \{0\}\}$;
. $AK \subset K \Rightarrow A^{\top}K^{\star} \subset K^{\star}$.

Proof with the Ingredients (2/2)

$$Q(n)^\top = A(0)^\top \cdots A(n)^\top =: A(0)^\top \cdots A(n_0 - 1)^\top R(n)^\top$$

2. The rows of Q converge in direction to a vector W

. $u_{n+1} = R(n)^\top A(n)^\top z$ with $u_{n_0} = z \in K^\star \backslash \{0\}$ is Cauchy:

$$\begin{aligned}
d_{K^\star}(u_{n+m}, u_n) &= d_{K^\star}(R(n)^\top A(n+1)^\top \cdots A(n+m)^\top z, R(n)^\top z) \\
&\leq L^{n-n_0} d_{K^\star}(A(n+1)^\top \cdots A(n+m)^\top z, z) \\
&\leq L^{n-n_0} \Delta(A) \to 0
\end{aligned}$$

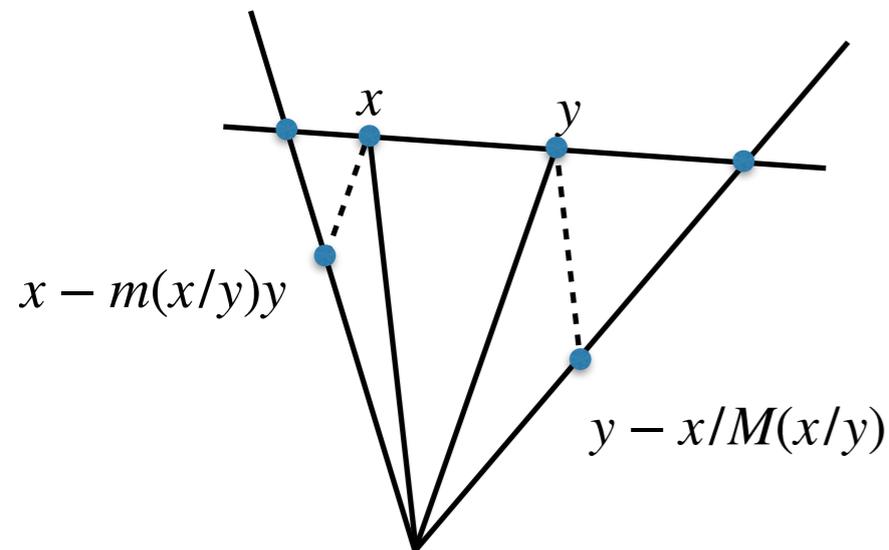. the set of directions is compact $\to$ limit vector Z

. $W = A(0)^\top \cdots A(n_0 - 1)^\top Z$

Diameter

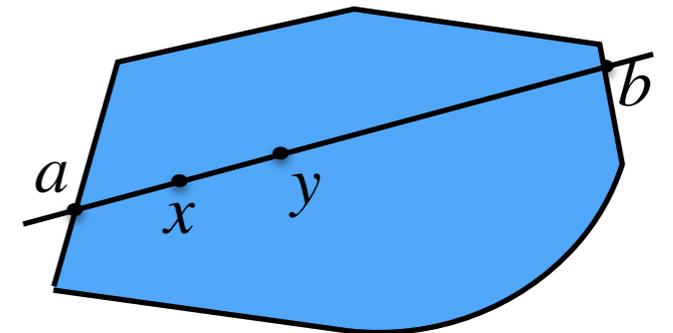A matrix s.t. $AK \subset K$

Def. Diameter of A : $\Delta_K(A) = \max(d_K(Ax, Ay) \mid x, y \text{ in } K\}$.

x y

$x - m(x/y)y$

$y - x/M(x/y)$

$$d_K(x, y) := \log \frac{M(x/y)}{m(x/y)}$$

Prop. $\Delta_K(A)$ finite iff $AK \subset K^\circ$.

a b

x y

$$d_K(x, y) = \log\left(\frac{\overline{bx}}{\overline{by}} \frac{\overline{ay}}{\overline{ax}} \right)$$

Transposition

Yet another expression for d_K:

$$d_K(x, y) = \sup_{f, g \in K^\star} \log \frac{(f, x)(g, y)}{(f, y)(g, x)} .$$

$$x \leq_K y \Leftrightarrow (f, x) \leq (f, y), \quad f \in K^\star$$

Ex.: $K = \mathbb{R}_+^d = K^\star$, $d_K(x, y) = \log \max_{i, j} \frac{x_i y_j}{y_i x_j}$
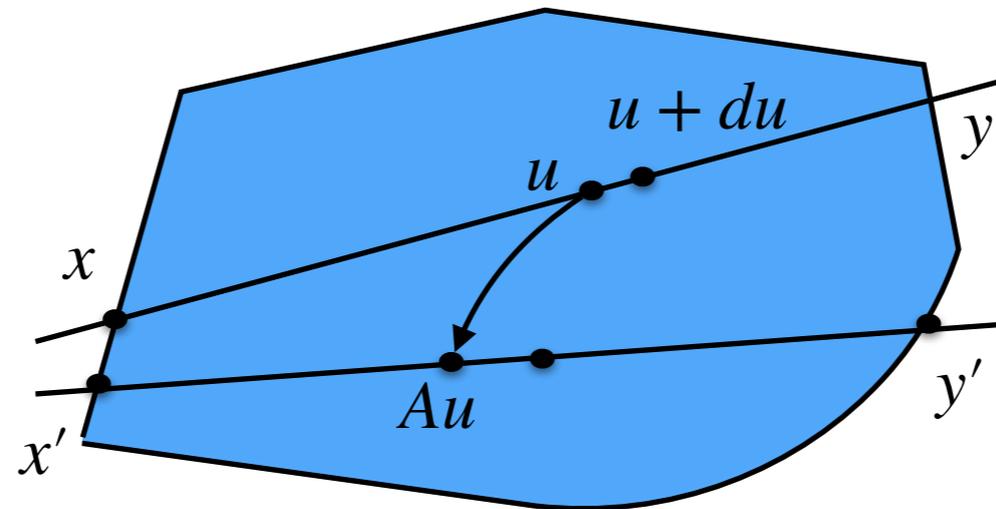
extends to the computation of d_K for polyhedral cones

Application:

$$\Delta_K(A) = \sup_{x, y \in K, f, g \in K^\star} \log \frac{(f, Ax)(g, Ay)}{(f, Ay)(g, Ax)}$$

$$= \sup_{x, y \in K, f, g \in K^\star} \log \frac{(A^\top f, x)(A^\top g, y)}{(A^\top f, y)(A^\top g, x)} = \Delta_{K^\star}(A^\top) .$$

[Eveson-Nussbaum 1995]

Birkhoff's Contraction Ratio

Thm. For any u, v in K, $d_K(Au, Av) \leq \lambda_K(A)\, d_K(u, v)$, with
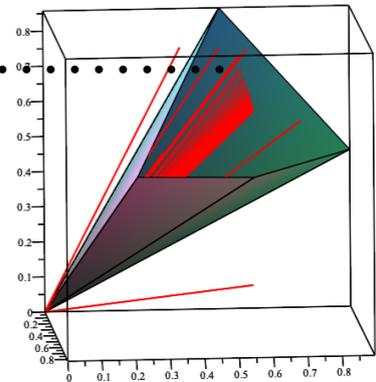
$$\lambda_K(A) = \tanh \frac{\Delta_K(A)}{4} \,.$$

[Birkhoff 1957]

Conclusion

Summary

Depends on $A(n)$ only

Depends also on U_0

0. Check $\lambda_1 > |\lambda_i|$, $i \neq 1$;
1. Construct a cone $K \subset \mathbb{R}_+^d$ s.t. $A(K \setminus \{0\}) \subset K°$;
2. Compute m s.t. for all $n \geq m$, $A(n)K \subset K$;
3. For $i = 1,2,\ldots$ do
    1. $U_i = A(i)U_{i-1}$;
    2. If $U_i \notin \mathbb{R}_+^d$ return false
   3. If $i \geq m$ and $U_i \in K$ return true

Constructs a formula $\forall n \geq m, U_n \in K \subset \mathbb{R}_+^d$ and proves it by induction

Thm. If $\lambda_1 > |\lambda_i|, i \neq 1$, λ_1 simple, then for arbitrary order d, the algorithm terminates for U_0 outside of a hyperplane.

A rather simple algorithm built on
the Perron-Frobenius theory for cones;
the generalized power method
handles many `simple' inequalities, plus extensions.

Possible Future Work

Find an approach to invariant cones of the type encountered in the Melczer-Mezzarobba rec.
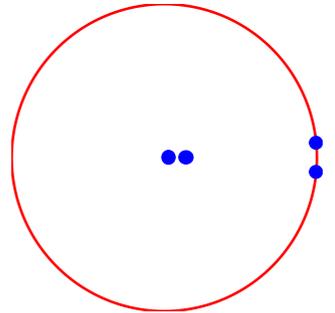
More classes where sufficient conditions ensure that a sequence of cones will capture cases with a limit invariant, but not contracted, cone.

Cones that behave more uniformly wrt parameters for the contraction index and the initial conditions.

Questions?