Inequalities - JNCF 26

Demo Part II

```
> with(LinearAlgebra):
> Digits:=20:
```

```
lowpol:=proc(p,x)
```

Generalized Power Method on an Example

Start from Apéry's recurrence

```
> rec := (n + 2)^3*u(n + 2) - (2*n + 3)*(17*n^2 + 51*n + 39)*u(n
  + 1) + (n + 1)^3*u(n);
```

$$rec := (n+2)^3\, u(n+2) - (2\,n+3)\,\left(17\,n^2+51\,n+39\right)\, u(n+1) + (n+1)^3\, u(n) \quad \textbf{(1.1)}$$

Construct A(n)

```
> pol:=eval(rec,u=(k->X^(k-n)));
```

$$pol := (n+2)^3\, X^2 - (2\,n+3)\,\left(17\,n^2+51\,n+39\right) X + (n+1)^3 \quad \textbf{(1.2)}$$

```
> An:=Transpose(CompanionMatrix(pol/lcoeff(pol,X),X));
```

$$An := \begin{bmatrix} 0 & 1 \\ -\frac{(n+1)^3}{(n+2)^3} & \frac{(2\,n+3)\,\left(17\,n^2+51\,n+39\right)}{(n+2)^3} \end{bmatrix} \quad \textbf{(1.3)}$$

```
> A:=map(limit,An,n=infinity);
```

$$A := \begin{bmatrix} 0 & 1 \\ -1 & 34 \end{bmatrix} \quad \textbf{(1.4)}$$

Compute the matrix factorial A(n)A(n-1)...A(0) for n=1000

```
> M:=evalf(eval(An,n=0));
```

$$M := \begin{bmatrix} 0. & 1. \\ -0.12500000000000000000 & 14.625000000000000000 \end{bmatrix} \quad \textbf{(1.5)}$$

```
> for i to 1000 do M:=eval(An,n=i).M; M:=M/M[1,1] od:
> M;
```

$$\begin{bmatrix} 1.0000000000000000000 & -116.88026220900885108 \\ 33.919735457270614365 & -3964.5475743060041487 \end{bmatrix} \quad \textbf{(1.6)}$$

"Recognize" the values:

```
> lcoeff(pol,n);
```

$$X^2 - 34\,X + 1 \qquad (1.7)$$

```
> solve(%,X);
```

$$17 + 12\sqrt{2},\ 17 - 12\sqrt{2} \qquad (1.8)$$

```
> evalf([%]);
```

$$[33.970562748477140586,\ 0.029437251522859414] \qquad (1.9)$$

```
> 1/(6/Zeta(3)-5);
```

$$\frac{1}{\frac{6}{\zeta(3)} - 5} \qquad (1.10)$$

```
> evalf(%);
```

$$-116.88026220900885094 \qquad (1.11)$$

Running Example

Aim of this example: prove that the sequence defined by

```
> rec:=(16*n+1)*u(n+3)-(32*n-2)*u(n+2)+(20*n-4)*u(n+1)-(5*n-3)*u
  (n)=0;
```

$$rec := (16\,n + 1)\,u(n+3) - (32\,n - 2)\,u(n+2) + (20\,n - 4)\,u(n+1) - (5\,n - 3)\,u(n) = 0 \qquad (2.1)$$

and

```
> ini:={u(0)=5,u(1)=5,u(2)=1};
```

$$ini := \{u(0) = 5,\ u(1) = 5,\ u(2) = 1\} \qquad (2.2)$$

is positive.

Limit matrix and its eigenvalues

```
> pol:=eval(op(1,rec),u=(k->X^(k-n)));
```

$$pol := (16\,n + 1)\,X^3 - (32\,n - 2)\,X^2 + (20\,n - 4)\,X - 5\,n + 3 \qquad (2.1.1)$$

```
> ch:=lcoeff(pol,n);
```

$$ch := 16\,X^3 - 32\,X^2 + 20\,X - 5 \qquad (2.1.2)$$

```
> An:=Transpose(CompanionMatrix(pol/lcoeff(pol,X),X));
```

$$An := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{-5\,n+3}{16\,n+1} & -\frac{20\,n-4}{16\,n+1} & -\frac{-32\,n+2}{16\,n+1} \end{bmatrix} \qquad (2.1.3)$$

```
> A:=map(limit,An,n=infinity);
```

(2.1.4)

$$A := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{5}{16} & -\frac{5}{4} & 2 \end{bmatrix} \qquad \textbf{(2.1.4)}$$

Eigenvalues of the limit matrix:

```
> eigs:=[fsolve(ch,X,complex)];
```

$$eigs := [0.42571087295564390688 - 0.30140628765072823665\,I,\ 0.42571087295564390688 + 0.30140628765072823665\,I,\ 1.1485782540887121862] \qquad \textbf{(2.1.5)}$$

Sort them by decreasing modulus:

```
> eigs:=sort(eigs,proc(u,v) abs(u)>=abs(v) end);
```

$$eigs := [1.1485782540887121862,\ 0.42571087295564390688 - 0.30140628765072823665\,I,\ 0.42571087295564390688 + 0.30140628765072823665\,I] \qquad \textbf{(2.1.6)}$$

```
> map(abs,eigs);
```

$$[1.1485782540887121862,\ 0.52160856740284655670,\ 0.52160856740284655670] \qquad \textbf{(2.1.7)}$$

Prepare intervals for proofs by interval analysis later.

```
> intvls:=[lambda[1]=RealBox(eigs[1],10^(-10)),beta=RealBox(Im
  (eigs[2]),10^(-10)),alpha=RealBox(Re(eigs[2]),10^(-10)),c=
  RealBox(0,1),s=RealBox(0,1)];
```

$$intvls := [\lambda_1 = \langle \text{RealBox: } 1.14858 \pm 1e\text{-}10\rangle,\ \beta = \langle \text{RealBox: } -0.301406 \pm 1e\text{-}10\rangle,\ \alpha = \langle \text{RealBox: } 0.425711 \pm 1e\text{-}10\rangle,\ c = \langle \text{RealBox: } 0 \pm 1\rangle,\ s = \langle \text{RealBox: } 0 \pm 1\rangle] \qquad \textbf{(2.1.8)}$$

Contraction index – first version

Symbolic basis – pure computer algebra:

```
> basis:=Matrix(3,3,(i,j)->lambda[j]^(i-1));
```

$$basis := \begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{bmatrix} \qquad \textbf{(2.2.1)}$$

Write lambda_2 = alpha + i beta

```
> basis:=subs(lambda[2]=alpha+I*beta,lambda[3]=alpha-I*beta,
  basis);
```

$$basis := \begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \alpha + I\,\beta & \alpha - I\,\beta \\ \lambda_1^2 & (\alpha + I\,\beta)^2 & (\alpha - I\,\beta)^2 \end{bmatrix} \qquad \textbf{(2.2.2)}$$

Extend the first vector so that the cone becomes positive

```
> basis:=basis.DiagonalMatrix([3/2,1,1]);
```

$$basis := \begin{bmatrix} \frac{3}{2} & 1 & 1 \\ \frac{3\lambda_1}{2} & \alpha + I\beta & \alpha - I\beta \\ \frac{3\lambda_1^2}{2} & (\alpha + I\beta)^2 & (\alpha - I\beta)^2 \end{bmatrix} \tag{2.2.3}$$

```
> incone:=basis.Vector([1,c+I*s,c-I*s]);
```

$$incone := \begin{bmatrix} \frac{3}{2} + 2c \\ \frac{3\lambda_1}{2} + (\alpha + I\beta)(c + I s) + (\alpha - I\beta)(c - I s) \\ \frac{3\lambda_1^2}{2} + (\alpha + I\beta)^2 (c + I s) + (\alpha - I\beta)^2 (c - I s) \end{bmatrix} \tag{2.2.4}$$

```
> iterate:=basis^(-1).An.incone;
```

$iterate :=$ **(2.2.5)**

$$\begin{bmatrix} -\frac{2(-5n+3)\left(\frac{3}{2}+2 \ldots\right.}{3\left(\alpha^2 - 2\alpha\lambda_1 + \beta^2 + \lambda_1^2\right)(1 \ldots} \\ -\frac{I\left(I\beta - \alpha + \lambda_1\right)(-5n+3)\left(\frac{3}{2} \ldots\right.}{2\beta\left(\alpha^2 - 2\alpha\lambda_1 + \beta^2 + \lambda_1^2\right)(16n \ldots} \\ -\frac{I\left(I\beta + \alpha - \lambda_1\right)(-5n+3)\left(\frac{3}{2} \ldots\right.}{2\beta\left(\alpha^2 - 2\alpha\lambda_1 + \beta^2 + \lambda_1^2\right)(16 \ldots} \end{bmatrix}$$

We need to prove that the first entry is positive and that it is larger than the modulus of the other two. Since the other two are conjugates, this last inequality reduces to the square of the first entry being larger than the product of the other two.

```
> pp:=normal(iterate[1]);
```

$$pp := -\frac{1}{3\left(\alpha^2 - 2\alpha\lambda_1 + \beta^2 + \lambda_1^2\right)(16n+1)}\left(64\alpha^3 c n - 192\alpha^2\beta n s\right. \tag{2.2.6}$$

$$- 192\alpha\beta^2 c n + 64\beta^3 n s + 4\alpha^3 c - 12\alpha^2\beta s - 128\alpha^2 c n - 48\alpha^2 n\lambda_1$$

$$- 12 \alpha \beta^2 c + 256 \alpha \beta n s + 96 \alpha n \lambda_1^2 + 4 \beta^3 s + 128 \beta^2 c n - 48 \beta^2 n \lambda_1 + 8 \alpha^2 c - 3 \alpha^2 \lambda_1 - 16 \alpha \beta s + 80 \alpha c n + 6 \alpha \lambda_1^2 - 8 \beta^2 c - 3 \beta^2 \lambda_1 - 80 \beta n s - 96 n \lambda_1^2 - 16 \alpha c + 16 \beta s - 20 n c + 60 n \lambda_1 + 6 \lambda_1^2 + 12 c - 15 n - 12 \lambda_1 + 9\Big)$$

```
> denom(pp);
```

$$3 \left(\alpha^2 - 2 \alpha \lambda_1 + \beta^2 + \lambda_1^2\right) (16 n + 1) \qquad \textbf{(2.2.7)}$$

```
> eval(%,intvls);
```

$$(\langle \text{RealBox: } 1.84015 \pm 2.06999\text{e-}09\rangle)\ (16 n + 1) \qquad \textbf{(2.2.8)}$$

```
> signum(%) assuming n>=0;
```

$$\langle \text{RealBox: } 1 \pm 0\rangle \qquad \textbf{(2.2.9)}$$

This shows that the denominator is positive.

```
> pp:=collect(numer(pp),[n,c,s]);
```

$$pp := \Big(\big(-64 \alpha^3 + 192 \alpha \beta^2 + 128 \alpha^2 - 128 \beta^2 - 80 \alpha + 20\big) c + \big(192 \alpha^2 \beta - 64 \beta^3 - 256 \alpha \beta + 80 \beta\big) s + 48 \alpha^2 \lambda_1 - 96 \alpha \lambda_1^2 + 48 \beta^2 \lambda_1 + 96 \lambda_1^2 - 60 \lambda_1 + 15\Big) n + \big(-4 \alpha^3 + 12 \alpha \beta^2 - 8 \alpha^2 + 8 \beta^2 + 16 \alpha - 12\big) c + \big(12 \alpha^2 \beta - 4 \beta^3 + 16 \alpha \beta - 16 \beta\big) s + 3 \alpha^2 \lambda_1 - 6 \alpha \lambda_1^2 + 3 \beta^2 \lambda_1 - 6 \lambda_1^2 + 12 \lambda_1 - 9 \qquad \textbf{(2.2.10)}$$

```
> eval(pp,intvls);
```

$$\langle \text{RealBox: } -5.56462 \pm 7.97977\rangle + (\langle \text{RealBox: } 33.8169 \pm 1.32959\text{e-}07\rangle)\ n \qquad \textbf{(2.2.11)}$$

we lower bound each coordinate

```
> lowpol(%,n);
```

$$-13.54438730058745665 + 33.81688192988078349\,3\, n \qquad \textbf{(2.2.12)}$$

since the leading coefficient is positive, the original polynomial is positive for any n larger than the largest positive root fo this new polynomial:

```
> solve(%,n);
```

$$0.4005214711596328840\,1 \qquad \textbf{(2.2.13)}$$

so it is positive for $n \geq 1$.

We now turn to the second inequality

```
> pp:=normal(evalc(iterate[1]^2-iterate[2]*iterate[3]));
```

$$pp := -\Big(36864 \alpha^8 c^2 n^2 - 147456 \alpha^7 \beta c n^2 s - 73728 \alpha^7 c^2 n^2 \lambda_1 - 65536 \alpha^6 \beta^2 c^2 n^2 + 147456 \alpha^6 \beta^2 n^2 s^2 + 221184 \alpha^6 \beta c n^2 s \lambda_1 + 36864 \alpha^6 c^2 n^2 \lambda_1^2 + 245760 \alpha^5 \beta^3 c n^2 s - 73728 \alpha^5 \beta^2 c^2 n^2 \lambda_1 - 147456 \alpha^5 \beta^2 n^2 s^2 \lambda_1 + 319488 \alpha^4 \beta^4 c^2 n^2 - 294912 \alpha^4 \beta^4 n^2 s^2 + 368640 \alpha^4 \beta^3 c n^2 s \lambda_1 + 294912 \alpha^4 \beta^2 c^2 n^2 \lambda_1^2 - 110592 \alpha^4 \beta^2 n^2 s^2 \lambda_1^2 - 73728 \alpha^4 \beta c n^2 s \lambda_1^3 - 1163264 \alpha^3 \beta^5 c n^2 s + 73728 \alpha^3 \beta^4 c^2 n^2 \lambda_1 - 294912 \alpha^3 \beta^4 n^2 s^2 \lambda_1 - 884736 \alpha^3 \beta^3 c n^2 s \lambda_1^2 - 221184 \alpha^3 \beta^2 c^2 n^2 \lambda_1^3$$

$$+ 73728\,\alpha^3\beta^2 n^2 s^2 \lambda_1^3 - 589824\,\alpha^2\beta^6 c^2 n^2 + 540672\,\alpha^2\beta^6 n^2 s^2$$
$$+ 73728\,\alpha^2\beta^5 c\,n^2 s\,\lambda_1 - 258048\,\alpha^2\beta^4 c^2 n^2 \lambda_1^2 + 516096\,\alpha^2\beta^4 n^2 s^2 \lambda_1^2$$
$$+ 442368\,\alpha^2\beta^3 c\,n^2 s\,\lambda_1^3 + 36864\,\alpha^2\beta^2 c^2 n^2 \lambda_1^4 + 36864\,\alpha^2\beta^2 n^2 s^2 \lambda_1^4$$
$$+ 540672\,\alpha\,\beta^7 c\,n^2 s + 73728\,\alpha\,\beta^6 c^2 n^2 \lambda_1 - 147456\,\alpha\,\beta^6 n^2 s^2 \lambda_1$$
$$+ 294912\,\alpha\,\beta^5 c\,n^2 s\,\lambda_1^2 + 73728\,\alpha\,\beta^4 c^2 n^2 \lambda_1^3 - 221184\,\alpha\,\beta^4 n^2 s^2 \lambda_1^3$$
$$+ 36864\,\beta^8 c^2 n^2 - 65536\,\beta^8 n^2 s^2 - 73728\,\beta^7 c\,n^2 s\,\lambda_1 + 73728\,\beta^6 c^2 n^2 \lambda_1^2$$
$$+ 36864\,\beta^6 n^2 s^2 \lambda_1^2 - 73728\,\beta^5 c\,n^2 s\,\lambda_1^3 + 36864\,\beta^4 c^2 n^2 \lambda_1^4 + 36864\,\beta^4 n^2 s^2$$
$$\lambda_1^4 + 4608\,\alpha^8 c^2 n - 18432\,\alpha^7\beta\,c\,n\,s - 147456\,\alpha^7 c^2 n^2 - 9216\,\alpha^7 c^2 n\,\lambda_1$$
$$- 8192\,\alpha^6\beta^2 c^2 n + 18432\,\alpha^6\beta^2 n\,s^2 + 589824\,\alpha^6\beta\,c\,n^2 s$$
$$+ 27648\,\alpha^6\beta\,c\,n\,s\,\lambda_1 + 294912\,\alpha^6 c^2 n^2 \lambda_1 + 4608\,\alpha^6 c^2 n\,\lambda_1^2$$
$$+ 30720\,\alpha^5\beta^3 c\,n\,s + 409600\,\alpha^5\beta^2 c^2 n^2 - 9216\,\alpha^5\beta^2 c^2 n\,\lambda_1$$
$$+ 98304\,\alpha^5\beta^2 c\,n^2 \lambda_1 - 589824\,\alpha^5\beta^2 n^2 s^2 - 18432\,\alpha^5\beta^2 n\,s^2 \lambda_1$$
$$- 1032192\,\alpha^5\beta\,c\,n^2 s\,\lambda_1 - 147456\,\alpha^5 c^2 n^2 \lambda_1^2 + 55296\,\alpha^5 c\,n^2 \lambda_1^3$$
$$+ 39936\,\alpha^4\beta^4 c^2 n - 36864\,\alpha^4\beta^4 n\,s^2 - 1310720\,\alpha^4\beta^3 c\,n^2 s$$
$$+ 46080\,\alpha^4\beta^3 c\,n\,s\,\lambda_1 - 294912\,\alpha^4\beta^3 n^2 s\,\lambda_1 - 442368\,\alpha^4\beta^2 c^2 n^2 \lambda_1$$
$$+ 36864\,\alpha^4\beta^2 c^2 n\,\lambda_1^2 - 196608\,\alpha^4\beta^2 c\,n^2 \lambda_1^2 + 884736\,\alpha^4\beta^2 n^2 s^2 \lambda_1$$
$$- 13824\,\alpha^4\beta^2 n\,s^2 \lambda_1^2 + 294912\,\alpha^4\beta\,c\,n^2 s\,\lambda_1^2 - 9216\,\alpha^4\beta\,c\,n\,s\,\lambda_1^3$$
$$- 110592\,\alpha^4\beta\,n^2 s\,\lambda_1^3 - 110592\,\alpha^4 c\,n^2 \lambda_1^4 - 145408\,\alpha^3\beta^5 c\,n\,s$$
$$- 901120\,\alpha^3\beta^4 c^2 n^2 + 9216\,\alpha^3\beta^4 c^2 n\,\lambda_1 - 196608\,\alpha^3\beta^4 c\,n^2 \lambda_1$$
$$+ 983040\,\alpha^3\beta^4 n^2 s^2 - 36864\,\alpha^3\beta^4 n\,s^2 \lambda_1 + 884736\,\alpha^3\beta^3 c\,n^2 s\,\lambda_1$$
$$- 110592\,\alpha^3\beta^3 c\,n\,s\,\lambda_1^2 + 589824\,\alpha^3\beta^3 n^2 s\,\lambda_1^2 - 27648\,\alpha^3\beta^2 c^2 n\,\lambda_1^3$$
$$+ 9216\,\alpha^3\beta^2 n\,s^2 \lambda_1^3 + 147456\,\alpha^3\beta\,c\,n^2 s\,\lambda_1^3 + 165888\,\alpha^3\beta\,n^2 s\,\lambda_1^4$$
$$+ 55296\,\alpha^3 c\,n^2 \lambda_1^5 - 73728\,\alpha^2\beta^6 c^2 n + 67584\,\alpha^2\beta^6 n\,s^2$$
$$+ 2031616\,\alpha^2\beta^5 c\,n^2 s + 9216\,\alpha^2\beta^5 c\,n\,s\,\lambda_1 - 196608\,\alpha^2\beta^5 n^2 s\,\lambda_1$$
$$+ 294912\,\alpha^2\beta^4 c^2 n^2 \lambda_1 - 32256\,\alpha^2\beta^4 c^2 n\,\lambda_1^2 + 589824\,\alpha^2\beta^4 c\,n^2 \lambda_1^2$$
$$- 294912\,\alpha^2\beta^4 n^2 s^2 \lambda_1 + 64512\,\alpha^2\beta^4 n\,s^2 \lambda_1^2 + 294912\,\alpha^2\beta^3 c\,n^2 s\,\lambda_1^2$$
$$+ 55296\,\alpha^2\beta^3 c\,n\,s\,\lambda_1^3 - 110592\,\alpha^2\beta^3 n^2 s\,\lambda_1^3 + 147456\,\alpha^2\beta^2 c^2 n^2 \lambda_1^3$$
$$+ 4608\,\alpha^2\beta^2 c^2 n\,\lambda_1^4 + 55296\,\alpha^2\beta^2 c\,n^2 \lambda_1^4 - 294912\,\alpha^2\beta^2 n^2 s^2 \lambda_1^3$$
$$+ 4608\,\alpha^2\beta^2 n\,s^2 \lambda_1^4 + 67584\,\alpha\,\beta^7 c\,n\,s + 638976\,\alpha\,\beta^6 c^2 n^2$$

$$+ 9216\,\alpha\,\beta^6\,c^2\,n\,\lambda_1 - 294912\,\alpha\,\beta^6\,c\,n^2\,\lambda_1 - 524288\,\alpha\,\beta^6\,n^2\,s^2$$
$$- 18432\,\alpha\,\beta^6\,n\,s^2\,\lambda_1 - 442368\,\alpha\,\beta^5\,c\,n^2\,s\,\lambda_1 + 36864\,\alpha\,\beta^5\,c\,n\,s\,\lambda_1^2$$
$$- 196608\,\alpha\,\beta^5\,n^2\,s\,\lambda_1^2 + 147456\,\alpha\,\beta^4\,c^2\,n^2\,\lambda_1^2 + 9216\,\alpha\,\beta^4\,c^2\,n\,\lambda_1^3$$
$$- 55296\,\alpha\,\beta^4\,c\,n^2\,\lambda_1^3 - 27648\,\alpha\,\beta^4\,n\,s^2\,\lambda_1^3 - 442368\,\alpha\,\beta^3\,c\,n^2\,s\,\lambda_1^3$$
$$- 55296\,\alpha\,\beta^3\,n^2\,s\,\lambda_1^4 + 55296\,\alpha\,\beta^2\,c\,n^2\,\lambda_1^5 - 55296\,\alpha\,\beta\,n^2\,s\,\lambda_1^6 + 4608\,\beta^8\,c^2\,n$$
$$- 8192\,\beta^8\,n\,s^2 - 262144\,\beta^7\,c\,n^2\,s - 9216\,\beta^7\,c\,n\,s\,\lambda_1 + 98304\,\beta^7\,n^2\,s\,\lambda_1$$
$$- 147456\,\beta^6\,c^2\,n^2\,\lambda_1 + 9216\,\beta^6\,c^2\,n\,\lambda_1^2 + 4608\,\beta^6\,n\,s^2\,\lambda_1^2 - 9216\,\beta^5\,c\,n\,s\,\lambda_1^3$$
$$- 147456\,\beta^4\,c^2\,n^2\,\lambda_1^3 + 4608\,\beta^4\,c^2\,n\,\lambda_1^4 - 55296\,\beta^4\,c\,n^2\,\lambda_1^4 + 4608\,\beta^4\,n\,s^2\,\lambda_1^4$$
$$- 55296\,\beta^2\,c\,n^2\,\lambda_1^6 + 144\,\alpha^8\,c^2 - 576\,\alpha^7\,\beta\,c\,s - 288\,\alpha^7\,c^2\,\lambda_1 - 256\,\alpha^6\,\beta^2\,c^2$$
$$+ 576\,\alpha^6\,\beta^2\,s^2 + 864\,\alpha^6\,\beta\,c\,s\,\lambda_1 + 239616\,\alpha^6\,c^2\,n^2 + 144\,\alpha^6\,c^2\,\lambda_1^2$$
$$+ 960\,\alpha^5\,\beta^3\,c\,s - 288\,\alpha^5\,\beta^2\,c^2\,\lambda_1 + 12288\,\alpha^5\,\beta^2\,c\,n\,\lambda_1 - 576\,\alpha^5\,\beta^2\,s^2\,\lambda_1$$
$$- 866304\,\alpha^5\,\beta\,c\,n^2\,s - 479232\,\alpha^5\,c^2\,n^2\,\lambda_1 - 110592\,\alpha^5\,c\,n^2\,\lambda_1^2 + 6912\,\alpha^5\,c\,n$$
$$\lambda_1^3 + 1248\,\alpha^4\,\beta^4\,c^2 - 1152\,\alpha^4\,\beta^4\,s^2 + 1440\,\alpha^4\,\beta^3\,c\,s\,\lambda_1 - 36864\,\alpha^4\,\beta^3\,n\,s\,\lambda_1$$
$$- 573440\,\alpha^4\,\beta^2\,c^2\,n^2 + 1152\,\alpha^4\,\beta^2\,c^2\,\lambda_1^2 - 196608\,\alpha^4\,\beta^2\,c\,n^2\,\lambda_1$$
$$- 24576\,\alpha^4\,\beta^2\,c\,n\,\lambda_1^2 + 774144\,\alpha^4\,\beta^2\,n^2\,s^2 - 36864\,\alpha^4\,\beta^2\,n^2\,\lambda_1^2 - 432\,\alpha^4\,\beta^2\,s^2$$
$$\lambda_1^2 + 1640448\,\alpha^4\,\beta\,c\,n^2\,s\,\lambda_1 - 288\,\alpha^4\,\beta\,c\,s\,\lambda_1^3 + 221184\,\alpha^4\,\beta\,n^2\,s\,\lambda_1^2$$
$$- 13824\,\alpha^4\,\beta\,n\,s\,\lambda_1^3 + 239616\,\alpha^4\,c^2\,n^2\,\lambda_1^2 + 110592\,\alpha^4\,c\,n^2\,\lambda_1^3 - 13824\,\alpha^4\,c\,n$$
$$\lambda_1^4 - 4544\,\alpha^3\,\beta^5\,c\,s + 288\,\alpha^3\,\beta^4\,c^2\,\lambda_1 - 24576\,\alpha^3\,\beta^4\,c\,n\,\lambda_1 - 1152\,\alpha^3\,\beta^4\,s^2\,\lambda_1$$
$$+ 1519616\,\alpha^3\,\beta^3\,c\,n^2\,s - 3456\,\alpha^3\,\beta^3\,c\,s\,\lambda_1^2 + 393216\,\alpha^3\,\beta^3\,n^2\,s\,\lambda_1$$
$$+ 73728\,\alpha^3\,\beta^3\,n\,s\,\lambda_1^2 + 681984\,\alpha^3\,\beta^2\,c^2\,n^2\,\lambda_1 - 864\,\alpha^3\,\beta^2\,c^2\,\lambda_1^3$$
$$+ 589824\,\alpha^3\,\beta^2\,c\,n^2\,\lambda_1^2 - 1456128\,\alpha^3\,\beta^2\,n^2\,s^2\,\lambda_1 + 147456\,\alpha^3\,\beta^2\,n^2\,\lambda_1^3$$
$$+ 288\,\alpha^3\,\beta^2\,s^2\,\lambda_1^3 - 681984\,\alpha^3\,\beta\,c\,n^2\,s\,\lambda_1^2 - 110592\,\alpha^3\,\beta\,n^2\,s\,\lambda_1^3$$
$$+ 20736\,\alpha^3\,\beta\,n\,s\,\lambda_1^4 + 110592\,\alpha^3\,c\,n^2\,\lambda_1^4 + 6912\,\alpha^3\,c\,n\,\lambda_1^5 - 2304\,\alpha^2\,\beta^6\,c^2$$
$$+ 2112\,\alpha^2\,\beta^6\,s^2 + 288\,\alpha^2\,\beta^5\,c\,s\,\lambda_1 - 24576\,\alpha^2\,\beta^5\,n\,s\,\lambda_1 + 776192\,\alpha^2\,\beta^4\,c^2\,n^2$$
$$- 1008\,\alpha^2\,\beta^4\,c^2\,\lambda_1^2 + 73728\,\alpha^2\,\beta^4\,c\,n\,\lambda_1^2 - 765952\,\alpha^2\,\beta^4\,n^2\,s^2$$
$$- 73728\,\alpha^2\,\beta^4\,n^2\,\lambda_1^2 + 2016\,\alpha^2\,\beta^4\,s^2\,\lambda_1^2 - 1363968\,\alpha^2\,\beta^3\,c\,n^2\,s\,\lambda_1$$
$$+ 1728\,\alpha^2\,\beta^3\,c\,s\,\lambda_1^3 - 1155072\,\alpha^2\,\beta^3\,n^2\,s\,\lambda_1^2 - 13824\,\alpha^2\,\beta^3\,n\,s\,\lambda_1^3$$
$$- 202752\,\alpha^2\,\beta^2\,c^2\,n^2\,\lambda_1^2 + 144\,\alpha^2\,\beta^2\,c^2\,\lambda_1^4 - 110592\,\alpha^2\,\beta^2\,c\,n^2\,\lambda_1^3$$
$$+ 6912\,\alpha^2\,\beta^2\,c\,n\,\lambda_1^4 + 589824\,\alpha^2\,\beta^2\,n^2\,s^2\,\lambda_1^2 - 147456\,\alpha^2\,\beta^2\,n^2\,\lambda_1^4$$

$$+ 144 \alpha^2 \beta^2 s^2 \lambda_1^4 - 92160 \alpha^2 \beta c n^2 s \lambda_1^3 - 442368 \alpha^2 \beta n^2 s \lambda_1^4$$
$$- 110592 \alpha^2 c n^2 \lambda_1^5 + 20736 \alpha^2 n^2 \lambda_1^6 + 2112 \alpha \beta^7 c s + 288 \alpha \beta^6 c^2 \lambda_1$$
$$- 36864 \alpha \beta^6 c n \lambda_1 - 576 \alpha \beta^6 s^2 \lambda_1 - 1021952 \alpha \beta^5 c n^2 s + 1152 \alpha \beta^5 c s \lambda_1^2$$
$$+ 393216 \alpha \beta^5 n^2 s \lambda_1 - 24576 \alpha \beta^5 n s \lambda_1^2 - 387072 \alpha \beta^4 c^2 n^2 \lambda_1$$
$$+ 288 \alpha \beta^4 c^2 \lambda_1^3 - 872448 \alpha \beta^4 c n^2 \lambda_1^2 - 6912 \alpha \beta^4 c n \lambda_1^3$$
$$+ 92160 \alpha \beta^4 n^2 s^2 \lambda_1 + 147456 \alpha \beta^4 n^2 \lambda_1^3 - 864 \alpha \beta^4 s^2 \lambda_1^3$$
$$+ 497664 \alpha \beta^3 c n^2 s \lambda_1^2 + 331776 \alpha \beta^3 n^2 s \lambda_1^3 - 6912 \alpha \beta^3 n s \lambda_1^4$$
$$- 92160 \alpha \beta^2 c^2 n^2 \lambda_1^3 - 331776 \alpha \beta^2 c n^2 \lambda_1^4 + 6912 \alpha \beta^2 c n \lambda_1^5$$
$$+ 92160 \alpha \beta^2 n^2 s^2 \lambda_1^3 + 331776 \alpha \beta n^2 s \lambda_1^5 - 6912 \alpha \beta n s \lambda_1^6 - 41472 \alpha n^2 \lambda_1^7$$
$$+ 144 \beta^8 c^2 - 256 \beta^8 s^2 - 288 \beta^7 c s \lambda_1 + 12288 \beta^7 n s \lambda_1 - 114688 \beta^6 c^2 n^2$$
$$+ 288 \beta^6 c^2 \lambda_1^2 + 196608 \beta^6 c n^2 \lambda_1 + 163840 \beta^6 n^2 s^2 - 36864 \beta^6 n^2 \lambda_1^2$$
$$+ 144 \beta^6 s^2 \lambda_1^2 + 92160 \beta^5 c n^2 s \lambda_1 - 288 \beta^5 c s \lambda_1^3 + 196608 \beta^5 n^2 s \lambda_1^2$$
$$+ 147456 \beta^4 c^2 n^2 \lambda_1^2 + 144 \beta^4 c^2 \lambda_1^4 + 221184 \beta^4 c n^2 \lambda_1^3 - 6912 \beta^4 c n \lambda_1^4$$
$$+ 144 \beta^4 s^2 \lambda_1^4 + 92160 \beta^3 c n^2 s \lambda_1^3 + 221184 \beta^2 c n^2 \lambda_1^5 - 6912 \beta^2 c n \lambda_1^6$$
$$+ 20736 \beta^2 n^2 \lambda_1^6 + 20736 n^2 \lambda_1^8 + 576 \alpha^7 c^2 - 2304 \alpha^6 \beta c s - 31104 \alpha^6 c^2 n$$
$$- 1152 \alpha^6 c^2 \lambda_1 - 1600 \alpha^5 \beta^2 c^2 + 384 \alpha^5 \beta^2 c \lambda_1 + 2304 \alpha^5 \beta^2 s^2$$
$$+ 111744 \alpha^5 \beta c n s + 4032 \alpha^5 \beta c s \lambda_1 - 207360 \alpha^5 c^2 n^2 + 62208 \alpha^5 c^2 n \lambda_1$$
$$+ 576 \alpha^5 c^2 \lambda_1^2 + 69120 \alpha^5 c n^2 \lambda_1 + 216 \alpha^5 c \lambda_1^3 + 5120 \alpha^4 \beta^3 c s$$
$$- 1152 \alpha^4 \beta^3 s \lambda_1 + 73728 \alpha^4 \beta^2 c^2 n + 1728 \alpha^4 \beta^2 c^2 \lambda_1 - 768 \alpha^4 \beta^2 c \lambda_1^2$$
$$- 99072 \alpha^4 \beta^2 n s^2 - 4608 \alpha^4 \beta^2 n \lambda_1^2 - 3456 \alpha^4 \beta^2 s^2 \lambda_1 + 599040 \alpha^4 \beta c n^2 s$$
$$- 210816 \alpha^4 \beta c n s \lambda_1 - 1152 \alpha^4 \beta c s \lambda_1^2 - 138240 \alpha^4 \beta n^2 s \lambda_1 - 432 \alpha^4 \beta s$$
$$\lambda_1^3 + 414720 \alpha^4 c^2 n^2 \lambda_1 - 31104 \alpha^4 c^2 n \lambda_1^2 + 82944 \alpha^4 c n^2 \lambda_1^2 - 432 \alpha^4 c \lambda_1^4$$
$$+ 3520 \alpha^3 \beta^4 c^2 - 768 \alpha^3 \beta^4 c \lambda_1 - 3840 \alpha^3 \beta^4 s^2 - 195840 \alpha^3 \beta^3 c n s$$
$$- 3456 \alpha^3 \beta^3 c s \lambda_1 + 2304 \alpha^3 \beta^3 s \lambda_1^2 + 368640 \alpha^3 \beta^2 c^2 n^2$$
$$- 86400 \alpha^3 \beta^2 c^2 n \lambda_1 - 368640 \alpha^3 \beta^2 n^2 s^2 + 185472 \alpha^3 \beta^2 n s^2 \lambda_1$$
$$+ 18432 \alpha^3 \beta^2 n \lambda_1^3 - 1175040 \alpha^3 \beta c n^2 s \lambda_1 + 86400 \alpha^3 \beta c n s \lambda_1^2$$
$$- 576 \alpha^3 \beta c s \lambda_1^3 - 235008 \alpha^3 \beta n^2 s \lambda_1^2 + 648 \alpha^3 \beta s \lambda_1^4 - 207360 \alpha^3 c^2 n^2 \lambda_1^2$$
$$- 304128 \alpha^3 c n^2 \lambda_1^3 + 216 \alpha^3 c \lambda_1^5 - 7936 \alpha^2 \beta^5 c s - 768 \alpha^2 \beta^5 s \lambda_1$$
$$- 102016 \alpha^2 \beta^4 c^2 n - 1152 \alpha^2 \beta^4 c^2 \lambda_1 + 2304 \alpha^2 \beta^4 c \lambda_1^2 + 99584 \alpha^2 \beta^4 n s^2$$

$$- 9216 \alpha^2 \beta^4 n \lambda_1^2 + 1152 \alpha^2 \beta^4 s^2 \lambda_1 - 691200 \alpha^2 \beta^3 c n^2 s$$
$$+ 172800 \alpha^2 \beta^3 c n s \lambda_1 - 1152 \alpha^2 \beta^3 c s \lambda_1^2 + 107520 \alpha^2 \beta^3 n^2 s \lambda_1$$
$$- 432 \alpha^2 \beta^3 s \lambda_1^3 - 391680 \alpha^2 \beta^2 c^2 n^2 \lambda_1 + 24192 \alpha^2 \beta^2 c^2 n \lambda_1^2 - 576 \alpha^2 \beta^2 c^2$$
$$\lambda_1^3 - 569856 \alpha^2 \beta^2 c n^2 \lambda_1^2 + 216 \alpha^2 \beta^2 c \lambda_1^4 + 737280 \alpha^2 \beta^2 n^2 s^2 \lambda_1$$
$$- 147456 \alpha^2 \beta^2 n^2 \lambda_1^3 - 73728 \alpha^2 \beta^2 n s^2 \lambda_1^2 - 18432 \alpha^2 \beta^2 n \lambda_1^4$$
$$+ 1152 \alpha^2 \beta^2 s^2 \lambda_1^3 + 552960 \alpha^2 \beta c n^2 s \lambda_1^2 + 12672 \alpha^2 \beta c n s \lambda_1^3$$
$$+ 815616 \alpha^2 \beta n^2 s \lambda_1^3 + 82944 \alpha^2 c n^2 \lambda_1^4 - 82944 \alpha^2 n^2 \lambda_1^5 + 2592 \alpha^2 n \lambda_1^6$$
$$- 2496 \alpha \beta^6 c^2 - 1152 \alpha \beta^6 c \lambda_1 + 2048 \alpha \beta^6 s^2 + 134784 \alpha \beta^5 c n s$$
$$+ 1728 \alpha \beta^5 c s \lambda_1 - 768 \alpha \beta^5 s \lambda_1^2 - 243200 \alpha \beta^4 c^2 n^2 + 49536 \alpha \beta^4 c^2 n \lambda_1$$
$$- 576 \alpha \beta^4 c^2 \lambda_1^2 + 422400 \alpha \beta^4 c n^2 \lambda_1 - 216 \alpha \beta^4 c \lambda_1^3 + 286720 \alpha \beta^4 n^2 s^2$$
$$- 12672 \alpha \beta^4 n s^2 \lambda_1 + 18432 \alpha \beta^4 n \lambda_1^3 + 391680 \alpha \beta^3 c n^2 s \lambda_1$$
$$- 61056 \alpha \beta^3 c n s \lambda_1^2 + 1728 \alpha \beta^3 c s \lambda_1^3 + 520704 \alpha \beta^3 n^2 s \lambda_1^2 - 216 \alpha \beta^3 s$$
$$\lambda_1^4 + 161280 \alpha \beta^2 c^2 n^2 \lambda_1^2 + 12672 \alpha \beta^2 c^2 n \lambda_1^3 + 580608 \alpha \beta^2 c n^2 \lambda_1^3$$
$$+ 216 \alpha \beta^2 c \lambda_1^5 - 368640 \alpha \beta^2 n^2 s^2 \lambda_1^2 + 294912 \alpha \beta^2 n^2 \lambda_1^4$$
$$- 12672 \alpha \beta^2 n s^2 \lambda_1^3 + 23040 \alpha \beta c n^2 s \lambda_1^3 - 373248 \alpha \beta n^2 s \lambda_1^4 - 216 \alpha \beta s$$
$$\lambda_1^6 + 69120 \alpha c n^2 \lambda_1^5 + 165888 \alpha n^2 \lambda_1^6 - 5184 \alpha n \lambda_1^7 + 1024 \beta^7 c s$$
$$+ 384 \beta^7 s \lambda_1 + 14336 \beta^6 c^2 n + 576 \beta^6 c^2 \lambda_1 - 22528 \beta^6 n s^2 - 4608 \beta^6 n \lambda_1^2$$
$$+ 184320 \beta^5 c n^2 s - 12672 \beta^5 c n s \lambda_1 - 245760 \beta^5 n^2 s \lambda_1 + 23040 \beta^4 c^2 n^2 \lambda_1$$
$$- 18432 \beta^4 c^2 n \lambda_1^2 + 576 \beta^4 c^2 \lambda_1^3 + 102912 \beta^4 c n^2 \lambda_1^2 - 216 \beta^4 c \lambda_1^4$$
$$- 147456 \beta^4 n^2 \lambda_1^3 - 184320 \beta^3 c n^2 s \lambda_1^2 - 12672 \beta^3 c n s \lambda_1^3 - 69120 \beta^3 n^2 s$$
$$\lambda_1^3 + 23040 \beta^2 c^2 n^2 \lambda_1^3 - 290304 \beta^2 c n^2 \lambda_1^4 - 216 \beta^2 c \lambda_1^6 - 82944 \beta^2 n^2 \lambda_1^5$$
$$+ 2592 \beta^2 n \lambda_1^6 - 69120 \beta n^2 s \lambda_1^5 - 82944 n^2 \lambda_1^7 + 2592 n \lambda_1^8 - 576 \alpha^6 c^2$$
$$+ 1152 \alpha^5 \beta c s + 60768 \alpha^5 c^2 n + 1152 \alpha^5 c^2 \lambda_1 - 17280 \alpha^5 c n^2$$
$$- 9504 \alpha^5 c n \lambda_1 + 432 \alpha^5 c \lambda_1^2 + 448 \alpha^4 \beta^2 c^2 + 768 \alpha^4 \beta^2 c \lambda_1 - 144 \alpha^4 \beta^2 \lambda_1^2$$
$$- 169920 \alpha^4 \beta c n s - 1152 \alpha^4 \beta c s \lambda_1 + 34560 \alpha^4 \beta n^2 s + 19008 \alpha^4 \beta n s \lambda_1$$
$$- 864 \alpha^4 \beta s \lambda_1^2 + 103680 \alpha^4 c^2 n^2 - 121536 \alpha^4 c^2 n \lambda_1 - 576 \alpha^4 c^2 \lambda_1^2$$
$$- 103680 \alpha^4 c n^2 \lambda_1 - 8640 \alpha^4 c n \lambda_1^2 - 432 \alpha^4 c \lambda_1^3 - 1792 \alpha^3 \beta^3 c s$$
$$- 1536 \alpha^3 \beta^3 s \lambda_1 - 108032 \alpha^3 \beta^2 c^2 n + 1152 \alpha^3 \beta^2 c^2 \lambda_1 + 30720 \alpha^3 \beta^2 c n^2$$
$$- 2304 \alpha^3 \beta^2 c \lambda_1^2 + 96768 \alpha^3 \beta^2 n s^2 - 1152 \alpha^3 \beta^2 s^2 \lambda_1 + 576 \alpha^3 \beta^2 \lambda_1^3$$

$$- 207360 \alpha^{3} \beta c n^{2} s + 327456 \alpha^{3} \beta c n s \lambda_{1} - 1152 \alpha^{3} \beta c s \lambda_{1}^{2}$$

$$+ 224640 \alpha^{3} \beta n^{2} s \lambda_{1} + 26784 \alpha^{3} \beta n s \lambda_{1}^{2} + 432 \alpha^{3} \beta s \lambda_{1}^{3}$$

$$- 207360 \alpha^{3} c^{2} n^{2} \lambda_{1} + 60768 \alpha^{3} c^{2} n \lambda_{1}^{2} + 120960 \alpha^{3} c n^{2} \lambda_{1}^{2} + 36288 \alpha^{3} c n \lambda_{1}^{3}$$

$$- 432 \alpha^{3} c \lambda_{1}^{4} - 3520 \alpha^{2} \beta^{4} c^{2} + 2048 \alpha^{2} \beta^{4} s^{2} - 288 \alpha^{2} \beta^{4} \lambda_{1}^{2}$$

$$+ 202560 \alpha^{2} \beta^{3} c n s - 2304 \alpha^{2} \beta^{3} c s \lambda_{1} - 57600 \alpha^{2} \beta^{3} n^{2} s$$

$$- 14784 \alpha^{2} \beta^{3} n s \lambda_{1} + 4512 \alpha^{2} \beta^{3} s \lambda_{1}^{2} - 126720 \alpha^{2} \beta^{2} c^{2} n^{2}$$

$$+ 109152 \alpha^{2} \beta^{2} c^{2} n \lambda_{1} - 2304 \alpha^{2} \beta^{2} c^{2} \lambda_{1}^{2} + 197760 \alpha^{2} \beta^{2} c n^{2} \lambda_{1}$$

$$+ 73440 \alpha^{2} \beta^{2} c n \lambda_{1}^{2} + 432 \alpha^{2} \beta^{2} c \lambda_{1}^{3} + 57600 \alpha^{2} \beta^{2} n^{2} s^{2} + 92160 \alpha^{2} \beta^{2} n^{2} \lambda_{1}^{2}$$

$$- 193536 \alpha^{2} \beta^{2} n s^{2} \lambda_{1} + 2304 \alpha^{2} \beta^{2} s^{2} \lambda_{1}^{2} - 576 \alpha^{2} \beta^{2} \lambda_{1}^{4}$$

$$+ 414720 \alpha^{2} \beta c n^{2} s \lambda_{1} - 145152 \alpha^{2} \beta c n s \lambda_{1}^{2} + 1152 \alpha^{2} \beta c s \lambda_{1}^{3}$$

$$- 414720 \alpha^{2} \beta n^{2} s \lambda_{1}^{2} - 101088 \alpha^{2} \beta n s \lambda_{1}^{3} + 1728 \alpha^{2} \beta s \lambda_{1}^{4}$$

$$+ 103680 \alpha^{2} c^{2} n^{2} \lambda_{1}^{2} + 120960 \alpha^{2} c n^{2} \lambda_{1}^{3} - 8640 \alpha^{2} c n \lambda_{1}^{4} + 432 \alpha^{2} c \lambda_{1}^{5}$$

$$+ 134784 \alpha^{2} n^{2} \lambda_{1}^{4} + 81 \alpha^{2} \lambda_{1}^{6} + 5248 \alpha \beta^{5} c s - 1536 \alpha \beta^{5} s \lambda_{1}$$

$$+ 91296 \alpha \beta^{4} c^{2} n - 74880 \alpha \beta^{4} c n^{2} - 58080 \alpha \beta^{4} c n \lambda_{1} + 3408 \alpha \beta^{4} c \lambda_{1}^{2}$$

$$- 75264 \alpha \beta^{4} n s^{2} - 1152 \alpha \beta^{4} s^{2} \lambda_{1} + 576 \alpha \beta^{4} \lambda_{1}^{3} + 161280 \alpha \beta^{3} c n^{2} s$$

$$- 109152 \alpha \beta^{3} c n s \lambda_{1} + 3456 \alpha \beta^{3} c s \lambda_{1}^{2} - 197760 \alpha \beta^{3} n^{2} s \lambda_{1}$$

$$- 67296 \alpha \beta^{3} n s \lambda_{1}^{2} - 1296 \alpha \beta^{3} s \lambda_{1}^{3} + 92160 \alpha \beta^{2} c^{2} n^{2} \lambda_{1} - 36000 \alpha \beta^{2} c^{2} n \lambda_{1}^{2}$$

$$+ 1152 \alpha \beta^{2} c^{2} \lambda_{1}^{3} - 124800 \alpha \beta^{2} c n^{2} \lambda_{1}^{2} - 74304 \alpha \beta^{2} c n \lambda_{1}^{3} + 1296 \alpha \beta^{2} c \lambda_{1}^{4}$$

$$- 115200 \alpha \beta^{2} n^{2} s^{2} \lambda_{1} - 184320 \alpha \beta^{2} n^{2} \lambda_{1}^{3} + 96768 \alpha \beta^{2} n s^{2} \lambda_{1}^{2}$$

$$- 1152 \alpha \beta^{2} s^{2} \lambda_{1}^{3} - 207360 \alpha \beta c n^{2} s \lambda_{1}^{2} - 12384 \alpha \beta c n s \lambda_{1}^{3}$$

$$+ 17280 \alpha \beta n^{2} s \lambda_{1}^{3} + 45792 \alpha \beta n s \lambda_{1}^{4} - 1296 \alpha \beta s \lambda_{1}^{5} - 103680 \alpha c n^{2} \lambda_{1}^{4}$$

$$- 9504 \alpha c n \lambda_{1}^{5} - 269568 \alpha n^{2} \lambda_{1}^{5} - 162 \alpha \lambda_{1}^{7} - 448 \beta^{6} c^{2} - 768 \beta^{6} c \lambda_{1}$$

$$- 2048 \beta^{6} s^{2} - 144 \beta^{6} \lambda_{1}^{2} - 59648 \beta^{5} c n s - 1152 \beta^{5} c s \lambda_{1} + 30720 \beta^{5} n^{2} s$$

$$+ 33792 \beta^{5} n s \lambda_{1} - 768 \beta^{5} s \lambda_{1}^{2} + 35840 \beta^{4} c^{2} n^{2} - 12384 \beta^{4} c^{2} n \lambda_{1}$$

$$+ 576 \beta^{4} c^{2} \lambda_{1}^{2} - 120960 \beta^{4} c n^{2} \lambda_{1} - 12000 \beta^{4} c n \lambda_{1}^{2} - 864 \beta^{4} c \lambda_{1}^{3}$$

$$- 44800 \beta^{4} n^{2} s^{2} + 92160 \beta^{4} n^{2} \lambda_{1}^{2} + 48384 \beta^{3} c n s \lambda_{1}^{2} - 1152 \beta^{3} c s \lambda_{1}^{3}$$

$$- 107520 \beta^{3} n^{2} s \lambda_{1}^{2} + 9504 \beta^{3} n s \lambda_{1}^{3} - 46080 \beta^{2} c^{2} n^{2} \lambda_{1}^{2} - 12384 \beta^{2} c^{2} n \lambda_{1}^{3}$$

$$+ 138240 \beta^{2} c n^{2} \lambda_{1}^{3} + 37152 \beta^{2} c n \lambda_{1}^{4} - 864 \beta^{2} c \lambda_{1}^{5} + 57600 \beta^{2} n^{2} s^{2} \lambda_{1}^{2}$$

$$- 12672 \beta^{2} n^{2} \lambda_{1}^{4} + 81 \beta^{2} \lambda_{1}^{6} + 138240 \beta n^{2} s \lambda_{1}^{4} + 9504 \beta n s \lambda_{1}^{5} - 17280 c n^{2}$$

$\lambda_1^5 + 134784\, n^2 \lambda_1^6 + 81\, \lambda_1^8 - 1440\, \alpha^5 c^2 + 9288\, \alpha^5 c\, n - 864\, \alpha^5 c\, \lambda_1$
$+ 5184\, \alpha^4 \beta\, c\, s - 18576\, \alpha^4 \beta\, n\, s + 1728\, \alpha^4 \beta\, s\, \lambda_1 - 53568\, \alpha^4 c^2 n$
$+ 2880\, \alpha^4 c^2 \lambda_1 + 34560\, \alpha^4 c\, n^2 + 17712\, \alpha^4 c\, n\, \lambda_1 + 2592\, \alpha^4 c\, \lambda_1^2$
$+ 2560\, \alpha^3 \beta^2 c^2 - 16512\, \alpha^3 \beta^2 c\, n - 4608\, \alpha^3 \beta^2 s^2 + 107136\, \alpha^3 \beta\, c\, n\, s$
$- 11232\, \alpha^3 \beta\, c\, s\, \lambda_1 - 69120\, \alpha^3 \beta\, n^2 s - 44712\, \alpha^3 \beta\, n\, s\, \lambda_1 - 4320\, \alpha^3 \beta\, s\, \lambda_1^2$
$- 28800\, \alpha^3 c^2 n^2 + 107136\, \alpha^3 c^2 n\, \lambda_1 - 1440\, \alpha^3 c^2 \lambda_1^2 + 17280\, \alpha^3 c\, n^2 \lambda_1$
$- 27000\, \alpha^3 c\, n\, \lambda_1^2 - 3456\, \alpha^3 c\, \lambda_1^3 - 4800\, \alpha^2 \beta^3 c\, s + 30960\, \alpha^2 \beta^3 n\, s$
$- 1344\, \alpha^2 \beta^3 s\, \lambda_1 + 72192\, \alpha^2 \beta^2 c^2 n - 3744\, \alpha^2 \beta^2 c^2 \lambda_1 - 61440\, \alpha^2 \beta^2 c\, n^2$
$- 38712\, \alpha^2 \beta^2 c\, n\, \lambda_1 + 672\, \alpha^2 \beta^2 c\, \lambda_1^2 - 23040\, \alpha^2 \beta^2 n^2 \lambda_1 - 23040\, \alpha^2 \beta^2 n\, s^2$
$- 12672\, \alpha^2 \beta^2 n\, \lambda_1^2 + 9216\, \alpha^2 \beta^2 s^2 \lambda_1 + 576\, \alpha^2 \beta^2 \lambda_1^3 + 28800\, \alpha^2 \beta\, c\, n^2 s$
$- 214272\, \alpha^2 \beta\, c\, n\, s\, \lambda_1 + 6912\, \alpha^2 \beta\, c\, s\, \lambda_1^2 + 51840\, \alpha^2 \beta\, n^2 s\, \lambda_1$
$+ 108864\, \alpha^2 \beta\, n\, s\, \lambda_1^2 + 4320\, \alpha^2 \beta\, s\, \lambda_1^3 + 57600\, \alpha^2 c^2 n^2 \lambda_1 - 53568\, \alpha^2 c^2 n\, \lambda_1^2$
$- 103680\, \alpha^2 c\, n^2 \lambda_1^2 - 27000\, \alpha^2 c\, n\, \lambda_1^3 + 2592\, \alpha^2 c\, \lambda_1^4 - 116640\, \alpha^2 n^2 \lambda_1^3$
$- 17496\, \alpha^2 n\, \lambda_1^4 + 324\, \alpha^2 \lambda_1^5 + 1952\, \alpha\, \beta^4 c^2 + 40248\, \alpha\, \beta^4 c\, n$
$- 5280\, \alpha\, \beta^4 c\, \lambda_1 + 3584\, \alpha\, \beta^4 s^2 - 83328\, \alpha\, \beta^3 c\, n\, s + 3744\, \alpha\, \beta^3 c\, s\, \lambda_1$
$+ 53760\, \alpha\, \beta^3 n^2 s + 47160\, \alpha\, \beta^3 n\, s\, \lambda_1 - 864\, \alpha\, \beta^3 s\, \lambda_1^2 + 22400\, \alpha\, \beta^2 c^2 n^2$
$- 61056\, \alpha\, \beta^2 c^2 n\, \lambda_1 + 3168\, \alpha\, \beta^2 c^2 \lambda_1^2 + 1920\, \alpha\, \beta^2 c\, n^2 \lambda_1 + 20616\, \alpha\, \beta^2 c\, n$
$\lambda_1^2 + 46080\, \alpha\, \beta^2 n^2 \lambda_1^2 + 46080\, \alpha\, \beta^2 n\, s^2 \lambda_1 + 25344\, \alpha\, \beta^2 n\, \lambda_1^3 - 4608\, \alpha\, \beta^2 s^2$
$\lambda_1^2 - 1152\, \alpha\, \beta^2 \lambda_1^4 - 57600\, \alpha\, \beta\, c\, n^2 s\, \lambda_1 + 107136\, \alpha\, \beta\, c\, n\, s\, \lambda_1^2 - 864\, \alpha\, \beta\, c\, s$
$\lambda_1^3 + 103680\, \alpha\, \beta\, n^2 s\, \lambda_1^2 - 9288\, \alpha\, \beta\, n\, s\, \lambda_1^3 - 2592\, \alpha\, \beta\, s\, \lambda_1^4 - 28800\, \alpha\, c^2 n^2 \lambda_1^2$
$+ 17280\, \alpha\, c\, n^2 \lambda_1^3 + 17712\, \alpha\, c\, n\, \lambda_1^4 - 864\, \alpha\, c\, \lambda_1^5 + 233280\, \alpha\, n^2 \lambda_1^4$
$+ 34992\, \alpha\, n\, \lambda_1^5 - 648\, \alpha\, \lambda_1^6 + 256\, \beta^5 c\, s - 16512\, \beta^5 n\, s + 3072\, \beta^5 s\, \lambda_1$
$- 23744\, \beta^4 c^2 n - 864\, \beta^4 c^2 \lambda_1 + 26880\, \beta^4 c\, n^2 + 35448\, \beta^4 c\, n\, \lambda_1 + 1536\, \beta^4 c$
$\lambda_1^2 - 23040\, \beta^4 n^2 \lambda_1 + 17920\, \beta^4 n\, s^2 - 12672\, \beta^4 n\, \lambda_1^2 + 576\, \beta^4 \lambda_1^3$
$- 22400\, \beta^3 c\, n^2 s - 2304\, \beta^3 c\, s\, \lambda_1^2 + 67200\, \beta^3 n^2 s\, \lambda_1 + 28224\, \beta^3 n\, s\, \lambda_1^2$
$+ 864\, \beta^3 s\, \lambda_1^3 + 30528\, \beta^2 c^2 n\, \lambda_1^2 - 864\, \beta^2 c^2 \lambda_1^3 - 61440\, \beta^2 c\, n^2 \lambda_1^2$
$- 36288\, \beta^2 c\, n\, \lambda_1^3 + 67680\, \beta^2 n^2 \lambda_1^3 - 23040\, \beta^2 n\, s^2 \lambda_1^2 + 936\, \beta^2 n\, \lambda_1^4 + 324\, \beta^2$
$\lambda_1^5 + 28800\, \beta\, c\, n^2 s\, \lambda_1^2 - 86400\, \beta\, n^2 s\, \lambda_1^3 - 36288\, \beta\, n\, s\, \lambda_1^4 + 864\, \beta\, s\, \lambda_1^5$
$+ 34560\, c\, n^2 \lambda_1^4 + 9288\, c\, n\, \lambda_1^5 - 116640\, n^2 \lambda_1^5 - 17496\, n\, \lambda_1^6 + 324\, \lambda_1^7$

$$+ 648 \alpha^5 c - 1296 \alpha^4 \beta s + 4032 \alpha^4 c^2 - 22896 \alpha^4 c n - 3024 \alpha^4 c \lambda_1$$
$$- 1152 \alpha^3 \beta^2 c - 8064 \alpha^3 \beta c s + 45792 \alpha^3 \beta n s + 5400 \alpha^3 \beta s \lambda_1$$
$$+ 23040 \alpha^3 c^2 n - 8064 \alpha^3 c^2 \lambda_1 - 21600 \alpha^3 c n^2 + 11232 \alpha^3 c n \lambda_1$$
$$+ 2376 \alpha^3 c \lambda_1^2 + 2160 \alpha^2 \beta^3 s - 4864 \alpha^2 \beta^2 c^2 + 40704 \alpha^2 \beta^2 c n$$
$$+ 4872 \alpha^2 \beta^2 c \lambda_1 + 12384 \alpha^2 \beta^2 n \lambda_1 + 2304 \alpha^2 \beta^2 s^2 - 1152 \alpha^2 \beta^2 \lambda_1^2$$
$$- 23040 \alpha^2 \beta c n s + 16128 \alpha^2 \beta c s \lambda_1 + 21600 \alpha^2 \beta n^2 s - 57024 \alpha^2 \beta n s \lambda_1$$
$$- 5184 \alpha^2 \beta s \lambda_1^2 + 3600 \alpha^2 c^2 n^2 - 46080 \alpha^2 c^2 n \lambda_1 + 4032 \alpha^2 c^2 \lambda_1^2$$
$$+ 21600 \alpha^2 c n^2 \lambda_1 + 23328 \alpha^2 c n \lambda_1^2 + 2376 \alpha^2 c \lambda_1^3 + 58320 \alpha^2 n^2 \lambda_1^2$$
$$+ 34182 \alpha^2 n \lambda_1^3 - 324 \alpha^2 \lambda_1^4 + 2808 \alpha \beta^4 c + 6272 \alpha \beta^3 c s - 35616 \alpha \beta^3 n s$$
$$- 3336 \alpha \beta^3 s \lambda_1 - 17920 \alpha \beta^2 c^2 n + 3456 \alpha \beta^2 c^2 \lambda_1 + 16800 \alpha \beta^2 c n^2$$
$$- 18912 \alpha \beta^2 c n \lambda_1 - 3768 \alpha \beta^2 c \lambda_1^2 - 24768 \alpha \beta^2 n \lambda_1^2 - 4608 \alpha \beta^2 s^2 \lambda_1$$
$$+ 2304 \alpha \beta^2 \lambda_1^3 + 46080 \alpha \beta c n s \lambda_1 - 8064 \alpha \beta c s \lambda_1^2 - 43200 \alpha \beta n^2 s \lambda_1$$
$$- 23328 \alpha \beta n s \lambda_1^2 - 648 \alpha \beta s \lambda_1^3 - 7200 \alpha c^2 n^2 \lambda_1 + 23040 \alpha c^2 n \lambda_1^2$$
$$+ 21600 \alpha c n^2 \lambda_1^2 + 11232 \alpha c n \lambda_1^3 - 3024 \alpha c \lambda_1^4 - 116640 \alpha n^2 \lambda_1^3$$
$$- 68364 \alpha n \lambda_1^4 + 648 \alpha \lambda_1^5 - 1152 \beta^5 s + 1344 \beta^4 c^2 - 17808 \beta^4 c n$$
$$- 840 \beta^4 c \lambda_1 + 12384 \beta^4 n \lambda_1 - 1792 \beta^4 s^2 - 1152 \beta^4 \lambda_1^2 + 17920 \beta^3 c n s$$
$$- 16800 \beta^3 n^2 s - 26880 \beta^3 n s \lambda_1 - 1344 \beta^3 s \lambda_1^2 - 2800 \beta^2 c^2 n^2$$
$$- 1728 \beta^2 c^2 \lambda_1^2 + 16800 \beta^2 c n^2 \lambda_1 + 40704 \beta^2 c n \lambda_1^2 + 1728 \beta^2 c \lambda_1^3$$
$$- 45360 \beta^2 n^2 \lambda_1^2 - 14202 \beta^2 n \lambda_1^3 + 2304 \beta^2 s^2 \lambda_1^2 - 900 \beta^2 \lambda_1^4$$
$$- 23040 \beta c n s \lambda_1^2 + 21600 \beta n^2 s \lambda_1^2 + 34560 \beta n s \lambda_1^3 + 1728 \beta s \lambda_1^4$$
$$+ 3600 c^2 n^2 \lambda_1^2 - 21600 c n^2 \lambda_1^3 - 22896 c n \lambda_1^4 + 648 c \lambda_1^5 + 58320 n^2 \lambda_1^4$$
$$+ 34182 n \lambda_1^5 - 324 \lambda_1^6 + 1296 \alpha^4 c - 2592 \alpha^3 \beta s - 3456 \alpha^3 c^2$$
$$+ 17280 \alpha^3 c n + 864 \alpha^3 c \lambda_1 - 2304 \alpha^2 \beta^2 c + 864 \alpha^2 \beta^2 \lambda_1 + 3456 \alpha^2 \beta c s$$
$$- 17280 \alpha^2 \beta n s + 1728 \alpha^2 \beta s \lambda_1 - 4320 \alpha^2 c^2 n + 6912 \alpha^2 c^2 \lambda_1$$
$$+ 5400 \alpha^2 c n^2 - 17280 \alpha^2 c n \lambda_1 - 4320 \alpha^2 c \lambda_1^2 - 16200 \alpha^2 n^2 \lambda_1$$
$$- 30132 \alpha^2 n \lambda_1^2 - 810 \alpha^2 \lambda_1^3 + 2016 \alpha \beta^3 s + 2688 \alpha \beta^2 c^2 - 13440 \alpha \beta^2 c n$$
$$- 96 \alpha \beta^2 c \lambda_1 - 1728 \alpha \beta^2 \lambda_1^2 - 6912 \alpha \beta c s \lambda_1 + 34560 \alpha \beta n s \lambda_1$$
$$+ 4320 \alpha \beta s \lambda_1^2 + 8640 \alpha c^2 n \lambda_1 - 3456 \alpha c^2 \lambda_1^2 - 10800 \alpha c n^2 \lambda_1$$
$$- 17280 \alpha c n \lambda_1^2 + 864 \alpha c \lambda_1^3 + 32400 \alpha n^2 \lambda_1^2 + 60264 \alpha n \lambda_1^3 + 1620 \alpha \lambda_1^4$$

$$+ 1008\,\beta^4 c + 864\,\beta^4 \lambda_1 - 2688\,\beta^3 c\,s + 13440\,\beta^3 n\,s + 2688\,\beta^3 s\,\lambda_1$$
$$+ 3360\,\beta^2 c^2 n - 4200\,\beta^2 c\,n^2 - 13440\,\beta^2 c\,n\,\lambda_1 - 2304\,\beta^2 c\,\lambda_1^2$$
$$+ 12600\,\beta^2 n^2 \lambda_1 + 23436\,\beta^2 n\,\lambda_1^2 + 1494\,\beta^2 \lambda_1^3 + 3456\,\beta\,c\,s\,\lambda_1^2 - 17280\,\beta\,n\,s$$
$$\lambda_1^2 - 3456\,\beta\,s\,\lambda_1^3 - 4320\,c^2 n\,\lambda_1^2 + 5400\,c\,n^2 \lambda_1^2 + 17280\,c\,n\,\lambda_1^3 + 1296\,c\,\lambda_1^4$$
$$- 16200\,n^2 \lambda_1^3 - 30132\,n\,\lambda_1^4 - 810\,\lambda_1^5 - 2592\,\alpha^3 c + 2592\,\alpha^2 \beta\,s + 1296\,\alpha^2 c^2$$
$$- 6480\,\alpha^2 c\,n + 2592\,\alpha^2 c\,\lambda_1 + 2025\,\alpha^2 n^2 + 12960\,\alpha^2 n\,\lambda_1 + 2268\,\alpha^2 \lambda_1^2$$
$$+ 2016\,\alpha\,\beta^2 c - 5184\,\alpha\,\beta\,s\,\lambda_1 - 2592\,\alpha\,c^2 \lambda_1 + 12960\,\alpha\,c\,n\,\lambda_1 + 2592\,\alpha\,c\,\lambda_1^2$$
$$- 4050\,\alpha\,n^2 \lambda_1 - 25920\,\alpha\,n\,\lambda_1^2 - 4536\,\alpha\,\lambda_1^3 - 2016\,\beta^3 s - 1008\,\beta^2 c^2$$
$$+ 5040\,\beta^2 c\,n + 2016\,\beta^2 c\,\lambda_1 - 1575\,\beta^2 n^2 - 10080\,\beta^2 n\,\lambda_1 - 1764\,\beta^2 \lambda_1^2$$
$$+ 2592\,\beta\,s\,\lambda_1^2 + 1296\,c^2 \lambda_1^2 - 6480\,c\,n\,\lambda_1^2 - 2592\,c\,\lambda_1^3 + 2025\,n^2 \lambda_1^2$$
$$+ 12960\,n\,\lambda_1^3 + 2268\,\lambda_1^4 + 1944\,\alpha^2 c - 2430\,\alpha^2 n - 1944\,\alpha^2 \lambda_1 - 3888\,\alpha\,c\,\lambda_1$$
$$+ 4860\,\alpha\,n\,\lambda_1 + 3888\,\alpha\,\lambda_1^2 - 1512\,\beta^2 c + 1890\,\beta^2 n + 1512\,\beta^2 \lambda_1 + 1944\,c\,\lambda_1^2$$
$$- 2430\,n\,\lambda_1^2 - 1944\,\lambda_1^3 + 729\,\alpha^2 - 1458\,\alpha\,\lambda_1 - 567\,\beta^2 + 729\,\lambda_1^2\Big) \Big/$$
$$\left(144\,\beta^2 (16\,n + 1)^2 \left(\alpha^2 - 2\,\alpha\,\lambda_1 + \beta^2 + \lambda_1^2\right)^2\right)$$

This can be simplified by exploiting the fact that c^2+s^2=1. The actual computation uses Gröbner bases, hidden from the user.

```
> pp:=simplify(pp,{c^2+s^2=1});
```

$$pp := \Bigg(-36864\,c^2 \left(n + \frac{1}{16}\right)^2 \alpha^8 + 73728\,c \left(n + \frac{1}{16}\right) \Bigg(\left(2\,n\,s + \frac{1}{8}\,s\right)\beta \tag{2.2.15}$$
$$+ c\left(\left(n + \frac{1}{16}\right)\lambda_1 + 2\,n - \frac{1}{8}\right)\Bigg)\alpha^7 + \Bigg(212992 \left(n + \frac{1}{16}\right)^2 \Bigg(c^2$$
$$- \frac{9}{13}\Bigg)\beta^2 - 221184\,c \left(n + \frac{1}{16}\right)\left(\left(n + \frac{1}{16}\right)\lambda_1 + \frac{8\,n}{3} - \frac{1}{6}\right) s\,\beta$$
$$- 36864 \left(\left(n + \frac{1}{16}\right)^2 \lambda_1^2 + \left(8\,n^2 - \frac{1}{32}\right)\lambda_1 + \frac{13\,n^2}{2} - \frac{27\,n}{32} - \frac{1}{64}\right) c^2\Bigg)$$
$$\alpha^6 + \Bigg(-245760\,c \left(n + \frac{1}{16}\right)^2 s\,\beta^3 - 73728 \Bigg(\left(n + \frac{1}{16}\right)\Bigg(c^2 + \frac{4}{3}\,c$$
$$- 2\Bigg)\lambda_1 + \frac{122\left(n - \frac{1}{16}\right)\left(c^2 - \frac{36}{61}\right)}{9}\Bigg)\left(n + \frac{1}{16}\right)\beta^2 + 1032192\,c\,\Bigg(\Bigg($$

$$-\frac{1}{256}+n^{2}\Big)\lambda_{1}+\frac{47\,n^{2}}{56}-\frac{97\,n}{896}-\frac{1}{896}\Big)\,s\,\beta+147456\,c\Big($$

$$-\frac{3\left(n+\frac{1}{16}\right)^{2}\lambda_{1}^{3}}{8}+\left(n-\frac{1}{16}\right)\left(n+\frac{1}{16}\right)\left(c+\frac{3}{4}\right)\lambda_{1}^{2}+\Big(\Big(\frac{13\,c}{4}$$

$$-\frac{15}{32}\Big)n^{2}+\left(-\frac{27\,c}{64}+\frac{33}{512}\right)n-\frac{c}{128}+\frac{3}{512}\Big)\lambda_{1}+\Big(\frac{45\,c}{32}$$

$$+\frac{15}{128}\Big)n^{2}+\left(-\frac{211\,c}{512}-\frac{129}{2048}\right)n+\frac{5\,c}{512}-\frac{9}{2048}\Big)\Big)\alpha^{5}+\Big($$

$$-614400\left(n+\frac{1}{16}\right)^{2}\left(c^{2}-\frac{12}{25}\right)\beta^{4}-368640\left(n+\frac{1}{16}\right)\Big(\Big(c-\frac{4}{5}\Big)\Big(n$$

$$+\frac{1}{16}\Big)\lambda_{1}-\frac{32\,c\left(n-\frac{1}{16}\right)}{9}\Big)\,s\,\beta^{3}+\Big(-405504\Big(c^{2}-\frac{16}{33}\,c$$

$$-\frac{4}{11}\Big)\left(n+\frac{1}{16}\right)^{2}\lambda_{1}^{2}+1327104\left(c^{2}+\frac{4}{27}\,c-\frac{2}{3}\right)\left(n-\frac{1}{16}\right)\Big(n$$

$$+\frac{1}{16}\Big)\lambda_{1}+\left(1347584\,c^{2}-774144\right)n^{2}+\left(-172800\,c^{2}+99072\right)n$$

$$-448\,c^{2}\Big)\beta^{2}+73728\Big(\left(n+\frac{1}{16}\right)^{2}\left(c+\frac{3}{2}\right)\lambda_{1}^{3}+\Big(\left(-4\,c-3\right)n^{2}+\frac{c}{64}$$

$$+\frac{3}{256}\Big)\lambda_{1}^{2}+\Big(\Big(-\frac{89\,c}{4}+\frac{15}{8}\Big)n^{2}+\left(\frac{183\,c}{64}-\frac{33}{128}\right)n+\frac{c}{64}$$

$$-\frac{3}{128}\Big)\lambda_{1}+\left(-\frac{65\,c}{8}-\frac{15}{32}\right)n^{2}+\left(\frac{295\,c}{128}+\frac{129}{512}\right)n-\frac{9\,c}{128}+\frac{9}{512}\Big)$$

$$s\,\beta-239616\,c\Big(-\frac{6\left(n+\frac{1}{16}\right)^{2}\lambda_{1}^{4}}{13}+\left(\frac{6\,n^{2}}{13}-\frac{3}{1664}\right)\lambda_{1}^{3}+\Big(\Big(c$$

$$+\frac{9}{26}\Big)n^{2}+\left(-\frac{27\,c}{208}-\frac{15}{416}\right)n-\frac{c}{416}+\frac{9}{832}\Big)\lambda_{1}^{2}+\Big(\Big(\frac{45\,c}{26}$$

$$-\frac{45}{104}\Big)n^{2}+\left(-\frac{211\,c}{416}+\frac{123}{1664}\right)n+\frac{5\,c}{416}-\frac{21}{1664}\Big)\lambda_{1}+\Big(\frac{45\,c}{104}$$

$$+ \frac{15}{104}\Big) n^2 + \Big(-\frac{93\,c}{416} - \frac{159}{1664}\Big) n + \frac{7\,c}{416} + \frac{9}{1664}\Big)\Big) \alpha^4$$

$$+ \Bigg(1163264\,c \Big(n + \frac{1}{16}\Big)^2 s\,\beta^5 - 368640 \Bigg(\Big(c + \frac{2}{3}\Big) \Big(n + \frac{1}{16}\Big) \Big(c$$

$$- \frac{6}{5}\Big) \lambda_1 - \frac{46 \left(c^2 - \frac{12}{23}\right) \left(n - \frac{1}{16}\right)}{9}\Bigg) \Big(n + \frac{1}{16}\Big) \beta^4 + 884736 \Bigg(\Big(c$$

$$- \frac{2}{3}\Big) \Big(n + \frac{1}{16}\Big)^2 \lambda_1^2 + \Big(\Big(-c - \frac{4}{9}\Big) n^2 + \frac{c}{256} + \frac{1}{576}\Big) \lambda_1$$

$$- \frac{371\,c \left(n^2 - \frac{765}{5936}\,n - \frac{1}{848}\right)}{216}\Bigg) s\,\beta^3 + \Bigg(294912 \Big(c^2 - \frac{3}{4}\Big) \Big(n + \frac{1}{16}\Big)^2$$

$$\lambda_1^3 + (-589824\,n^2\,c + 2304\,c)\,\lambda_1^2 + ((-2138112\,c^2 + 1456128)\,n^2$$

$$+ (271872\,c^2 - 185472)\,n - 2304\,c^2 + 1152)\,\lambda_1 + (-737280\,c^2 - 30720\,c$$

$$+ 368640)\,n^2 + (204800\,c^2 + 16512\,c - 96768)\,n - 7168\,c^2 + 1152\,c$$

$$+ 4608\Bigg) \beta^2 - 147456 \Bigg(\frac{9 \left(n + \frac{1}{16}\right)^2 \lambda_1^4}{8} + \Big(n - \frac{1}{16}\Big) \Big(n + \frac{1}{16}\Big) \Big(c$$

$$- \frac{3}{4}\Big) \lambda_1^3 + \Big(\Big(-\frac{37\,c}{8} - \frac{51}{32}\Big) n^2 + \Big(\frac{75\,c}{128} + \frac{93}{512}\Big) n - \frac{c}{128} - \frac{15}{512}\Big)$$

$$\lambda_1^2 + \Big(\Big(-\frac{255\,c}{32} + \frac{195}{128}\Big) n^2 + \Big(-\frac{621}{2048} + \frac{1137\,c}{512}\Big) n - \frac{39\,c}{512}$$

$$+ \frac{75}{2048}\Big) \lambda_1 + \Big(-\frac{45\,c}{32} - \frac{15}{32}\Big) n^2 + \Big(\frac{93\,c}{128} + \frac{159}{512}\Big) n - \frac{9}{512} - \frac{7\,c}{128}\Big)$$

$$s\,\beta + 207360\,c \Bigg(-\frac{4 \left(n + \frac{1}{16}\right)^2 \lambda_1^5}{15} + \Big(\frac{1}{480} - \frac{8\,n^2}{15}\Big) \lambda_1^4 + \Big(\frac{1}{60}$$

$$+ \frac{22}{15}\,n^2 - \frac{7}{40}\,n\Big) \lambda_1^3 + \Big(\Big(c - \frac{7}{12}\Big) n^2 + \Big(-\frac{211\,c}{720} + \frac{25}{192}\Big) n + \frac{c}{144}$$

$$-\frac{11}{960}\Big)\lambda_1^2+\Big(\Big(c-\frac{1}{12}\Big)n^2+\Big(-\frac{31\,c}{60}-\frac{13}{240}\Big)n+\frac{7\,c}{180}-\frac{1}{240}\Big)\lambda_1$$

$$+\frac{5\left(n-\frac{1}{5}\right)\left(n-\frac{3}{5}\right)\left(c+\frac{3}{4}\right)}{36}\Big)\Big)\alpha^3+\Big(1130496\left(c^2-\frac{11}{23}\right)\Big(n$$

$$+\frac{1}{16}\Big)^2\beta^6-73728\Big(\Big(n+\frac{1}{16}\Big)\Big(c-\frac{8}{3}\Big)\lambda_1+\frac{248\,c\left(n-\frac{1}{16}\right)}{9}\Big)\Big(n$$

$$+\frac{1}{16}\Big)s\,\beta^5+\Big(774144\Big(n+\frac{1}{16}\Big)^2\Big(c^2-\frac{16}{21}c-\frac{4}{7}\Big)\lambda_1^2+((\,$$

$$-589824\,c^2+294912)\,n^2+2304\,c^2-1152)\,\lambda_1+(-1542144\,c^2$$

$$+765952)\,n^2+(201600\,c^2-99584)\,n+5568\,c^2-2048\Big)\beta^4-442368\Big(\Big(c$$

$$-\frac{1}{4}\Big)\Big(n+\frac{1}{16}\Big)^2\lambda_1^3+\frac{2\left(n-\frac{1}{16}\right)\left(n+\frac{1}{16}\right)\left(c-\frac{47}{12}\right)\lambda_1^2}{3}+\Big(\Big(\,$$

$$-\frac{37\,c}{12}+\frac{35}{144}\Big)n^2+\Big(\frac{25\,c}{64}-\frac{77}{2304}\Big)n-\frac{c}{192}-\frac{7}{2304}\Big)\lambda_1+\Big(\,$$

$$-\frac{25\,c}{16}-\frac{25}{192}\Big)n^2+\Big(\frac{1055\,c}{2304}+\frac{215}{3072}\Big)n-\frac{25\,c}{2304}+\frac{5}{1024}\Big)s\,\beta^3+\Big(\,$$

$$-55296\,(c-2)\Big(n+\frac{1}{16}\Big)^2\lambda_1^4-442368\Big(c^2-\frac{1}{4}c-1\Big)\Big(n-\frac{1}{16}\Big)\Big(n$$

$$+\frac{1}{16}\Big)\lambda_1^3+((792576\,c^2+569856\,c-681984)\,n^2+(-97920\,c^2$$

$$-73440\,c+86400)\,n+4608\,c^2-672\,c-1152)\,\lambda_1^2+((1128960\,c^2$$

$$-197760\,c-714240)\,n^2+(-302688\,c^2+38712\,c+181152)\,n+12960\,c^2$$

$$-4872\,c-10080)\,\lambda_1+184320\Big(\Big(c-\frac{5}{12}\Big)n^2+\Big(-\frac{31\,c}{60}+\frac{1}{6}\Big)n$$

$$+ \frac{7c}{180} - \frac{1}{60}\Big)\Big(c + \frac{3}{4}\Big)\Big)\beta^2 + 92160\Bigg(\Big(-\frac{3}{160} + \frac{24n^2}{5}\Big)\lambda_1^4 + \Big(\Big(c$$

$$- \frac{177}{20}\Big)n^2 + \Big(-\frac{11c}{80} + \frac{351}{320}\Big)n - \frac{c}{80} - \frac{3}{64}\Big)\lambda_1^3 - 6\Big(n - \frac{1}{5}\Big)\Big(n$$

$$- \frac{1}{16}\Big)\Big(c - \frac{3}{4}\Big)\lambda_1^2 + \Big(\Big(-\frac{9c}{2} - \frac{9}{16}\Big)n^2 + \Big(\frac{99}{160} + \frac{93c}{40}\Big)n - \frac{7c}{40}$$

$$- \frac{3}{160}\Big)\lambda_1 - \frac{5\left(n - \frac{1}{5}\right)\left(n - \frac{3}{5}\right)\left(c + \frac{3}{4}\right)}{16}\Bigg)s\,\beta - 20736\Big(n$$

$$+ \frac{1}{16}\Big)^2\lambda_1^6 + \big((110592c + 82944)n^2 - 432c - 324\big)\lambda_1^5 + \big((-82944c$$

$$- 134784)n^2 + (8640c + 17496)n - 2592c + 324\big)\lambda_1^4 + \big((-120960c$$

$$+ 116640)n^2 + (27000c - 34182)n - 2376c + 810\big)\lambda_1^3 + \big((-103680c^2$$

$$+ 103680c - 58320)n^2 + (53568c^2 - 23328c + 30132)n - 4032c^2$$

$$+ 4320c - 2268\big)\lambda_1^2 - 57600\Big(n - \frac{1}{5}\Big)\Big(n - \frac{3}{5}\Big)\Big(c - \frac{3}{8}\Big)\Big(c + \frac{3}{4}\Big)\lambda_1$$

$$- 3600\Big(n - \frac{3}{5}\Big)^2\Big(c + \frac{3}{4}\Big)^2\Big)\alpha^2 + \Bigg(-540672c\Big(n + \frac{1}{16}\Big)^2 s\,\beta^7$$

$$- 221184\Big(n + \frac{1}{16}\Big)\Bigg(\Big(c^2 - \frac{4}{3}c - \frac{2}{3}\Big)\Big(n + \frac{1}{16}\Big)\lambda_1$$

$$+ \frac{142\left(n - \frac{1}{16}\right)\left(c^2 - \frac{32}{71}\right)}{27}\Bigg)\beta^6 - 294912\Bigg(\Big(c - \frac{2}{3}\Big)\Big(n + \frac{1}{16}\Big)^2\lambda_1^2$$

$$+ \Big(\Big(-\frac{3c}{2} + \frac{4}{3}\Big)n^2 + \frac{3c}{512} - \frac{1}{192}\Big)\lambda_1$$

$$- \frac{499c\left(n^2 - \frac{1053}{7984}n - \frac{41}{7984}\right)}{144}\Bigg)s\,\beta^5 + \Big(-294912\Big(n + \frac{1}{16}\Big)^2\Big(c^2$$

$$- \frac{3}{16} c - \frac{1}{4} \Big) \lambda_1^3 + \big(\big(-147456 c^2 + 872448 c \big) n^2 + 576 c^2 - 3408 c \big) \lambda_1^2$$

$$+ \big(\big(479232 c^2 - 422400 c - 92160 \big) n^2 + \big(-62208 c^2 + 58080 c$$

$$+ 12672 \big) n - 1152 c^2 + 5280 c + 1152 \big) \lambda_1 + \big(529920 c^2 + 74880 c$$

$$- 286720 \big) n^2 + \big(-166560 c^2 - 40248 c + 75264 \big) n + 1632 c^2 - 2808 c$$

$$- 3584 \big) \beta^4 + 442368 \left(\frac{\left(n + \frac{1}{16} \right)^2 \lambda_1^4}{8} + \left(n - \frac{1}{16} \right) \left(n + \frac{1}{16} \right) \Big(c \right.$$

$$- \frac{3}{4} \Big) \lambda_1^3 + \left(\left(- \frac{9 c}{8} - \frac{113}{96} \right) n^2 + \left(\frac{53 c}{384} + \frac{701}{4608} \right) n - \frac{c}{128} + \frac{1}{512} \right)$$

$$\lambda_1^2 + \left(\left(- \frac{85 c}{96} + \frac{515}{1152} \right) n^2 + \left(- \frac{655}{6144} + \frac{379 c}{1536} \right) n - \frac{13 c}{1536} \right.$$

$$+ \frac{139}{18432} \Big) \lambda_1 + \left(- \frac{35 c}{96} - \frac{35}{288} \right) n^2 + \left(\frac{217 c}{1152} + \frac{371}{4608} \right) n - \frac{7}{1536}$$

$$- \frac{49 c}{3456} \Big) s \beta^3 + \left(-55296 c \left(n + \frac{1}{16} \right)^2 \lambda_1^5 + \big(\big(331776 c - 294912 \big) n^2 \right.$$

$$- 1296 c + 1152 \big) \lambda_1^4 + \big(\big(184320 c^2 - 580608 c + 92160 \big) n^2 + \big(-25344 c^2$$

$$+ 74304 c - 12672 \big) n - 2304 c^2 - 1152 \big) \lambda_1^3 + \big(\big(-529920 c^2 + 124800 c$$

$$+ 322560 \big) n^2 + \big(132768 c^2 - 20616 c - 72000 \big) n - 7776 c^2 + 3768 c$$

$$+ 6336 \big) \lambda_1^2 - 207360 \left(c + \frac{3}{4} \right) \left(\left(c - \frac{20}{27} \right) n^2 + \left(- \frac{31 c}{60} + \frac{8}{27} \right) n \right.$$

$$+ \frac{7 c}{180} - \frac{4}{135} \Big) \lambda_1 - 22400 c \left(n - \frac{1}{5} \right) \left(n - \frac{3}{5} \right) \left(c + \frac{3}{4} \right) \Big) \beta^2$$

$$- 23040 \left(- \frac{12 \left(n + \frac{1}{16}\right)^2 \lambda_1^5}{5} + \left(- \frac{9}{160} + \frac{72 n^2}{5} \right) \lambda_1^4 + \left(- \frac{9}{80} \right.\right.$$

$$\left. - \frac{81}{5} n^2 + \frac{159}{80} n \right) \lambda_1^3 + \left(n - \frac{3}{5}\right) \left(n + \frac{1}{16}\right) \left(c + \frac{3}{4}\right) \lambda_1^2 + \left(\left(-9 c \right.\right.$$

$$\left.\left. + \frac{9}{2} \right) n^2 + \left(- \frac{81}{80} + \frac{93 c}{20} \right) n - \frac{7 c}{20} + \frac{3}{16} \right) \lambda_1$$

$$\left. - \frac{5 \left(n - \frac{1}{5}\right) \left(n - \frac{3}{5}\right) \left(c + \frac{3}{4}\right)}{2} \right) \lambda_1 s \beta + 28800 \left(\left(- \frac{12 n}{5} - \frac{3}{20} \right) \right.$$

$$\lambda_1^3 + \left(\frac{24 n}{5} - \frac{3}{10} \right) \lambda_1^2 + \left(-3 n + \frac{3}{5} \right) \lambda_1 + \left(n - \frac{3}{5}\right) \left(c + \frac{3}{4}\right) \Bigg) \lambda_1 \Bigg(\Bigg($$

$$- \frac{3 n}{5} - \frac{3}{80} \Bigg) \lambda_1^3 + \left(\frac{6 n}{5} - \frac{3}{40} \right) \lambda_1^2 + \left(n - \frac{1}{5}\right) \left(c - \frac{3}{4}\right) \lambda_1$$

$$\left. + \frac{\left(n - \frac{3}{5}\right) \left(c + \frac{3}{4}\right)}{4} \right) \Bigg) \alpha - 102400 \left(c + \frac{4}{5}\right) \left(c - \frac{4}{5}\right) \Bigg(n$$

$$+ \frac{1}{16} \Bigg)^2 \beta^8 + 73728 \left(n + \frac{1}{16}\right) s \Bigg(\left(n + \frac{1}{16}\right) \left(c - \frac{4}{3}\right) \lambda_1$$

$$+ \frac{32 c \left(n - \frac{1}{16}\right)}{9} \Bigg) \beta^7 + \Bigg(-36864 c^2 \left(n + \frac{1}{16}\right)^2 \lambda_1^2 + ((147456 c^2$$

$$- 196608 c) n^2 - 576 c^2 + 768 c) \lambda_1 + (278528 c^2 - 163840) n^2 + ($$

$$-36864 c^2 + 22528) n - 1600 c^2 + 2048) \beta^6 + 73728 \Bigg(c \left(n + \frac{1}{16}\right)^2 \lambda_1^3 + \Bigg($$

$$- \frac{8 n^2}{3} + \frac{1}{96} \Bigg) \lambda_1^2 - \frac{5 \left(n - \frac{1}{5}\right) \left(n + \frac{1}{16}\right) \left(c - \frac{8}{3}\right) \lambda_1}{4} + \Bigg(- \frac{5 c}{2}$$

$$- \frac{5}{12} \Bigg) n^2 + \left(\frac{233 c}{288} + \frac{43}{192} \right) n - \frac{c}{288} + \frac{1}{64} \Bigg) s \beta^5 + \Bigg(55296 \Bigg(c$$

$$- \frac{2}{3} \Bigg) \left(n + \frac{1}{16}\right)^2 \lambda_1^4 + 147456 \left(n - \frac{1}{16}\right) \left(n + \frac{1}{16}\right) \left(c^2 - \frac{3}{2} c + 1\right) \lambda_1^3$$

$$
\begin{aligned}
&+ \left(\left(-147456\, c^2 - 102912\, c - 92160 \right) n^2 + \left(18432\, c^2 + 12000\, c \right.\right. \\
&\left.+ 12672 \right) n - 576\, c^2 - 1536\, c + 1152 \Big) \lambda_1^2 + \left(\left(-23040\, c^2 + 120960\, c \right.\right. \\
&\left.+ 23040 \right) n^2 + \left(12384\, c^2 - 35448\, c - 12384 \right) n + 864\, c^2 + 840\, c - 864 \Big) \lambda_1 \\
&+ \left(-80640\, c^2 - 26880\, c + 44800 \right) n^2 + \left(41664\, c^2 + 17808\, c - 17920 \right) n \\
&- 3136\, c^2 - 1008\, c + 1792 \Big) \beta^4 - 92160 \left(n - \frac{1}{5} \right) \left(\left(n + \frac{1}{16} \right) \left(c - \frac{3}{4} \right) \right. \\
&\lambda_1^3 - 2 \left(c + \frac{7}{12} \right) \left(n - \frac{1}{16} \right) \lambda_1^2 + \left(\frac{35\, n}{48} - \frac{7}{48} \right) \lambda_1 \\
&\left. - \frac{35 \left(n - \frac{3}{5} \right) \left(c + \frac{3}{4} \right)}{144} \right) s\, \beta^3 + \left(55296 \left(c - \frac{3}{8} \right) \left(n + \frac{1}{16} \right)^2 \lambda_1^6 + \left(\left(\right.\right.\right. \\
&-221184\, c + 82944 \right) n^2 + 864\, c - 324 \Big) \lambda_1^5 + \Big(900 + \left(290304\, c + 12672 \right) n^2 \\
&+ \left(-37152\, c - 936 \right) n \Big) \lambda_1^4 + \left(\left(-23040\, c^2 - 138240\, c - 67680 \right) n^2 \right. \\
&+ \left(12384\, c^2 + 36288\, c + 14202 \right) n + 864\, c^2 - 1728\, c - 1494 \Big) \lambda_1^3 \\
&+ 103680 \left(c + \frac{3}{4} \right) \left(\left(c - \frac{17}{108} \right) n^2 + \left(-\frac{31\, c}{60} - \frac{11}{2160} \right) n + \frac{7\, c}{180} \right. \\
&\left. - \frac{1}{144} \right) \lambda_1^2 - 16800 \left(n - \frac{1}{5} \right) \left(n - \frac{3}{5} \right) \left(c + \frac{3}{4} \right) \lambda_1 + 2800 \left(n \right. \\
&\left. - \frac{3}{5} \right)^2 \left(c + \frac{3}{4} \right)^2 \Big) \beta^2 - 28800 \left(n - \frac{1}{5} \right) \left(\left(\left(-\frac{12\, n}{5} - \frac{3}{20} \right) \lambda_1^3 \right.\right. \\
&\left.\left. + \left(\frac{24\, n}{5} - \frac{3}{10} \right) \lambda_1^2 + \left(-3\, n + \frac{3}{5} \right) \lambda_1 + \left(n - \frac{3}{5} \right) \left(c + \frac{3}{4} \right) \right) \lambda_1^2\, s\, \beta \\
&- 3600 \left(\left(\left(-\frac{12\, n}{5} - \frac{3}{20} \right) \lambda_1^3 + \left(\frac{24\, n}{5} - \frac{3}{10} \right) \lambda_1^2 + \left(-3\, n + \frac{3}{5} \right) \lambda_1 \right.\right. \\
&\left.\left. + \left(n - \frac{3}{5} \right) \left(c + \frac{3}{4} \right) \right)^2 \lambda_1^2 \right) \Big/ \left(36864\, \beta^2 \left(n + \frac{1}{16} \right)^2 \left(\alpha^2 - 2\, \alpha\, \lambda_1 \right.\right. \\
&\left.\left. + \beta^2 + \lambda_1^2 \right)^2 \right)
\end{aligned}
$$

```
> denom(pp);
```

$$144\,\beta^2\,(16\,n+1)^2\left(\alpha^2-2\,\alpha\,\lambda_1+\beta^2+\lambda_1^2\right)^2 \tag{2.2.16}$$

>0.

```
> pp:=numer(pp):
> collect(pp,[n,c,s]);
```

$$\begin{aligned}
&-729\,\lambda_1^2-3888\,\alpha\,\lambda_1^2+1944\,\alpha^2\,\lambda_1+567\,\beta^2-729\,\alpha^2+\big(2430\,\lambda_1^2+25920\,\alpha\,\lambda_1^2\\
&\quad-12960\,\alpha^2\,\lambda_1-1890\,\beta^2+2430\,\alpha^2+\big(-4608\,\alpha^8+9216\,\alpha^7\,\lambda_1\\
&\quad+26624\,\alpha^6\,\beta^2-4608\,\alpha^6\,\lambda_1^2-9216\,\alpha^5\,\beta^2\,\lambda_1-76800\,\alpha^4\,\beta^4-50688\,\alpha^4\,\beta^2\,\lambda_1^2\\
&\quad-46080\,\alpha^3\,\beta^4\,\lambda_1+36864\,\alpha^3\,\beta^2\,\lambda_1^3+141312\,\alpha^2\,\beta^6+96768\,\alpha^2\,\beta^4\,\lambda_1^2\\
&\quad-27648\,\alpha\,\beta^6\,\lambda_1-36864\,\alpha\,\beta^4\,\lambda_1^3-12800\,\beta^8-4608\,\beta^6\,\lambda_1^2+31104\,\alpha^6\\
&\quad-62208\,\alpha^5\,\lambda_1-172800\,\alpha^4\,\beta^2+31104\,\alpha^4\,\lambda_1^2+271872\,\alpha^3\,\beta^2\,\lambda_1\\
&\quad+201600\,\alpha^2\,\beta^4-97920\,\alpha^2\,\beta^2\,\lambda_1^2-62208\,\alpha\,\beta^4\,\lambda_1-25344\,\alpha\,\beta^2\,\lambda_1^3\\
&\quad-36864\,\beta^6+18432\,\beta^4\,\lambda_1^2-60768\,\alpha^5+121536\,\alpha^4\,\lambda_1+204800\,\alpha^3\,\beta^2\\
&\quad-60768\,\alpha^3\,\lambda_1^2-302688\,\alpha^2\,\beta^2\,\lambda_1-166560\,\alpha\,\beta^4+132768\,\alpha\,\beta^2\,\lambda_1^2\\
&\quad+12384\,\beta^4\,\lambda_1+12384\,\beta^2\,\lambda_1^3+53568\,\alpha^4-107136\,\alpha^3\,\lambda_1-95232\,\alpha^2\,\beta^2\\
&\quad+53568\,\alpha^2\,\lambda_1^2+107136\,\alpha\,\beta^2\,\lambda_1+41664\,\beta^4-53568\,\beta^2\,\lambda_1^2-23040\,\alpha^3\\
&\quad+46080\,\alpha^2\,\lambda_1+17920\,\alpha\,\beta^2-23040\,\alpha\,\lambda_1^2+4320\,\alpha^2-8640\,\alpha\,\lambda_1-3360\,\beta^2\\
&\quad+4320\,\lambda_1^2\big)\,c^2+\big(6480\,\lambda_1^2+17280\,\alpha\,\lambda_1^2+17280\,\alpha^2\,\lambda_1-5040\,\beta^2+6480\,\alpha^2\\
&\quad-17280\,\alpha^3+\big(18432\,\alpha^7\,\beta-27648\,\alpha^6\,\beta\,\lambda_1-30720\,\alpha^5\,\beta^3-46080\,\alpha^4\,\beta^3\,\lambda_1\\
&\quad+9216\,\alpha^4\,\beta\,\lambda_1^3+145408\,\alpha^3\,\beta^5+110592\,\alpha^3\,\beta^3\,\lambda_1^2-9216\,\alpha^2\,\beta^5\,\lambda_1\\
&\quad-55296\,\alpha^2\,\beta^3\,\lambda_1^3-67584\,\alpha\,\beta^7-36864\,\alpha\,\beta^5\,\lambda_1^2+9216\,\beta^7\,\lambda_1+9216\,\beta^5\,\lambda_1^3\\
&\quad-111744\,\alpha^5\,\beta+210816\,\alpha^4\,\beta\,\lambda_1+195840\,\alpha^3\,\beta^3-86400\,\alpha^3\,\beta\,\lambda_1^2\\
&\quad-172800\,\alpha^2\,\beta^3\,\lambda_1-12672\,\alpha^2\,\beta\,\lambda_1^3-134784\,\alpha\,\beta^5+61056\,\alpha\,\beta^3\,\lambda_1^2\\
&\quad+12672\,\beta^5\,\lambda_1+12672\,\beta^3\,\lambda_1^3+169920\,\alpha^4\,\beta-327456\,\alpha^3\,\beta\,\lambda_1\\
&\quad-202560\,\alpha^2\,\beta^3+145152\,\alpha^2\,\beta\,\lambda_1^2+109152\,\alpha\,\beta^3\,\lambda_1+12384\,\alpha\,\beta\,\lambda_1^3\\
&\quad+59648\,\beta^5-48384\,\beta^3\,\lambda_1^2-107136\,\alpha^3\,\beta+214272\,\alpha^2\,\beta\,\lambda_1+83328\,\alpha\,\beta^3\\
&\quad-107136\,\alpha\,\beta\,\lambda_1^2+23040\,\alpha^2\,\beta-46080\,\alpha\,\beta\,\lambda_1-17920\,\beta^3+23040\,\beta\,\lambda_1^2\big)\,s\\
&\quad-6912\,\alpha^5\,\lambda_1^3+13824\,\alpha^4\,\lambda_1^4-6912\,\alpha^3\,\lambda_1^5+6912\,\beta^4\,\lambda_1^4+6912\,\beta^2\,\lambda_1^6\\
&\quad+9504\,\alpha^5\,\lambda_1+8640\,\alpha^4\,\lambda_1^2-36288\,\alpha^3\,\lambda_1^3+8640\,\alpha^2\,\lambda_1^4+9504\,\alpha\,\lambda_1^5\\
&\quad-17712\,\alpha^4\,\lambda_1+16512\,\alpha^3\,\beta^2+27000\,\alpha^3\,\lambda_1^2+27000\,\alpha^2\,\lambda_1^3-40248\,\alpha\,\beta^4
\end{aligned} \tag{2.2.17}$$

$$- 17712\,\alpha\,\lambda_1^4 - 11232\,\alpha^3\,\lambda_1 - 40704\,\alpha^2\,\beta^2 - 23328\,\alpha^2\,\lambda_1^2 - 11232\,\alpha\,\lambda_1^3$$
$$+ 13440\,\alpha\,\beta^2 - 20616\,\alpha\,\beta^2\,\lambda_1^2 - 12288\,\alpha^5\,\beta^2\,\lambda_1 + 58080\,\alpha\,\beta^4\,\lambda_1$$
$$+ 36864\,\alpha\,\beta^6\,\lambda_1 + 6912\,\alpha\,\beta^4\,\lambda_1^3 + 18912\,\alpha\,\beta^2\,\lambda_1 + 74304\,\alpha\,\beta^2\,\lambda_1^3$$
$$+ 24576\,\alpha^4\,\beta^2\,\lambda_1^2 - 73440\,\alpha^2\,\beta^2\,\lambda_1^2 + 24576\,\alpha^3\,\beta^4\,\lambda_1 - 6912\,\alpha\,\beta^2\,\lambda_1^5$$
$$- 6912\,\alpha^2\,\beta^2\,\lambda_1^4 + 38712\,\alpha^2\,\beta^2\,\lambda_1 - 73728\,\alpha^2\,\beta^4\,\lambda_1^2 + 12000\,\beta^4\,\lambda_1^2$$
$$- 37152\,\beta^2\,\lambda_1^4 + 36288\,\beta^2\,\lambda_1^3 - 40704\,\beta^2\,\lambda_1^2 - 35448\,\beta^4\,\lambda_1 - 12960\,\alpha\,\lambda_1$$
$$- 9288\,\alpha^5 + 22896\,\alpha^4 + 17808\,\beta^4 - 9288\,\lambda_1^5 + 22896\,\lambda_1^4 - 17280\,\lambda_1^3$$
$$+ 13440\,\beta^2\,\lambda_1 \Big)\,c + \Big(36864\,\alpha^4\,\beta^3\,\lambda_1 + 13824\,\alpha^4\,\beta\,\lambda_1^3 - 73728\,\alpha^3\,\beta^3\,\lambda_1^2$$
$$- 20736\,\alpha^3\,\beta\,\lambda_1^4 + 24576\,\alpha^2\,\beta^5\,\lambda_1 + 13824\,\alpha^2\,\beta^3\,\lambda_1^3 + 24576\,\alpha\,\beta^5\,\lambda_1^2$$
$$+ 6912\,\alpha\,\beta^3\,\lambda_1^4 + 6912\,\alpha\,\beta\,\lambda_1^6 - 12288\,\beta^7\,\lambda_1 - 19008\,\alpha^4\,\beta\,\lambda_1 - 26784\,\alpha^3\,\beta$$
$$\lambda_1^2 + 14784\,\alpha^2\,\beta^3\,\lambda_1 + 101088\,\alpha^2\,\beta\,\lambda_1^3 + 67296\,\alpha\,\beta^3\,\lambda_1^2 - 45792\,\alpha\,\beta\,\lambda_1^4$$
$$- 33792\,\beta^5\,\lambda_1 - 9504\,\beta^3\,\lambda_1^3 - 9504\,\beta\,\lambda_1^5 + 18576\,\alpha^4\,\beta + 44712\,\alpha^3\,\beta\,\lambda_1$$
$$- 30960\,\alpha^2\,\beta^3 - 108864\,\alpha^2\,\beta\,\lambda_1^2 - 47160\,\alpha\,\beta^3\,\lambda_1 + 9288\,\alpha\,\beta\,\lambda_1^3 + 16512\,\beta^5$$
$$- 28224\,\beta^3\,\lambda_1^2 + 36288\,\beta\,\lambda_1^4 - 45792\,\alpha^3\,\beta + 57024\,\alpha^2\,\beta\,\lambda_1 + 35616\,\alpha\,\beta^3$$
$$+ 23328\,\alpha\,\beta\,\lambda_1^2 + 26880\,\beta^3\,\lambda_1 - 34560\,\beta\,\lambda_1^3 + 17280\,\alpha^2\,\beta - 34560\,\alpha\,\beta\,\lambda_1$$
$$- 13440\,\beta^3 + 17280\,\beta\,\lambda_1^2\Big)\,s - 4608\,\beta^4\,\lambda_1^4 - 2592\,\beta^2\,\lambda_1^6 + 17496\,\alpha^2\,\lambda_1^4$$
$$- 34992\,\alpha\,\lambda_1^5 - 96768\,\alpha^3\,\beta^2 - 34182\,\alpha^2\,\lambda_1^3 + 75264\,\alpha\,\beta^4 + 68364\,\alpha\,\lambda_1^4$$
$$+ 23040\,\alpha^2\,\beta^2 + 30132\,\alpha^2\,\lambda_1^2 - 60264\,\alpha\,\lambda_1^3 - 18432\,\alpha^6\,\beta^2 + 36864\,\alpha^4\,\beta^4$$
$$- 67584\,\alpha^2\,\beta^6 - 2592\,\alpha^2\,\lambda_1^6 + 5184\,\alpha\,\lambda_1^7 + 99072\,\alpha^4\,\beta^2 - 99584\,\alpha^2\,\beta^4$$
$$- 27648\,\alpha^3\,\beta^2\,\lambda_1^3 - 185472\,\alpha^3\,\beta^2\,\lambda_1 - 72000\,\alpha\,\beta^2\,\lambda_1^2 + 18432\,\alpha^5\,\beta^2\,\lambda_1$$
$$+ 12672\,\alpha\,\beta^4\,\lambda_1 + 18432\,\alpha\,\beta^6\,\lambda_1 + 9216\,\alpha\,\beta^4\,\lambda_1^3 - 46080\,\alpha\,\beta^2\,\lambda_1$$
$$- 12672\,\alpha\,\beta^2\,\lambda_1^3 + 18432\,\alpha^4\,\beta^2\,\lambda_1^2 + 86400\,\alpha^2\,\beta^2\,\lambda_1^2 + 36864\,\alpha^3\,\beta^4\,\lambda_1$$
$$+ 13824\,\alpha^2\,\beta^2\,\lambda_1^4 + 181152\,\alpha^2\,\beta^2\,\lambda_1 - 55296\,\alpha^2\,\beta^4\,\lambda_1^2 + 12672\,\beta^4\,\lambda_1^2$$
$$- 936\,\beta^2\,\lambda_1^4 + 14202\,\beta^2\,\lambda_1^3 - 396\,\beta^2\,\lambda_1^2 - 12384\,\beta^4\,\lambda_1 - 4860\,\alpha\,\lambda_1$$
$$+ 22528\,\beta^6 + 8192\,\beta^8 - 17920\,\beta^4 - 2592\,\lambda_1^8 + 17496\,\lambda_1^6 - 34182\,\lambda_1^5$$
$$+ 30132\,\lambda_1^4 - 12960\,\lambda_1^3 + 10080\,\beta^2\,\lambda_1\Big)\,n + \Big(-144\,\alpha^8 + 288\,\alpha^7\,\lambda_1$$
$$+ 832\,\alpha^6\,\beta^2 - 144\,\alpha^6\,\lambda_1^2 - 288\,\alpha^5\,\beta^2\,\lambda_1 - 2400\,\alpha^4\,\beta^4 - 1584\,\alpha^4\,\beta^2\,\lambda_1^2$$
$$- 1440\,\alpha^3\,\beta^4\,\lambda_1 + 1152\,\alpha^3\,\beta^2\,\lambda_1^3 + 4416\,\alpha^2\,\beta^6 + 3024\,\alpha^2\,\beta^4\,\lambda_1^2 - 864\,\alpha\,\beta^6\,\lambda_1$$
$$- 1152\,\alpha\,\beta^4\,\lambda_1^3 - 400\,\beta^8 - 144\,\beta^6\,\lambda_1^2 - 576\,\alpha^7 + 1152\,\alpha^6\,\lambda_1 + 3904\,\alpha^5\,\beta^2$$

$$- 576 \alpha^5 \lambda_1^2 - 5184 \alpha^4 \beta^2 \lambda_1 - 7360 \alpha^3 \beta^4 + 2304 \alpha^2 \beta^4 \lambda_1 + 1728 \alpha^2 \beta^2 \lambda_1^3$$
$$+ 4544 \alpha \beta^6 + 576 \alpha \beta^4 \lambda_1^2 - 576 \beta^6 \lambda_1 - 576 \beta^4 \lambda_1^3 + 576 \alpha^6 - 1152 \alpha^5 \lambda_1$$
$$- 448 \alpha^4 \beta^2 + 576 \alpha^4 \lambda_1^2 - 2304 \alpha^3 \beta^2 \lambda_1 + 5568 \alpha^2 \beta^4 + 4608 \alpha^2 \beta^2 \lambda_1^2$$
$$- 1152 \alpha \beta^4 \lambda_1 - 2304 \alpha \beta^2 \lambda_1^3 - 1600 \beta^6 - 576 \beta^4 \lambda_1^2 + 1440 \alpha^5$$
$$- 2880 \alpha^4 \lambda_1 - 7168 \alpha^3 \beta^2 + 1440 \alpha^3 \lambda_1^2 + 12960 \alpha^2 \beta^2 \lambda_1 + 1632 \alpha \beta^4$$
$$- 7776 \alpha \beta^2 \lambda_1^2 + 864 \beta^4 \lambda_1 + 864 \beta^2 \lambda_1^3 - 4032 \alpha^4 + 8064 \alpha^3 \lambda_1$$
$$+ 7168 \alpha^2 \beta^2 - 4032 \alpha^2 \lambda_1^2 - 8064 \alpha \beta^2 \lambda_1 - 3136 \beta^4 + 4032 \beta^2 \lambda_1^2 + 3456 \alpha^3$$
$$- 6912 \alpha^2 \lambda_1 - 2688 \alpha \beta^2 + 3456 \alpha \lambda_1^2 - 1296 \alpha^2 + 2592 \alpha \lambda_1 + 1008 \beta^2$$
$$- 1296 \lambda_1^2\Big) c^2 + \Big(-1944 \lambda_1^2 - 2592 \alpha \lambda_1^2 - 2592 \alpha^2 \lambda_1 + 1512 \beta^2 - 1944 \alpha^2$$
$$+ 2592 \alpha^3 + \Big(576 \alpha^7 \beta - 864 \alpha^6 \beta \lambda_1 - 960 \alpha^5 \beta^3 - 1440 \alpha^4 \beta^3 \lambda_1$$
$$+ 288 \alpha^4 \beta \lambda_1^3 + 4544 \alpha^3 \beta^5 + 3456 \alpha^3 \beta^3 \lambda_1^2 - 288 \alpha^2 \beta^5 \lambda_1 - 1728 \alpha^2 \beta^3 \lambda_1^3$$
$$- 2112 \alpha \beta^7 - 1152 \alpha \beta^5 \lambda_1^2 + 288 \beta^7 \lambda_1 + 288 \beta^5 \lambda_1^3 + 2304 \alpha^6 \beta$$
$$- 4032 \alpha^5 \beta \lambda_1 - 5120 \alpha^4 \beta^3 + 1152 \alpha^4 \beta \lambda_1^2 + 3456 \alpha^3 \beta^3 \lambda_1 + 576 \alpha^3 \beta \lambda_1^3$$
$$+ 7936 \alpha^2 \beta^5 + 1152 \alpha^2 \beta^3 \lambda_1^2 - 1728 \alpha \beta^5 \lambda_1 - 1728 \alpha \beta^3 \lambda_1^3 - 1024 \beta^7$$
$$- 1152 \alpha^5 \beta + 1152 \alpha^4 \beta \lambda_1 + 1792 \alpha^3 \beta^3 + 1152 \alpha^3 \beta \lambda_1^2 + 2304 \alpha^2 \beta^3 \lambda_1$$
$$- 1152 \alpha^2 \beta \lambda_1^3 - 5248 \alpha \beta^5 - 3456 \alpha \beta^3 \lambda_1^2 + 1152 \beta^5 \lambda_1 + 1152 \beta^3 \lambda_1^3$$
$$- 5184 \alpha^4 \beta + 11232 \alpha^3 \beta \lambda_1 + 4800 \alpha^2 \beta^3 - 6912 \alpha^2 \beta \lambda_1^2 - 3744 \alpha \beta^3 \lambda_1$$
$$+ 864 \alpha \beta \lambda_1^3 - 256 \beta^5 + 2304 \beta^3 \lambda_1^2 + 8064 \alpha^3 \beta - 16128 \alpha^2 \beta \lambda_1$$
$$- 6272 \alpha \beta^3 + 8064 \alpha \beta \lambda_1^2 - 3456 \alpha^2 \beta + 6912 \alpha \beta \lambda_1 + 2688 \beta^3 - 3456 \beta$$
$$\lambda_1^2\Big) s - 216 \alpha^5 \lambda_1^3 + 432 \alpha^4 \lambda_1^4 - 216 \alpha^3 \lambda_1^5 + 216 \beta^4 \lambda_1^4 + 216 \beta^2 \lambda_1^6 - 432 \alpha^5$$
$$\lambda_1^2 + 432 \alpha^4 \lambda_1^3 + 432 \alpha^3 \lambda_1^4 - 432 \alpha^2 \lambda_1^5 + 864 \beta^2 \lambda_1^5 + 864 \alpha^5 \lambda_1 - 2592 \alpha^4 \lambda_1^2$$
$$+ 3456 \alpha^3 \lambda_1^3 - 2592 \alpha^2 \lambda_1^4 + 864 \alpha \lambda_1^5 + 3024 \alpha^4 \lambda_1 + 1152 \alpha^3 \beta^2 - 2376 \alpha^3$$
$$\lambda_1^2 - 2376 \alpha^2 \lambda_1^3 - 2808 \alpha \beta^4 + 3024 \alpha \lambda_1^4 - 864 \alpha^3 \lambda_1 + 2304 \alpha^2 \beta^2$$
$$+ 4320 \alpha^2 \lambda_1^2 - 864 \alpha \lambda_1^3 - 2016 \alpha \beta^2 + 3768 \alpha \beta^2 \lambda_1^2 - 384 \alpha^5 \beta^2 \lambda_1$$
$$+ 5280 \alpha \beta^4 \lambda_1 + 2304 \alpha^3 \beta^2 \lambda_1^2 + 1152 \alpha \beta^6 \lambda_1 - 768 \alpha^4 \beta^2 \lambda_1 + 216 \alpha \beta^4 \lambda_1^3$$
$$+ 96 \alpha \beta^2 \lambda_1 - 432 \alpha^2 \beta^2 \lambda_1^3 + 768 \alpha^4 \beta^2 \lambda_1^2 - 672 \alpha^2 \beta^2 \lambda_1^2 - 3408 \alpha \beta^4 \lambda_1^2$$
$$+ 768 \alpha^3 \beta^4 \lambda_1 - 1296 \alpha \beta^2 \lambda_1^4 - 216 \alpha \beta^2 \lambda_1^5 - 216 \alpha^2 \beta^2 \lambda_1^4 - 4872 \alpha^2 \beta^2 \lambda_1$$
$$- 2304 \alpha^2 \beta^4 \lambda_1^2 + 864 \beta^4 \lambda_1^3 - 1536 \beta^4 \lambda_1^2 - 1728 \beta^2 \lambda_1^3 + 2304 \beta^2 \lambda_1^2$$
$$+ 768 \beta^6 \lambda_1 + 840 \beta^4 \lambda_1 + 3888 \alpha \lambda_1 - 648 \alpha^5 - 1296 \alpha^4 - 1008 \beta^4 - 648 \lambda_1^5$$

$$- 1296 \lambda_1^4 + 2592 \lambda_1^3 - 2016 \beta^2 \lambda_1 \Big) c + \Big(1152 \alpha^4 \beta^3 \lambda_1 + 432 \alpha^4 \beta \lambda_1^3$$
$$- 2304 \alpha^3 \beta^3 \lambda_1^2 - 648 \alpha^3 \beta \lambda_1^4 + 768 \alpha^2 \beta^5 \lambda_1 + 432 \alpha^2 \beta^3 \lambda_1^3 + 768 \alpha \beta^5 \lambda_1^2$$
$$+ 216 \alpha \beta^3 \lambda_1^4 + 216 \alpha \beta \lambda_1^6 - 384 \beta^7 \lambda_1 + 864 \alpha^4 \beta \lambda_1^2 + 1536 \alpha^3 \beta^3 \lambda_1$$
$$- 432 \alpha^3 \beta \lambda_1^3 - 4512 \alpha^2 \beta^3 \lambda_1^2 - 1728 \alpha^2 \beta \lambda_1^4 + 1536 \alpha \beta^5 \lambda_1 + 1296 \alpha \beta^3 \lambda_1^3$$
$$+ 1296 \alpha \beta \lambda_1^5 + 768 \beta^5 \lambda_1^2 - 1728 \alpha^4 \beta \lambda_1 + 4320 \alpha^3 \beta \lambda_1^2 + 1344 \alpha^2 \beta^3 \lambda_1$$
$$- 4320 \alpha^2 \beta \lambda_1^3 + 864 \alpha \beta^3 \lambda_1^2 + 2592 \alpha \beta \lambda_1^4 - 3072 \beta^5 \lambda_1 - 864 \beta^3 \lambda_1^3$$
$$- 864 \beta \lambda_1^5 + 1296 \alpha^4 \beta - 5400 \alpha^3 \beta \lambda_1 - 2160 \alpha^2 \beta^3 + 5184 \alpha^2 \beta \lambda_1^2$$
$$+ 3336 \alpha \beta^3 \lambda_1 + 648 \alpha \beta \lambda_1^3 + 1152 \beta^5 + 1344 \beta^3 \lambda_1^2 - 1728 \beta \lambda_1^4$$
$$+ 2592 \alpha^3 \beta - 1728 \alpha^2 \beta \lambda_1 - 2016 \alpha \beta^3 - 4320 \alpha \beta \lambda_1^2 - 2688 \beta^3 \lambda_1$$
$$+ 3456 \beta \lambda_1^3 - 2592 \alpha^2 \beta + 5184 \alpha \beta \lambda_1 + 2016 \beta^3 - 2592 \beta \lambda_1^2 \Big) s - 144 \beta^4$$
$$\lambda_1^4 - 81 \beta^2 \lambda_1^6 - 324 \alpha^2 \lambda_1^5 - 324 \beta^2 \lambda_1^5 + 324 \alpha^2 \lambda_1^4 - 648 \alpha \lambda_1^5 + 4608 \alpha^3 \beta^2$$
$$+ 810 \alpha^2 \lambda_1^3 - 3584 \alpha \beta^4 - 1620 \alpha \lambda_1^4 - 2304 \alpha^2 \beta^2 - 2268 \alpha^2 \lambda_1^2 + 4536 \alpha$$
$$\lambda_1^3 - 576 \alpha^6 \beta^2 + 1152 \alpha^4 \beta^4 - 2112 \alpha^2 \beta^6 - 81 \alpha^2 \lambda_1^6 + 162 \alpha \lambda_1^7$$
$$- 2304 \alpha^5 \beta^2 + 3840 \alpha^3 \beta^4 - 2048 \alpha \beta^6 + 648 \alpha \lambda_1^6 - 2048 \alpha^2 \beta^4 + \Big(-2025$$
$$\lambda_1^2 - 32400 \alpha \lambda_1^2 + 16200 \alpha^2 \lambda_1 + 1575 \beta^2 - 2025 \alpha^2 + \Big(-36864 \alpha^8$$
$$+ 73728 \alpha^7 \lambda_1 + 212992 \alpha^6 \beta^2 - 36864 \alpha^6 \lambda_1^2 - 73728 \alpha^5 \beta^2 \lambda_1$$
$$- 614400 \alpha^4 \beta^4 - 405504 \alpha^4 \beta^2 \lambda_1^2 - 368640 \alpha^3 \beta^4 \lambda_1 + 294912 \alpha^3 \beta^2 \lambda_1^3$$
$$+ 1130496 \alpha^2 \beta^6 + 774144 \alpha^2 \beta^4 \lambda_1^2 - 221184 \alpha \beta^6 \lambda_1 - 294912 \alpha \beta^4 \lambda_1^3$$
$$- 102400 \beta^8 - 36864 \beta^6 \lambda_1^2 + 147456 \alpha^7 - 294912 \alpha^6 \lambda_1 - 999424 \alpha^5 \beta^2$$
$$+ 147456 \alpha^5 \lambda_1^2 + 1327104 \alpha^4 \beta^2 \lambda_1 + 1884160 \alpha^3 \beta^4 - 589824 \alpha^2 \beta^4 \lambda_1$$
$$- 442368 \alpha^2 \beta^2 \lambda_1^3 - 1163264 \alpha \beta^6 - 147456 \alpha \beta^4 \lambda_1^2 + 147456 \beta^6 \lambda_1$$
$$+ 147456 \beta^4 \lambda_1^3 - 239616 \alpha^6 + 479232 \alpha^5 \lambda_1 + 1347584 \alpha^4 \beta^2 - 239616 \alpha^4$$
$$\lambda_1^2 - 2138112 \alpha^3 \beta^2 \lambda_1 - 1542144 \alpha^2 \beta^4 + 792576 \alpha^2 \beta^2 \lambda_1^2 + 479232 \alpha \beta^4 \lambda_1$$
$$+ 184320 \alpha \beta^2 \lambda_1^3 + 278528 \beta^6 - 147456 \beta^4 \lambda_1^2 + 207360 \alpha^5 - 414720 \alpha^4 \lambda_1$$
$$- 737280 \alpha^3 \beta^2 + 207360 \alpha^3 \lambda_1^2 + 1128960 \alpha^2 \beta^2 \lambda_1 + 529920 \alpha \beta^4$$
$$- 529920 \alpha \beta^2 \lambda_1^2 - 23040 \beta^4 \lambda_1 - 23040 \beta^2 \lambda_1^3 - 103680 \alpha^4 + 207360 \alpha^3 \lambda_1$$
$$+ 184320 \alpha^2 \beta^2 - 103680 \alpha^2 \lambda_1^2 - 207360 \alpha \beta^2 \lambda_1 - 80640 \beta^4 + 103680 \beta^2 \lambda_1^2$$
$$+ 28800 \alpha^3 - 57600 \alpha^2 \lambda_1 - 22400 \alpha \beta^2 + 28800 \alpha \lambda_1^2 - 3600 \alpha^2$$
$$+ 7200 \alpha \lambda_1 + 2800 \beta^2 - 3600 \lambda_1^2 \Big) c^2 - 36864 \beta^4 \lambda_1^4 - 20736 \beta^2 \lambda_1^6$$

$$+ 82944\,\alpha^{2}\,\lambda_{1}^{5} + 82944\,\beta^{2}\,\lambda_{1}^{5} - 134784\,\alpha^{2}\,\lambda_{1}^{4} + 269568\,\alpha\,\lambda_{1}^{5} + 368640\,\alpha^{3}\,\beta^{2}$$

$$+ 116640\,\alpha^{2}\,\lambda_{1}^{3} - 286720\,\alpha\,\beta^{4} - 233280\,\alpha\,\lambda_{1}^{4} - 57600\,\alpha^{2}\,\beta^{2} - 58320\,\alpha^{2}\,\lambda_{1}^{2}$$

$$+ 116640\,\alpha\,\lambda_{1}^{3} + \Big(-5400\,\lambda_{1}^{2} - 21600\,\alpha\,\lambda_{1}^{2} - 21600\,\alpha^{2}\,\lambda_{1} + 4200\,\beta^{2}$$

$$- 5400\,\alpha^{2} + 21600\,\alpha^{3} + \Big(147456\,\alpha^{7}\,\beta - 221184\,\alpha^{6}\,\beta\,\lambda_{1} - 245760\,\alpha^{5}\,\beta^{3}$$

$$- 368640\,\alpha^{4}\,\beta^{3}\,\lambda_{1} + 73728\,\alpha^{4}\,\beta\,\lambda_{1}^{3} + 1163264\,\alpha^{3}\,\beta^{5} + 884736\,\alpha^{3}\,\beta^{3}\,\lambda_{1}^{2}$$

$$- 73728\,\alpha^{2}\,\beta^{5}\,\lambda_{1} - 442368\,\alpha^{2}\,\beta^{3}\,\lambda_{1}^{3} - 540672\,\alpha\,\beta^{7} - 294912\,\alpha\,\beta^{5}\,\lambda_{1}^{2}$$

$$+ 73728\,\beta^{7}\,\lambda_{1} + 73728\,\beta^{5}\,\lambda_{1}^{3} - 589824\,\alpha^{6}\,\beta + 1032192\,\alpha^{5}\,\beta\,\lambda_{1}$$

$$+ 1310720\,\alpha^{4}\,\beta^{3} - 294912\,\alpha^{4}\,\beta\,\lambda_{1}^{2} - 884736\,\alpha^{3}\,\beta^{3}\,\lambda_{1} - 147456\,\alpha^{3}\,\beta\,\lambda_{1}^{3}$$

$$- 2031616\,\alpha^{2}\,\beta^{5} - 294912\,\alpha^{2}\,\beta^{3}\,\lambda_{1}^{2} + 442368\,\alpha\,\beta^{5}\,\lambda_{1} + 442368\,\alpha\,\beta^{3}\,\lambda_{1}^{3}$$

$$+ 262144\,\beta^{7} + 866304\,\alpha^{5}\,\beta - 1640448\,\alpha^{4}\,\beta\,\lambda_{1} - 1519616\,\alpha^{3}\,\beta^{3}$$

$$+ 681984\,\alpha^{3}\,\beta\,\lambda_{1}^{2} + 1363968\,\alpha^{2}\,\beta^{3}\,\lambda_{1} + 92160\,\alpha^{2}\,\beta\,\lambda_{1}^{3} + 1021952\,\alpha\,\beta^{5}$$

$$- 497664\,\alpha\,\beta^{3}\,\lambda_{1}^{2} - 92160\,\beta^{5}\,\lambda_{1} - 92160\,\beta^{3}\,\lambda_{1}^{3} - 599040\,\alpha^{4}\,\beta$$

$$+ 1175040\,\alpha^{3}\,\beta\,\lambda_{1} + 691200\,\alpha^{2}\,\beta^{3} - 552960\,\alpha^{2}\,\beta\,\lambda_{1}^{2} - 391680\,\alpha\,\beta^{3}\,\lambda_{1}$$

$$- 23040\,\alpha\,\beta\,\lambda_{1}^{3} - 184320\,\beta^{5} + 184320\,\beta^{3}\,\lambda_{1}^{2} + 207360\,\alpha^{3}\,\beta - 414720\,\alpha^{2}\,\beta\,\lambda_{1}$$

$$- 161280\,\alpha\,\beta^{3} + 207360\,\alpha\,\beta\,\lambda_{1}^{2} - 28800\,\alpha^{2}\,\beta + 57600\,\alpha\,\beta\,\lambda_{1} + 22400\,\beta^{3}$$

$$- 28800\,\beta\,\lambda_{1}^{2} \Big)\,s - 55296\,\alpha^{5}\,\lambda_{1}^{3} + 110592\,\alpha^{4}\,\lambda_{1}^{4} - 55296\,\alpha^{3}\,\lambda_{1}^{5} + 55296\,\beta^{4}\,\lambda_{1}^{4}$$

$$+ 55296\,\beta^{2}\,\lambda_{1}^{6} + 110592\,\alpha^{5}\,\lambda_{1}^{2} - 110592\,\alpha^{4}\,\lambda_{1}^{3} - 110592\,\alpha^{3}\,\lambda_{1}^{4} + 110592\,\alpha^{2}$$

$$\lambda_{1}^{5} - 221184\,\beta^{2}\,\lambda_{1}^{5} - 69120\,\alpha^{5}\,\lambda_{1} - 82944\,\alpha^{4}\,\lambda_{1}^{2} + 304128\,\alpha^{3}\,\lambda_{1}^{3} - 82944\,\alpha^{2}$$

$$\lambda_{1}^{4} - 69120\,\alpha\,\lambda_{1}^{5} + 103680\,\alpha^{4}\,\lambda_{1} - 30720\,\alpha^{3}\,\beta^{2} - 120960\,\alpha^{3}\,\lambda_{1}^{2} - 120960\,\alpha^{2}$$

$$\lambda_{1}^{3} + 74880\,\alpha\,\beta^{4} + 103680\,\alpha\,\lambda_{1}^{4} - 17280\,\alpha^{3}\,\lambda_{1} + 61440\,\alpha^{2}\,\beta^{2} + 103680\,\alpha^{2}\,\lambda_{1}^{2}$$

$$- 17280\,\alpha\,\lambda_{1}^{3} - 16800\,\alpha\,\beta^{2} + 124800\,\alpha\,\beta^{2}\,\lambda_{1}^{2} - 98304\,\alpha^{5}\,\beta^{2}\,\lambda_{1}$$

$$- 422400\,\alpha\,\beta^{4}\,\lambda_{1} - 589824\,\alpha^{3}\,\beta^{2}\,\lambda_{1}^{2} + 294912\,\alpha\,\beta^{6}\,\lambda_{1} + 196608\,\alpha^{4}\,\beta^{2}\,\lambda_{1}$$

$$+ 55296\,\alpha\,\beta^{4}\,\lambda_{1}^{3} - 1920\,\alpha\,\beta^{2}\,\lambda_{1} - 580608\,\alpha\,\beta^{2}\,\lambda_{1}^{3} + 110592\,\alpha^{2}\,\beta^{2}\,\lambda_{1}^{3}$$

$$+ 196608\,\alpha^{4}\,\beta^{2}\,\lambda_{1}^{2} + 569856\,\alpha^{2}\,\beta^{2}\,\lambda_{1}^{2} + 872448\,\alpha\,\beta^{4}\,\lambda_{1}^{2} + 196608\,\alpha^{3}\,\beta^{4}\,\lambda_{1}$$

$$+ 331776\,\alpha\,\beta^{2}\,\lambda_{1}^{4} - 55296\,\alpha\,\beta^{2}\,\lambda_{1}^{5} - 55296\,\alpha^{2}\,\beta^{2}\,\lambda_{1}^{4} - 197760\,\alpha^{2}\,\beta^{2}\,\lambda_{1}$$

$$- 589824\,\alpha^{2}\,\beta^{4}\,\lambda_{1}^{2} - 221184\,\beta^{4}\,\lambda_{1}^{3} - 102912\,\beta^{4}\,\lambda_{1}^{2} + 290304\,\beta^{2}\,\lambda_{1}^{4}$$

$$- 138240\,\beta^{2}\,\lambda_{1}^{3} + 61440\,\beta^{2}\,\lambda_{1}^{2} - 196608\,\beta^{6}\,\lambda_{1} + 120960\,\beta^{4}\,\lambda_{1} + 10800\,\alpha\,\lambda_{1}$$

$$+ 17280\,\alpha^{5} - 34560\,\alpha^{4} - 26880\,\beta^{4} + 17280\,\lambda_{1}^{5} - 34560\,\lambda_{1}^{4} + 21600\,\lambda_{1}^{3}$$

$$- 16800\,\beta^{2}\,\lambda_{1} \Big)\,c + \Big(294912\,\alpha^{4}\,\beta^{3}\,\lambda_{1} + 110592\,\alpha^{4}\,\beta\,\lambda_{1}^{3} - 589824\,\alpha^{3}\,\beta^{3}\,\lambda_{1}^{2}$$

$$- 165888 \alpha^3 \beta \lambda_1^4 + 196608 \alpha^2 \beta^5 \lambda_1 + 110592 \alpha^2 \beta^3 \lambda_1^3 + 196608 \alpha \beta^5 \lambda_1^2$$
$$+ 55296 \alpha \beta^3 \lambda_1^4 + 55296 \alpha \beta \lambda_1^6 - 98304 \beta^7 \lambda_1 - 221184 \alpha^4 \beta \lambda_1^2$$
$$- 393216 \alpha^3 \beta^3 \lambda_1 + 110592 \alpha^3 \beta \lambda_1^3 + 1155072 \alpha^2 \beta^3 \lambda_1^2 + 442368 \alpha^2 \beta \lambda_1^4$$
$$- 393216 \alpha \beta^5 \lambda_1 - 331776 \alpha \beta^3 \lambda_1^3 - 331776 \alpha \beta \lambda_1^5 - 196608 \beta^5 \lambda_1^2$$
$$+ 138240 \alpha^4 \beta \lambda_1 + 235008 \alpha^3 \beta \lambda_1^2 - 107520 \alpha^2 \beta^3 \lambda_1 - 815616 \alpha^2 \beta \lambda_1^3$$
$$- 520704 \alpha \beta^3 \lambda_1^2 + 373248 \alpha \beta \lambda_1^4 + 245760 \beta^5 \lambda_1 + 69120 \beta^3 \lambda_1^3 + 69120 \beta$$
$$\lambda_1^5 - 34560 \alpha^4 \beta - 224640 \alpha^3 \beta \lambda_1 + 57600 \alpha^2 \beta^3 + 414720 \alpha^2 \beta \lambda_1^2$$
$$+ 197760 \alpha \beta^3 \lambda_1 - 17280 \alpha \beta \lambda_1^3 - 30720 \beta^5 + 107520 \beta^3 \lambda_1^2 - 138240 \beta \lambda_1^4$$
$$+ 69120 \alpha^3 \beta - 51840 \alpha^2 \beta \lambda_1 - 53760 \alpha \beta^3 - 103680 \alpha \beta \lambda_1^2 - 67200 \beta^3 \lambda_1$$
$$+ 86400 \beta \lambda_1^3 - 21600 \alpha^2 \beta + 43200 \alpha \beta \lambda_1 + 16800 \beta^3 - 21600 \beta \lambda_1^2 \Big) s$$
$$- 147456 \alpha^6 \beta^2 + 294912 \alpha^4 \beta^4 - 540672 \alpha^2 \beta^6 - 20736 \alpha^2 \lambda_1^6 + 41472 \alpha \lambda_1^7$$
$$+ 589824 \alpha^5 \beta^2 - 983040 \alpha^3 \beta^4 + 524288 \alpha \beta^6 - 165888 \alpha \lambda_1^6$$
$$- 774144 \alpha^4 \beta^2 + 765952 \alpha^2 \beta^4 - 221184 \alpha^3 \beta^2 \lambda_1^3 + 1456128 \alpha^3 \beta^2 \lambda_1$$
$$+ 322560 \alpha \beta^2 \lambda_1^2 + 147456 \alpha^5 \beta^2 \lambda_1 - 92160 \alpha \beta^4 \lambda_1 + 147456 \alpha \beta^6 \lambda_1$$
$$- 884736 \alpha^4 \beta^2 \lambda_1 + 73728 \alpha \beta^4 \lambda_1^3 + 115200 \alpha \beta^2 \lambda_1 + 92160 \alpha \beta^2 \lambda_1^3$$
$$+ 442368 \alpha^2 \beta^2 \lambda_1^3 + 147456 \alpha^4 \beta^2 \lambda_1^2 - 681984 \alpha^2 \beta^2 \lambda_1^2 + 294912 \alpha^3 \beta^4 \lambda_1$$
$$+ 294912 \alpha^2 \beta^4 \lambda_1 - 294912 \alpha \beta^2 \lambda_1^4 + 110592 \alpha^2 \beta^2 \lambda_1^4 - 714240 \alpha^2 \beta^2 \lambda_1$$
$$- 442368 \alpha^2 \beta^4 \lambda_1^2 + 147456 \beta^4 \lambda_1^3 - 92160 \beta^4 \lambda_1^2 + 12672 \beta^2 \lambda_1^4 - 67680 \beta^2$$
$$\lambda_1^3 - 12240 \beta^2 \lambda_1^2 + 23040 \beta^4 \lambda_1 + 4050 \alpha \lambda_1 - 163840 \beta^6 + 65536 \beta^8$$
$$+ 44800 \beta^4 - 20736 \lambda_1^8 + 82944 \lambda_1^7 - 134784 \lambda_1^6 + 116640 \lambda_1^5 - 58320 \lambda_1^4$$
$$+ 16200 \lambda_1^3 - 12600 \beta^2 \lambda_1 \Big) n^2 - 864 \alpha^3 \beta^2 \lambda_1^3 + 1152 \alpha^3 \beta^2 \lambda_1 + 6336 \alpha \beta^2 \lambda_1^2$$
$$+ 576 \alpha^5 \beta^2 \lambda_1 + 1152 \alpha \beta^4 \lambda_1 + 576 \alpha \beta^6 \lambda_1 + 3456 \alpha^4 \beta^2 \lambda_1 + 288 \alpha \beta^4 \lambda_1^3$$
$$+ 4608 \alpha \beta^2 \lambda_1 - 1152 \alpha \beta^2 \lambda_1^3 - 1728 \alpha^2 \beta^2 \lambda_1^3 + 576 \alpha^4 \beta^2 \lambda_1^2 - 1152 \alpha^2 \beta^2$$
$$\lambda_1^2 + 1152 \alpha^3 \beta^4 \lambda_1 - 1152 \alpha^2 \beta^4 \lambda_1 + 1152 \alpha \beta^2 \lambda_1^4 + 432 \alpha^2 \beta^2 \lambda_1^4$$
$$- 10080 \alpha^2 \beta^2 \lambda_1 - 1728 \alpha^2 \beta^4 \lambda_1^2 - 576 \beta^4 \lambda_1^3 + 1152 \beta^4 \lambda_1^2 + 900 \beta^2 \lambda_1^4$$
$$- 1494 \beta^2 \lambda_1^3 - 540 \beta^2 \lambda_1^2 - 864 \beta^4 \lambda_1 + 1458 \alpha \lambda_1 + 2048 \beta^6 + 256 \beta^8$$
$$+ 1792 \beta^4 - 81 \lambda_1^8 - 324 \lambda_1^7 + 324 \lambda_1^6 + 810 \lambda_1^5 - 2268 \lambda_1^4 + 1944 \lambda_1^3$$
$$- 1512 \beta^2 \lambda_1$$

```
> eval(%,intvls);
```

⟨RealBox: -290.095 ± 687.995⟩ + (⟨RealBox: -401.143 ± 1462.62⟩) n + (**(2.2.18)**

$\langle\text{RealBox: } 1319.42 \pm 0.00310919\rangle)\ n^2$

```
> lowpol(%,n);
```

$$-978.090650521843941 + 1319.4140911151459964\, n^2 - 1863.764530859043871\, n \quad (2.2.19)$$

```
> fsolve(%,n);
```

$$-0.40733314930799776327,\ 1.8199029736031303753 \quad (2.2.20)$$

For n≥2, the cone is contracted.

Check first values

When does the sequence enter the cone?

```
> L:=gfun:-rectoproc({rec} union ini,u(n),list)(10);
```

$$L := \left[5, 5, 1, 3, \frac{84}{17}, \frac{3491}{561}, \frac{193118}{27489}, \frac{389756}{51051}, \frac{79660582}{9648639}, \frac{765516302}{85083453}, \frac{1047161107676}{105758732079}\right] \quad (2.2.1.1)$$

```
> normal(basis^(-1).Vector(L[2..4]));
```

$$\left[\begin{array}{c} \frac{2\left(5\,\alpha^2+5\,\beta^2 \ldots\right.}{3\left(\alpha^2-2\,\alpha\,\lambda \ldots\right.} \\ \frac{\frac{I}{2}\left(10\,I\,\alpha\,\beta\,\lambda_1-5\,I\,\beta\,\lambda_1^2-2\,I\,\alpha\,\beta-5\,\alpha^2\,\lambda_1+5\,\alpha \ldots\right.}{\beta\left(\alpha^2-2\,\alpha\,\lambda \ldots\right.} \\ \frac{\frac{I}{2}\left(-5\,\beta^2\,\lambda_1+10\,I\,\alpha\,\beta\,\lambda_1-5\,I\,\beta\,\lambda_1^2-2\,I\,\alpha\,\beta+5\,\text{o} \ldots\right.}{\beta\left(\alpha^2-2\,\alpha\,\lambda \ldots\right.} \end{array}\right] \quad (2.2.1.2)$$

```
> for i from 2 do vv:=normal(basis^(-1).Vector(L[i..i+2]));
  if LessThan(RealBox(0),eval(vv[1],intvls)) and LessThan
  (RealBox(0),eval(normal(vv[1]^2-vv[2]*vv[3]),intvls)) then
  break fi od; i;
```

$vv :=$

$$
vv := \left[\begin{array}{c}
\frac{2\left(5\,\alpha^2+5\,\beta^2 \ldots\right.}{3\left(\alpha^2-2\,\alpha\,\lambda \ldots\right.} \\
\frac{\frac{I}{2}\left(10\,I\,\alpha\,\beta\,\lambda_1-5\,I\,\beta\,\lambda_1^2-2\,I\,\alpha\,\beta-5\,\alpha^2\,\lambda_1+5\,\alpha \ldots\right.}{\beta\left(\alpha^2-2\,\alpha\,\lambda \ldots\right.} \\
\frac{\frac{I}{2}\left(-5\,\beta^2\,\lambda_1+10\,I\,\alpha\,\beta\,\lambda_1-5\,I\,\beta\,\lambda_1^2-2\,I\,\alpha\,\beta+5\,\alpha \ldots\right.}{\beta\left(\alpha^2-2\,\alpha\,\lambda \ldots\right.}
\end{array}\right], \left[\begin{array}{c}
\frac{2\left(1 \ldots\right.}{51 \ldots} \\
\frac{\frac{I}{34}\left(34\,I\,\alpha\,\beta\,\lambda_1-17\,I\,\beta\,\lambda_1^2-102\,I\,\alpha\,\beta-17\,\alpha^2\,\lambda \ldots\right.}{\beta} \\
\frac{\frac{I}{34}\left(-17\,\beta^2\,\lambda_1+34\,I\,\alpha\,\beta\,\lambda_1-17\,I\,\beta\,\lambda_1^2-102\,I\,\alpha \ldots\right.}{\beta}
\end{array}\right]
$$

3 **(2.2.1.3)**

PROOF COMPLETE!

Approximate cone

Construction of the cone

```
> convert(eigs[1],confrac,'K'); K;
```

$$[1, 6, 1, 2, 1, 2, 2, 4, 2, 1, 3, 3, 1633, 1, 7, 2, 11]$$

$$\left[1, \frac{7}{6}, \frac{8}{7}, \frac{23}{20}, \frac{31}{27}, \frac{85}{74}, \frac{201}{175}, \frac{889}{774}, \frac{1979}{1723}, \frac{2868}{2497}, \frac{10583}{9214}, \frac{34617}{30139}, \frac{56540144}{49226201}, \frac{56574761}{49256340}, \frac{452563471}{394020581}, \frac{961701703}{837297502}, \frac{11031282204}{9604293103}\right] \quad \textbf{(2.3.1.1)}$$

```
> lambda_app[1]:=%[2];
```

$$lambda\_app_1 := \frac{7}{6} \quad \textbf{(2.3.1.2)}$$

```
> eigs[2];
```

$$0.42571087295564390688 - 0.30140628765072823665\,I \quad \textbf{(2.3.1.3)}$$

```
> convert(Re(%),confrac,'K'):K;
```

$$\left[0, \frac{1}{2}, \frac{2}{5}, \frac{3}{7}, \frac{20}{47}, \frac{43}{101}, \frac{106}{249}, \frac{149}{350}, \frac{255}{599}, \frac{404}{949}, \frac{1063}{2497}, \frac{7845}{18428}, \frac{8908}{20925}, \frac{16753}{39353}, \frac{25661}{60278}, \frac{20956129}{49226201}, \frac{41937919}{98512680}, \frac{146769886}{344764241}, \frac{188707805}{443276921}, \frac{524185496}{1231318083}, \frac{712893301}{1674595004}, \frac{6227331904}{14628078115}\right] \quad \textbf{(2.3.1.4)}$$

```
> r2_app:=%[3];
```

$$r2\_app := \frac{2}{5} \quad \textbf{(2.3.1.5)}$$

```
> convert(Im(eigs[2]),confrac,'K'):K;
```

$$\left[-1, 0, -\frac{1}{3}, -\frac{3}{10}, -\frac{19}{63}, -\frac{22}{73}, -\frac{107}{355}, -\frac{343}{1138}, -\frac{793}{2631}, -\frac{5101}{16924}, -\frac{21197}{70327}, -\frac{26298}{87251}, -\frac{73793}{244829}, -\frac{395263}{1311396}, -\frac{20627469}{68437421}, -\frac{21022732}{69748817}, -\frac{62672933}{207935055}, -\frac{83695665}{277683872}, -\frac{732238253}{2429406031}, -\frac{815933918}{2707089903}, -\frac{8891577433}{29500305061}\right] \quad \textbf{(2.3.1.6)}$$

```
> i2_app:=%[3];
```

$$i2\_app := -\frac{1}{3} \quad \textbf{(2.3.1.7)}$$

```
> lambda_app[2]:=r2_app+I*i2_app;lambda_app[3]:=conjugate
  (lambda_app[2]);
```

$$lambda\_app_2 := \frac{2}{5} - \frac{I}{3}$$

$$lambda\_app_3 := \frac{2}{5} + \frac{I}{3} \quad \textbf{(2.3.1.8)}$$

```
> for i to 3 do Vt[i]:=Vector([seq(lambda_app[i]^j,j=0..2)])
  od;
```

$$Vt_1 := \begin{bmatrix} 1 \\ \frac{7}{6} \\ \frac{49}{36} \end{bmatrix}$$

$$Vt_2 := \begin{bmatrix} 1 \\ \frac{2}{5} - \frac{I}{3} \\ \frac{11}{225} - \frac{4\,I}{15} \end{bmatrix}$$

$$Vt_3 := \begin{bmatrix} 1 \\ \frac{2}{5} + \frac{I}{3} \\ \frac{11}{225} + \frac{4\,I}{15} \end{bmatrix} \quad \textbf{(2.3.1.9)}$$

```
> basis:=Matrix([3/2*Vt[1],Vt[2],Vt[3]]);
```

$$basis := \begin{bmatrix} \frac{3}{2} & 1 & 1 \\ \frac{7}{4} & \frac{2}{5} - \frac{I}{3} & \frac{2}{5} + \frac{I}{3} \\ \frac{49}{24} & \frac{11}{225} - \frac{4\,I}{15} & \frac{11}{225} + \frac{4\,I}{15} \end{bmatrix} \quad \textbf{(2.3.1.10)}$$

```
> incone:=basis.Vector([1,c+I*s,c-I*s]);
```

$incone :=$ **(2.3.1.11)**

$$\begin{bmatrix} \frac{3}{2} + 2\,c \quad \cdots \\ \frac{7}{4} + \left(\frac{2}{5} - \frac{I}{3}\right)(c + I\,s) + \left(\frac{2}{5} + \frac{I}{3}\right)(c - I\,s) \cdots \\ \frac{49}{24} + \left(\frac{11}{225} - \frac{4\,I}{15}\right)(c + I\,s) + \left(\frac{11}{225} + \frac{4\,I}{15}\right)(c - \cdots \end{bmatrix}$$

Check that it is contracted by A

```
> out:=evalc(basis^(-1).A.incone);
```

$out :=$ **(2.3.2.1)**

$$\begin{bmatrix} \frac{8681}{7548} - \frac{367\,c}{9435} - \frac{4\,s}{333} \quad \cdots \\ \frac{125}{10064} + \frac{5399\,c}{12580} + \frac{38\,s}{111} + I\left(\frac{575}{20128} - \frac{100477\,c}{377400} + \cdots \right. \\ \frac{125}{10064} + \frac{5399\,c}{12580} + \frac{38\,s}{111} + I\left(-\frac{575}{20128} + \frac{100477\,c}{377400} \cdots \right. \end{bmatrix}$$

```
> signum(eval(out[1],intvls));
```

⟨RealBox: 1 ± 0⟩ **(2.3.2.2)**

```
> evalc(out[1]^2-out[2]*out[3]);
```

(2.3.2.3)

$$\left(\frac{8681}{7548}-\frac{367\,c}{9435}-\frac{4\,s}{333}\right)^2+\left(\frac{575}{20128}-\frac{100477\,c}{377400}+\frac{467\,s}{1110}\right)\left(-\frac{575}{20128}+\frac{100477\,c}{377400}-\frac{467\,s}{1110}\right)-\left(\frac{125}{10064}+\frac{5399\,c}{12580}+\frac{38\,s}{111}\right)^2+\mathrm{I}\left(-\left(\frac{125}{10064}+\frac{5399\,c}{12580}+\frac{38\,s}{111}\right)\left(-\frac{575}{20128}+\frac{100477\,c}{377400}-\frac{467\,s}{1110}\right)-\left(\frac{575}{20128}-\frac{100477\,c}{377400}+\frac{467\,s}{1110}\right)\left(\frac{125}{10064}+\frac{5399\,c}{12580}+\frac{38\,s}{111}\right)\right) \quad \textbf{(2.3.2.3)}$$

```
> simplify(%,{c^2+s^2=1});
```

$$\frac{843137379091}{820401177600}+\frac{51918941339\,c^2}{1281876840000}+\frac{(-14514769725-11774967856\,s)\,c}{170916912000}-\frac{1209929\,s}{20107872} \quad \textbf{(2.3.2.4)}$$

```
> SMTLIB[Satisfiable]({%<0,c^2+s^2=1});
```

$$\mathit{false} \quad \textbf{(2.3.2.5)}$$

Or by interval analysis:

```
> eval(%%,intvls);
```

$$\langle\text{RealBox: } 1.02771 \pm 0.25449\rangle \quad \textbf{(2.3.2.6)}$$

Contraction index

```
> out:=basis^(-1).An.incone:
> normal(out[1]);
```

$$-\frac{4\,(4404\,n\,c+1360\,n\,s+19284\,c-130215\,n-9140\,s+18510)}{28305\,(16\,n+1)} \quad \textbf{(2.3.3.1)}$$

```
> collect(numer(%),n);
```

$$(-17616\,c-5440\,s+520860)\,n-77136\,c+36560\,s-74040 \quad \textbf{(2.3.3.2)}$$

```
> eval(%,intvls);
```

$$\langle\text{RealBox: -74040} \pm 113696\rangle+(\langle\text{RealBox: } 520860 \pm 23056\rangle)\,n \quad \textbf{(2.3.3.3)}$$

```
> lowpol(%,n);
```

$$-187736.000854492188+497803.9996948242187\,n \quad \textbf{(2.3.3.4)}$$

```
> fsolve(%,n);
```

$$0.3771283496508317063 6 \quad \textbf{(2.3.3.5)}$$

```
> evalc(out[1]^2-out[2]*out[3]):
> simplify(%,{c^2+s^2-1});
```

$$\frac{1}{5127507360000\,(16\,n+1)^2}\Big(\big(53164995931136\,c^2+(-90431753134080\,s-111473431488000)\,c-78984165120000\,s+1349019806545600\big)\,n^2+\big(-123251143973888\,c^2+(-476766164367360\,s-437765596200000)\,c-351000691680000\,s-370648005434800\big)\,n-79704661402624\,c^2+(61706669214720\,s-290235461292000)\,c+106156343640000\,s-235121125943525\Big) \quad \textbf{(2.3.3.6)}$$

```
> pp:=collect(normal(%*(16*n+1)^2),[n,c,s]);
```

$$pp := \left(\frac{207675765356\,c^2}{20029325625} + \left(-\frac{1385290336\,s}{78546375} - \frac{387060526}{17803845}\right)c - \frac{9679432\,s}{628371} + \frac{843137379091}{3204692100}\right)n^2 + \left(-\frac{481449781148\,c^2}{20029325625} + \left(-\frac{13795317256\,s}{148365375} - \frac{10955095}{128316}\right)c - \frac{731251441\,s}{10682307} - \frac{25043784151}{346453200}\right)n - \frac{18314490212\,c^2}{1178195625} + \left(\frac{16069445108\,s}{1335288375} - \frac{8062096147}{142430760}\right)c + \frac{884636197\,s}{42729228} - \frac{9404845037741}{205100294400} \quad \textbf{(2.3.3.7)}$$

```
> eval(%,intvls);
```

⟨RealBox: -45.8549 ± 104.886⟩ + (⟨RealBox: 263.095 ± 65.1495⟩) n^2 + (⟨RealBox: -72.2862 ± 270.85⟩) n **(2.3.3.8)**

```
> lowpol(%,n);
```

$$-150.7407422416859365 + 197.9451970219856202\,n^2 - 343.1358565864169781\,n \quad \textbf{(2.3.3.9)}$$

```
> fsolve(%,n);
```

$$-0.3632041257001672540 6, 2.0966933022438403165 \quad \textbf{(2.3.3.10)}$$

```
> signum(eval(collect(eval(pp,n=2),[c,s]),intvls));
```

⟨RealBox: 1 ± 0⟩ **(2.3.3.11)**

```
> signum(eval(collect(eval(pp,n=1),[c,s]),intvls));
```

⟨RealBox: 0 ± 1⟩ **(2.3.3.12)**

Check first values

When does the sequence enter the cone?

```
> L;
```

$$\left[5, 5, 1, 3, \frac{84}{17}, \frac{3491}{561}, \frac{193118}{27489}, \frac{389756}{51051}, \frac{79660582}{9648639}, \frac{765516302}{85083453}, \frac{1047161107676}{105758732079}\right] \quad \textbf{(2.3.4.1)}$$

```
> for i do vv:=basis^(-1).Vector(L[i..i+2]) until vv[1]>0
  and vv[1]^2>vv[2]*vv[3]; i;
```

$$vv := \begin{bmatrix} -\frac{80}{51} \\ \frac{125}{34} + \frac{245\,I}{34} \\ \frac{125}{34} - \frac{245\,I}{34} \end{bmatrix}$$

$$vv := \begin{bmatrix} \frac{6400}{1887} \\ -\frac{55}{1258} - \frac{9247\,I}{1258} \\ -\frac{55}{1258} + \frac{9247\,I}{1258} \end{bmatrix}$$

$$vv := \begin{bmatrix} \frac{86056}{32079} \\ -\frac{32335}{21386} - \frac{15559\,I}{21386} \\ -\frac{32335}{21386} + \frac{15559\,I}{21386} \end{bmatrix}$$

$$3 \qquad \textbf{(2.3.4.2)}$$

PROOF COMPLETE!

Polyhedral cone

Same basis, different boundary: |c|+|s|≤1.

Contraction index

```
> basis^(-1).An.incone:
> out:=map(evalc@normal,ScalarMultiply(%,16*n+1));
```

$out :=$ **(2.4.1.1)**

$$\begin{bmatrix} -\frac{5872}{9435}, \cdots \\ \frac{21596\,n\,c}{3145} + \frac{7686\,c}{3145} + \frac{125\,n}{629} + \frac{14275}{5032} + \frac{6(}{} \cdots \\ \frac{21596\,n\,c}{3145} + \frac{7686\,c}{3145} + \frac{125\,n}{629} + \frac{14275}{5032} + \frac{6(}{} \cdots \end{bmatrix}$$

```
> dir:=[c=1,s=0];
```

$$dir := [c = 1, s = 0] \qquad \textbf{(2.4.1.2)}$$

```
> eval(out[1],dir);
```

$$\frac{55916\,n}{3145} - \frac{50392}{9435} \qquad \textbf{(2.4.1.3)}$$

```
> eval(out[2],dir);
```

$$\frac{22221\,n}{3145}+\frac{132863}{25160}+\mathrm{I}\left(\frac{8221531}{754800}-\frac{358783\,n}{94350}\right) \qquad (2.4.1.4)$$

```
> %%-abs(coeff(%,I,0))-abs(coeff(%,I,1));
```

$$\frac{55916\,n}{3145}-\frac{50392}{9435}-\frac{\left|\frac{132863}{8}+22221\,n\right|}{3145}-\frac{\left|-\frac{8221531}{8}+358783\,n\right|}{94350} \qquad (2.4.1.5)$$

```
> signum(%) assuming n>=2;
```

$$1 \qquad (2.4.1.6)$$

Same for the other directions:

```
> DIRS:=[[0,1],[1,0],[0,-1],[-1,0]];
```

$$DIRS := [[0,1],[1,0],[0,-1],[-1,0]] \qquad (2.4.1.7)$$

```
> for dir in DIRS do
    an:=eval(out[1],[c=dir[1],s=dir[2]]);
    bn:=eval(out[2],[c=dir[1],s=dir[2]]);
    pol:=an-abs(coeff(bn,I,0))-abs(coeff(bn,I,1));
    for i while signum(eval(pol,n=i))<>1 do od;
    n0[dir]:=i-1
  od;
```

$$an := -\frac{7496}{5661}+\frac{103084\,n}{5661}$$

$$bn := \frac{33233}{15096}+\frac{10711\,n}{1887}+\mathrm{I}\left(\frac{709007}{150960}+\frac{135649\,n}{18870}\right)$$

$$pol := -\frac{7496}{5661}+\frac{103084\,n}{5661}-\frac{\left|\frac{33233}{8}+10711\,n\right|}{1887}-\frac{\left|\frac{709007}{8}+135649\,n\right|}{18870}$$

$$n0_{[0,1]} := 1$$

$$an := \frac{55916\,n}{3145}-\frac{50392}{9435}$$

$$bn := \frac{22221\,n}{3145}+\frac{132863}{25160}+\mathrm{I}\left(\frac{8221531}{754800}-\frac{358783\,n}{94350}\right)$$

$$pol := \frac{55916\,n}{3145}-\frac{50392}{9435}-\frac{\left|\frac{132863}{8}+22221\,n\right|}{3145}-\frac{\left|-\frac{8221531}{8}+358783\,n\right|}{94350}$$

$$n0_{[1,0]} := 1$$

$$an := -\frac{22120}{5661}+\frac{105260\,n}{5661}$$

$$bn := \frac{52417}{15096}-\frac{9961\,n}{1887}+\mathrm{I}\left(\frac{1260943}{150960}-\frac{118399\,n}{18870}\right)$$

$$pol := -\frac{22120}{5661} + \frac{105260\,n}{5661} - \frac{\left|-\frac{52417}{8} + 9961\,n\right|}{1887} - \frac{\left|-\frac{1260943}{8} + 118399\,n\right|}{18870}$$

$$n0_{[0,\,-1]} := 0$$

$$an := \frac{179492\,n}{9435} + \frac{344}{3145}$$

$$bn := -\frac{20971\,n}{3145} + \frac{9887}{25160} + \mathrm{I}\left(\frac{1628219}{754800} + \frac{445033\,n}{94350}\right)$$

$$pol := \frac{179492\,n}{9435} + \frac{344}{3145} - \frac{\left|-\frac{9887}{8} + 20971\,n\right|}{3145} - \frac{\left|\frac{1628219}{8} + 445033\,n\right|}{94350}$$

$$n0_{[-1,\,0]} := 0 \qquad \textbf{(2.4.1.8)}$$

```
> max(seq(n0[dir],dir=DIRS));
```

$$1 \qquad \textbf{(2.4.1.9)}$$

For n>=1 the polyhedral cone is contracted.

Check first values

When does the sequence enter the cone?

```
> L;
```

$$\left[5, 5, 1, 3, \frac{84}{17}, \frac{3491}{561}, \frac{193118}{27489}, \frac{389756}{51051}, \frac{79660582}{9648639}, \frac{765516302}{85083453}, \frac{1047161107676}{105758732079}\right] \qquad \textbf{(2.4.2.1)}$$

```
> for i do vv:=basis^(-1).Vector(L[i..i+2]) until vv[1]>abs
  (coeff(vv[2],I,0))+abs(coeff(vv[2],I,1)); i;
```

$$vv := \begin{bmatrix} -\frac{80}{51} \\ \frac{125}{34} + \frac{245\,\mathrm{I}}{34} \\ \frac{125}{34} - \frac{245\,\mathrm{I}}{34} \end{bmatrix}$$

$$vv := \begin{bmatrix} \frac{6400}{1887} \\ -\frac{55}{1258} - \frac{9247\,I}{1258} \\ -\frac{55}{1258} + \frac{9247\,I}{1258} \end{bmatrix}$$

$$vv := \begin{bmatrix} \frac{86056}{32079} \\ -\frac{32335}{21386} - \frac{15559\,I}{21386} \\ -\frac{32335}{21386} + \frac{15559\,I}{21386} \end{bmatrix}$$

$$3 \tag{2.4.2.2}$$

PROOF COMPLETE!

A parameterized inequality

From Aharonov (2017)

```
> F:=1-(1-a[1]*t)^mu[1]*(1-a[2]*t)^(1-mu[1]);
```

$$F := 1 - \left(-a_1\, t + 1\right)^{\mu_1} \left(-a_2\, t + 1\right)^{1-\mu_1} \tag{3.1}$$

```
> map(expand,series(F,t,6));
```

$$\left(\mu_1 a_1 - a_2 \mu_1 + a_2\right) t + \left(-\frac{1}{2} a_2^2 \mu_1^2 + \frac{1}{2} a_2^2 \mu_1 + \mu_1^2 a_1 a_2 - \mu_1 a_1 a_2 - \frac{1}{2} \mu_1^2 a_1^2 \right. \tag{3.2}$$
$$\left. + \frac{1}{2} \mu_1 a_1^2\right) t^2 + \left(\frac{1}{6} a_2^3 \mu_1 - \frac{1}{6} a_2^3 \mu_1^3 + \frac{1}{2} \mu_1^3 a_1 a_2^2 - \frac{1}{2} \mu_1^2 a_1 a_2^2 + a_2 \mu_1^2 a_1^2 \right.$$
$$\left. - \frac{1}{2} a_2 \mu_1^3 a_1^2 - \frac{1}{2} a_2 \mu_1 a_1^2 + \frac{1}{3} \mu_1 a_1^3 - \frac{1}{2} \mu_1^2 a_1^3 + \frac{1}{6} \mu_1^3 a_1^3\right) t^3 + \left(\frac{1}{12} a_2^4 \mu_1 \right.$$
$$+ \frac{1}{24} a_2^4 \mu_1^2 - \frac{1}{12} a_2^4 \mu_1^3 - \frac{1}{24} a_2^4 \mu_1^4 - \frac{1}{6} \mu_1^2 a_1 a_2^3 + \frac{1}{6} \mu_1^4 a_1 a_2^3 - \frac{1}{4} \mu_1^4 a_1^2 a_2^2$$
$$+ \frac{1}{2} \mu_1^3 a_1^2 a_2^2 - \frac{1}{4} \mu_1^2 a_1^2 a_2^2 - \frac{1}{3} a_2 \mu_1 a_1^3 + \frac{5}{6} a_2 \mu_1^2 a_1^3 - \frac{2}{3} a_2 \mu_1^3 a_1^3 + \frac{1}{6} a_2 \mu_1^4 a_1^3$$
$$\left. + \frac{1}{4} \mu_1 a_1^4 - \frac{11}{24} \mu_1^2 a_1^4 + \frac{1}{4} \mu_1^3 a_1^4 - \frac{1}{24} \mu_1^4 a_1^4\right) t^4 + \left(-\frac{1}{4} a_2 \mu_1 a_1^4 + \frac{17}{24} a_2 \mu_1^2 a_1^4 \right.$$
$$- \frac{17}{24} a_2 \mu_1^3 a_1^4 + \frac{7}{24} a_2 \mu_1^4 a_1^4 - \frac{1}{24} a_2 \mu_1^5 a_1^4 - \frac{1}{6} \mu_1^2 a_1^3 a_2^2 - \frac{1}{3} \mu_1^4 a_1^3 a_2^2 + \frac{1}{12} \mu_1^5$$
$$a_1^3 a_2^2 - \frac{1}{12} \mu_1^2 a_1^2 a_2^3 + \frac{1}{12} \mu_1^4 a_1^2 a_2^3 + \frac{5}{12} \mu_1^3 a_1^3 a_2^2 - \frac{1}{12} \mu_1^5 a_1^2 a_2^3 + \frac{1}{24} \mu_1^5 a_1 a_2^4$$

$$+\frac{1}{12}\mu_1^3 a_1^2 a_2^3 - \frac{1}{12}\mu_1^2 a_1 a_2^4 - \frac{1}{24}\mu_1^3 a_1 a_2^4 + \frac{1}{12}\mu_1^4 a_1 a_2^4 - \frac{5}{12}\mu_1^2 a_1^5 + \frac{1}{20} a_2^5 \mu_1 + \frac{1}{24} a_2^5 \mu_1^2 - \frac{1}{24} a_2^5 \mu_1^3 - \frac{1}{24} a_2^5 \mu_1^4 - \frac{1}{120} a_2^5 \mu_1^5 + \frac{1}{5}\mu_1 a_1^5 + \frac{7}{24}\mu_1^3 a_1^5 - \frac{1}{12}\mu_1^4 a_1^5 + \frac{1}{120}\mu_1^5 a_1^5\Big) t^5 + \mathrm{O}(t^6)$$

The theorem is that all these coefficients are positive for a[1]>a[2]>0,0<mu[1]<1.

```
> map(signum,%) assuming a[1]>a[2],a[2]>0,0<mu[1],mu[1]<1;
```

$$t + t^2 + t^3 + t^4 + \mathrm{signum}\Big(\frac{17}{24} a_2 \mu_1 a_1^4 - \frac{17}{24} a_2 \mu_1^2 a_1^4 + \frac{7}{24} a_2 \mu_1^3 a_1^4 - \frac{1}{24} a_2 \mu_1^4 a_1^4 + \frac{5}{12}\mu_1^2 a_1^3 a_2^2 + \frac{1}{12}\mu_1^4 a_1^3 a_2^2 + \frac{1}{12}\mu_1^2 a_1^2 a_2^3 - \frac{1}{12}\mu_1^4 a_1^2 a_2^3 - \frac{1}{3}\mu_1^3 a_1^3 a_2^2 + \frac{1}{12}\mu_1^3 a_1^2 a_2^3 - \frac{1}{24}\mu_1^2 a_1 a_2^4 + \frac{1}{12}\mu_1^3 a_1 a_2^4 + \frac{1}{24}\mu_1^4 a_1 a_2^4 + \frac{7}{24}\mu_1^2 a_1^5 + \frac{1}{24} a_2^5 \mu_1 - \frac{1}{24} a_2^5 \mu_1^2 - \frac{1}{24} a_2^5 \mu_1^3 - \frac{1}{120} a_2^5 \mu_1^4 - \frac{5}{12}\mu_1 a_1^5 - \frac{1}{12}\mu_1^3 a_1^5 + \frac{1}{120}\mu_1^4 a_1^5 + \frac{1}{20} a_2^5 + \frac{1}{5} a_1^5 - \frac{1}{4} a_1^4 a_2 - \frac{1}{12} a_1 a_2^4 \mu_1 - \frac{1}{12} a_1^2 a_2^3 \mu_1 - \frac{1}{6} a_1^3 a_2^2 \mu_1\Big) t^5 + \mathrm{signum}(\mathrm{O}(1))\, t^6 \quad (3.3)$$

```
> rec:=collect(op(1,gfun:-seriestorec(series(F,t,20),u(n))),u,
  factor);
```

$$rec := \Big\{n\,(n-1)\,a_2\,a_1\,u(n) - n\,(a_1 n + a_2 n - \mu_1 a_1 + a_2 \mu_1 + a_1)\,u(n+1) + n\,(n+2)\,u(n+2),\ u(0)=0,\ u(1) = \mu_1 a_1 - a_2 \mu_1 + a_2,\ u(2) = -\frac{\mu_1 (a_1 - a_2)^2 (-1+\mu_1)}{2}\Big\} \quad (3.4)$$

```
> pol:=eval(op(1,rec),u=(k->X^(k-n)));
```

$$pol := n\,(n-1)\,a_2\,a_1 - n\,(a_1 n + a_2 n - \mu_1 a_1 + a_2 \mu_1 + a_1)\,X + n\,(n+2)\,X^2 \quad (3.5)$$

```
> ch:=lcoeff(%,n);
```

$$ch := a_1 a_2 - (a_1 + a_2)\,X + X^2 \quad (3.6)$$

```
> solve(%,X);
```

$$a_2,\ a_1 \quad (3.7)$$

```
> An:=Transpose(CompanionMatrix(pol/lcoeff(pol,X),X));
```

$$An := \begin{bmatrix} 0 & 1 \\ -\frac{(n-1)\,a_2\,a_1}{n+2} & -\frac{-a_1 n - a_2 n + \mu_1 a_1 - a_2 \mu_1 - a_1}{n+2} \end{bmatrix} \quad (3.8)$$

```
> A:=map(limit,An,n=infinity);
```

(3.9)

$$A := \begin{bmatrix} 0 & 1 \\ -a_1 a_2 & a_1 + a_2 \end{bmatrix} \quad (3.9)$$

```
> basis:=Matrix([[1,1],[a[1],a[2]]]);
```

$$basis := \begin{bmatrix} 1 & 1 \\ a_1 & a_2 \end{bmatrix} \quad (3.10)$$

Vandergraft's cone – leads to variable contraction index

```
> incone:=basis.Vector([c[1],c[2]]);
```

$$incone := \begin{bmatrix} c_1 + c_2 \\ a_1 c_1 + a_2 c_2 \end{bmatrix} \quad (3.1.1)$$

```
> out:=basis^(-1).An.incone;
```

$out :=$ (3.1.2)

$$\begin{bmatrix} -\frac{(n-1) a_2 a_1 (c_1 + c_2)}{(a_1 - a_2)(n+2)} + \left(-\frac{a_2}{a_1 - a_2} - \frac{-a_1 n - a_2 n}{(a_1 - } \cdots \\ \frac{(n-1) a_2 a_1 (c_1 + c_2)}{(a_1 - a_2)(n+2)} + \left(\frac{a_1}{a_1 - a_2} + \frac{-a_1 n - a_2 n +}{(a_1 - a_2} \cdots \end{bmatrix}$$

```
> out:=normal(out);
```

$$out := \begin{bmatrix} \frac{n a_1 c_1 - a_1 c_1 \mu_1 - a_2 c_2 \mu_1 + a_1 c_1 + 2 a_2 c_2}{n+2} \\ \frac{n a_2 c_2 + a_1 c_1 \mu_1 + a_2 c_2 \mu_1 + a_1 c_1}{n+2} \end{bmatrix} \quad (3.1.3)$$

```
> hyps:=0<mu[1],mu[1]<1,a[1]>a[2],a[2]>0,c[1]>0,c[2]>-c[1],c
  [2]<c[1];
```

$$hyps := 0 < \mu_1, \mu_1 < 1, a_2 < a_1, 0 < a_2, 0 < c_1, -c_1 < c_2, c_2 < c_1 \quad (3.1.4)$$

out[1]>0

```
> numer(out[1]);
```

$$n a_1 c_1 - a_1 c_1 \mu_1 - a_2 c_2 \mu_1 + a_1 c_1 + 2 a_2 c_2 \quad (3.1.5)$$

```
> SMTLIB[Satisfiable]({%<=0,n>=1,hyps},timelimit=5);
```

$$false \quad (3.1.6)$$

-out[2]<out[1]

```
> normal(out[1]+out[2]);
```

$$a_1 c_1 + a_2 c_2 \quad (3.1.7)$$

```
> SMTLIB[Satisfiable]({%<=0,hyps});
```

(3.1.8)

$$false \quad (3.1.8)$$

out[2]<out[1]

```
> normal(out[1]-out[2]);
```

$$\frac{n a_1 c_1 - n a_2 c_2 - 2 a_1 c_1 \mu_1 - 2 a_2 c_2 \mu_1 + 2 a_2 c_2}{n+2} \quad (3.1.9)$$

```
> pp:=collect(numer(%),n);
```

$$pp := (a_1 c_1 - a_2 c_2)\, n - 2 a_1 c_1 \mu_1 - 2 a_2 c_2 \mu_1 + 2 a_2 c_2 \quad (3.1.10)$$

```
> SMTLIB[Satisfiable]({pp<=0,n>=1,hyps},timelimit=5);
```

$$true \quad (3.1.11)$$

```
> SMTLIB[Satisfiable]({pp<=0,n>=100,hyps},timelimit=5);
```

$$true \quad (3.1.12)$$

```
> SMTLIB[Satisfiable]({pp<=0,n>=1000,hyps},timelimit=5);
```

$$true \quad (3.1.13)$$

```
> SMTLIB[Satisfy]({pp<=0,n>=1000,hyps},timelimit=5);
```

$$\left\{n = 1001, a_1 = \frac{131073}{131072}, a_2 = 1, c_1 = 1, c_2 = \frac{511}{512}, \mu_1 = \frac{127}{128}\right\} \quad (3.1.14)$$

The problem appears: a[1]-a[2] become small. A contraction index that depends on the difference can be found:

```
> SMTLIB[Satisfiable]({pp<=0,n>(a[1]+a[2])/(a[1]-a[2])+1,hyps}
  );
```

$$false \quad (3.1.15)$$

and this is close to optimal:

```
> SMTLIB[Satisfiable]({pp<=0,n>(a[1]+a[2])/(a[1]-a[2]),hyps});
```

$$true \quad (3.1.16)$$

Initial conditions can only be checked for fixed a_1,a_2.

Better cone

```
> incone;
```

$$\begin{bmatrix} c_1 + c_2 \\ a_1 c_1 + a_2 c_2 \end{bmatrix} \quad (3.2.1)$$

Boundary: c_1 ≥ 0,c_1+c_2 ≥ 0, but do not demand c_1-c_2 ≥ 0.

```
> hyps:=0<mu[1],mu[1]<1,a[1]>a[2],a[2]>0,c[1]>0,c[1]+c[2]>0;
```

$$hyps := 0 < \mu_1, \mu_1 < 1, a_2 < a_1, 0 < a_2, 0 < c_1, 0 < c_1 + c_2 \quad (3.2.2)$$

Contraction index

```
> out:=basis^(-1).An.incone;
```

$$out := \quad (3.2.1.1)$$

$$\left[\begin{array}{c} -\frac{(n-1)\,a_2\,a_1\,(c_1+c_2)}{(a_1-a_2)\,(n+2)} + \left(-\frac{a_2}{a_1-a_2} - \frac{-a_1\,n\cdot}{}\cdots \\ \frac{(n-1)\,a_2\,a_1\,(c_1+c_2)}{(a_1-a_2)\,(n+2)} + \left(\frac{a_1}{a_1-a_2} + \frac{-a_1\,n-}{(}\cdots \end{array}\right]$$

```
> out:=map(normal,%);
```

$$out := \left[\begin{array}{c} \frac{n\,a_1\,c_1 - a_1\,c_1\,\mu_1 - a_2\,c_2\,\mu_1 + a_1\,c_1 + 2\,a_2\,c_2}{n+2} \\ \frac{n\,a_2\,c_2 + a_1\,c_1\,\mu_1 + a_2\,c_2\,\mu_1 + a_1\,c_1}{n+2} \end{array}\right] \quad (3.2.1.2)$$

```
> out:=(n+2)*out;
```

$$out := \left[\begin{array}{c} n\,a_1\,c_1 - a_1\,c_1\,\mu_1 - a_2\,c_2\,\mu_1 + a_1\,c_1 + 2\,a_2\,c_2 \\ n\,a_2\,c_2 + a_1\,c_1\,\mu_1 + a_2\,c_2\,\mu_1 + a_1\,c_1 \end{array}\right] \quad (3.2.1.3)$$

out[1]>0:

```
> SMTLIB[Satisfiable]({out[1]<=0,n>=1,hyps});
```

$$false \quad (3.2.1.4)$$

out[1]+out[2]>0

```
> SMTLIB[Satisfiable]({out[1]+out[2]<=0,n>=1,hyps});
```

$$false \quad (3.2.1.5)$$

Initial conditions

```
> L:=map(normal,gfun:-rectoproc(rec,u(n),list)(5)):
> L[1..4];
```

$$\left[0, \mu_1\,a_1 - a_2\,\mu_1 + a_2, -\frac{1}{2}\,a_2^2\,\mu_1^2 + \frac{1}{2}\,a_2^2\,\mu_1 + \mu_1^2\,a_1\,a_2 - \mu_1\,a_1\,a_2 - \frac{1}{2}\,\mu_1^2\,a_1^2 + \frac{1}{2}\,\mu_1\,a_1^2, \frac{1}{6}\,a_2^3\,\mu_1 - \frac{1}{6}\,a_2^3\,\mu_1^3 + \frac{1}{2}\,\mu_1^3\,a_1\,a_2^2 - \frac{1}{2}\,\mu_1^2\,a_1\,a_2^2 + a_2\,\mu_1^2\,a_1^2 - \frac{1}{2}\,a_2\,\mu_1^3\,a_1^2 - \frac{1}{2}\,a_2\,\mu_1\,a_1^2 + \frac{1}{3}\,\mu_1\,a_1^3 - \frac{1}{2}\,\mu_1^2\,a_1^3 + \frac{1}{6}\,\mu_1^3\,a_1^3\right] \quad (3.2.2.1)$$

```
> out:=normal(basis^(-1).Vector([L[3],L[4]]));
```

$$out := \quad (3.2.2.2)$$

$$\left[\begin{array}{c} \frac{(\mu_1^2\,a_1^2 - 2\,\mu_1^2\,a_1\,a_2 + a_2^2\,\mu_1^2 - 3\,\mu_1\,a_1^2 + 6\,\mu_1\,a_1\,a_2 - 3\,a_2^2}{6}\cdots \\ -\frac{(\mu_1^2\,a_1^2 - 2\,\mu_1^2\,a_1\,a_2 + a_2^2\,\mu_1^2 - a_1^2 + 2\,(}{6}\cdots \end{array}\right]$$

```
> out[1];
```

(3.2.2.3)

$$\frac{1}{6}\left(\left(\mu_1^2 a_1^2 - 2\mu_1^2 a_1 a_2 + a_2^2\mu_1^2 - 3\mu_1 a_1^2 + 6\mu_1 a_1 a_2 - 3 a_2^2\mu_1 + 2a_1^2 - 4a_1 a_2 + 2a_2^2\right)\mu_1\right) \tag{3.2.2.3}$$

```
> SMTLIB[Satisfiable]({out[1]<=0,hyps},timelimit=5);
```

$$\mathit{false} \tag{3.2.2.4}$$

```
> normal(out[1]+out[2]);
```

$$-\frac{1}{2}a_2^2\mu_1^2 + \frac{1}{2}a_2^2\mu_1 + \mu_1^2 a_1 a_2 - \mu_1 a_1 a_2 - \frac{1}{2}\mu_1^2 a_1^2 + \frac{1}{2}\mu_1 a_1^2 \tag{3.2.2.5}$$

```
> SMTLIB[Satisfiable]({%<=0,hyps},timelimit=5);
```

$$\mathit{false} \tag{3.2.2.6}$$

PROOF COMPLETE!

Turan's inequality by a sequence of cones

```
> x:=3/4;
```

$$x := \frac{3}{4} \tag{4.1}$$

```
> L:=expand([seq(LegendreP(n,x)^2-LegendreP(n-1,x)*LegendreP(n+1,
  x),n=1..40)]):
> rec:=gfun:-listtorec(L,u(n),[ogf])[1];
```

$$rec := \left\{(-32n^4 - 272n^3 - 816n^2 - 1024n - 448)\,u(n) + (40n^4 + 396n^3 + 1446n^2 + 2313n + 1370)\,u(n+1) + (-40n^4 - 564n^3 - 2958n^2 - 6831n - 5852)\,u(n+2) + (32n^4 + 496n^3 + 2832n^2 + 7040n + 6400)\,u(n+3), u(0) = \frac{7}{32}, u(1) = \frac{175}{1024}, u(2) = \frac{8211}{65536}\right\} \tag{4.2}$$

```
> rec2:=op(1,rec):
> collect(rec2,u,factor);
```

$$-16(2n+7)(n+1)(n+2)^2 u(n) + (2n+5)(n+2)(20n^2+108n+137)\,u(n+1) - (2n+7)(n+4)(20n^2+132n+209)\,u(n+2) + 16(n+5)(2n+5)(n+4)^2 u(n+3) \tag{4.3}$$

```
> eval(rec2,u=(k->X^(k-n)));
```

$$-32n^4 - 272n^3 - 816n^2 - 1024n - 448 + (40n^4 + 396n^3 + 1446n^2 + 2313n + 1370)X + (-40n^4 - 564n^3 - 2958n^2 - 6831n - 5852)X^2 + (32n^4 + 496n^3 + 2832n^2 + 7040n + 6400)X^3 \tag{4.4}$$

```
> charpol:=collect(%,X,factor);
```

$$charpol := 16(n+5)(2n+5)(n+4)^2 X^3 - (2n+7)(n+4)(20n^2+132n+209)X^2 + (2n+5)(n+2)(20n^2+108n+137)X - 16(2n+7)(n \tag{4.5}$$

$+ 1)\ (n+2)^2$

```
> An:=LinearAlgebra[Transpose](LinearAlgebra[CompanionMatrix](charpol/lcoeff(charpol,X),X));
```

$$An := \begin{bmatrix} 0 & 1 & \cdots \\ 0 & 0 & \cdots \\ \frac{(2n+7)(n+1)(n+2)^2}{(n+5)(2n+5)(n+4)^2} & -\frac{(n+2)(20n^2+108n+137)}{16(n+5)(n+4)^2} & \cdots \end{bmatrix} \quad \textbf{(4.6)}$$

```
> ch:=lcoeff(charpol,n);
```

$$ch := 32X^3 - 40X^2 + 40X - 32 \quad \textbf{(4.7)}$$

```
> factor(%);
```

$$8(X-1)(4X^2 - X + 4) \quad \textbf{(4.8)}$$

All 3 roots have modulus 1 for x in (-1,1).

```
> numer(normal(subs(n=1/t,charpol)));
```

$$6400X^3t^4 + 7040X^3t^3 - 5852X^2t^4 + 2832X^3t^2 - 6831X^2t^3 + 1370Xt^4 + 496X^3t - 2958X^2t^2 + 2313Xt^3 - 448t^4 + 32X^3 - 564X^2t + 1446Xt^2 - 1024t^3 - 40X^2 + 396Xt - 816t^2 + 40X - 272t - 32 \quad \textbf{(4.9)}$$

```
> subs(t=1/n,gfun:-algeqtoseries(%,t,X,2));
```

$$\left[RootOf(4\_Z^3 - 5\_Z^2 + 5\_Z - 4) + \frac{1}{n}\left(\frac{8\,RootOf(4\_Z^3 - 5\_Z^2 + 5\_Z - 4)^2}{7} - \frac{23\,RootOf(4\_Z^3 - 5\_Z^2 + 5\_Z - 4)}{7} + \frac{8}{7} \right) + O\left(\frac{1}{n^2}\right) \right] \quad \textbf{(4.10)}$$

```
> allvalues(op(%));
```

$$1 - \frac{1}{n} + O\left(\frac{1}{n^2}\right),\ \frac{1}{8} + \frac{3I\sqrt{7}}{8} + \frac{\frac{8\left(\frac{1}{8} + \frac{3I\sqrt{7}}{8}\right)^2}{7} + \frac{41}{56} - \frac{69I\sqrt{7}}{56}}{n} + O\left(\frac{1}{n^2}\right),\ \frac{1}{8} - \frac{3I\sqrt{7}}{8} + \frac{\frac{8\left(\frac{1}{8} - \frac{3I\sqrt{7}}{8}\right)^2}{7} + \frac{41}{56} + \frac{69I\sqrt{7}}{56}}{n} + O\left(\frac{1}{n^2}\right) \quad \textbf{(4.11)}$$

```
> eigs:=%[1..2];
```

$$eigs := 1 - \frac{1}{n} + O\left(\frac{1}{n^2}\right),\ \frac{1}{8} + \frac{3I\sqrt{7}}{8} \quad \textbf{(4.12)}$$

$$+\frac{\frac{8\left(\frac{1}{8}+\frac{3\,\mathrm{I}\sqrt{7}}{8}\right)^{2}}{7}+\frac{41}{56}-\frac{69\,\mathrm{I}\sqrt{7}}{56}}{n}+\mathrm{O}\left(\frac{1}{n^{2}}\right)$$

```
> asympt(evalc(eigs[2]),n);
```

$$\frac{1}{8}+\frac{3\,\mathrm{I}\sqrt{7}}{8}+\frac{-\frac{3}{8}-\frac{9\,\mathrm{I}\sqrt{7}}{8}}{n}+\mathrm{O}\left(\frac{1}{n^{2}}\right) \qquad \textbf{(4.13)}$$

```
> aeigs2:=map(evalc,asympt(eigs[2]*subs(I=-I,eigs[2]),n))
  assuming x>0,x<1;
```

$$aeigs2 := 1-\frac{6}{n}+\mathrm{O}\left(\frac{1}{n^{2}}\right) \qquad \textbf{(4.14)}$$

```
> aeigs2:=asympt(sqrt(aeigs2),n);
```

$$aeigs2 := 1-\frac{3}{n}+\mathrm{O}\left(\frac{1}{n^{2}}\right) \qquad \textbf{(4.15)}$$

1 positive, 2 conjugates of smaller modulus.

Construction of the cones

```
> V1:=2*Vector([seq(eigs[1]^i,i=0..2)]);
```

$$V1 := \begin{bmatrix} 2 \\ 2-\frac{2}{n}+2\,\mathrm{O}\left(\frac{1}{n^{2}}\right) \\ 2\left(1-\frac{1}{n}+\mathrm{O}\left(\frac{1}{n^{2}}\right)\right)^{2} \end{bmatrix} \qquad \textbf{(4.1.1)}$$

```
> V1:=map(asympt,%,n,2);
```

$$V1 := \begin{bmatrix} 2 \\ 2-\frac{2}{n}+\mathrm{O}\left(\frac{1}{n^{2}}\right) \\ 2-\frac{4}{n}+\mathrm{O}\left(\frac{1}{n^{2}}\right) \end{bmatrix} \qquad \textbf{(4.1.2)}$$

```
> V1:=map(convert,V1,polynom);
```

$$V1 := \begin{bmatrix} 2 \\ 2-\frac{2}{n} \\ 2-\frac{4}{n} \end{bmatrix} \qquad \textbf{(4.1.3)}$$

```
> V2:=Vector([seq(eigs[2]^i,i=0..2)]);
```

$V2 :=$ **(4.1.4)**

$$\left[\begin{array}{c} 1 \ \ldots \\ \frac{1}{8}+\frac{3\,I\sqrt{7}}{8}+\frac{\frac{8\left(\frac{1}{8}+\frac{3\,I\sqrt{7}}{8}\right)^2}{7}+\frac{41}{56}-\frac{69\,I\sqrt{7}}{56}}{n} \ \ldots \\ \left(\frac{1}{8}+\frac{3\,I\sqrt{7}}{8}+\frac{\frac{8\left(\frac{1}{8}+\frac{3\,I\sqrt{7}}{8}\right)^2}{7}+\frac{41}{56}-\frac{69\,I\sqrt{7}}{56}}{n}\right. \ \ldots \end{array}\right]$$

```
> V2:=map(asympt,%,n,2);
```

$V2 :=$ **(4.1.5)**

$$\left[\begin{array}{c} 1 \ \ldots \\ \frac{1}{8}+\frac{3\,I\sqrt{7}}{8}+\frac{\frac{8\left(\frac{1}{8}+\frac{3\,I\sqrt{7}}{8}\right)^2}{7}+\frac{41}{56}}{n} \ \ldots \\ \left(\frac{1}{8}+\frac{3\,I\sqrt{7}}{8}\right)^2+\frac{2\left(\frac{1}{8}+\frac{3\,I\sqrt{7}}{8}\right)\left(\frac{8\left(\frac{1}{8}+\frac{3\,I\sqrt{7}}{8}\right.}{7}\right.}{n} \ \ldots \end{array}\right]$$

```
> V2:=map(evalc,map(convert,%,polynom));
```

$$V2 := \left[\begin{array}{c} 1 \\ \frac{1}{8}-\frac{3}{8\,n}+I\left(\frac{3\sqrt{7}}{8}-\frac{9\sqrt{7}}{8\,n}\right) \\ -\frac{31}{32}+\frac{93}{16\,n}+I\left(\frac{3\sqrt{7}}{32}-\frac{9\sqrt{7}}{16\,n}\right) \end{array}\right] \quad \textbf{(4.1.6)}$$

```
> V2:=map(asympt,V2,n);
```

(4.1.7)

$$V2 := \left[\begin{array}{c} 1 \\ \frac{1}{8} + \frac{3\,I\sqrt{7}}{8} + \frac{-\frac{3}{8} - \frac{9\,I\sqrt{7}}{8}}{n} \\ -\frac{31}{32} + \frac{3\,I\sqrt{7}}{32} + \frac{\frac{93}{16} - \frac{9\,I\sqrt{7}}{16}}{n} \end{array}\right] \quad (4.1.7)$$

```
> V3:=subs(I=-I,V2);
```

$$V3 := \left[\begin{array}{c} 1 \\ \frac{1}{8} - \frac{3\,I\sqrt{7}}{8} + \frac{-\frac{3}{8} + \frac{9\,I\sqrt{7}}{8}}{n} \\ -\frac{31}{32} - \frac{3\,I\sqrt{7}}{32} + \frac{\frac{93}{16} + \frac{9\,I\sqrt{7}}{16}}{n} \end{array}\right] \quad (4.1.8)$$

```
> basis:=Matrix([V1,V2,V3]);
```

$$basis := \quad (4.1.9)$$

$$\left[\begin{array}{ccc} 2 & 1 & \cdots \\ 2 - \frac{2}{n} & \frac{1}{8} + \frac{3\,I\sqrt{7}}{8} + \frac{-\frac{3}{8} - \frac{9\,I\sqrt{7}}{8}}{n} & \frac{1}{8} - \cdots \\ 2 - \frac{4}{n} & -\frac{31}{32} + \frac{3\,I\sqrt{7}}{32} + \frac{\frac{93}{16} - \frac{9\,I\sqrt{7}}{16}}{n} & -\frac{31}{32} \cdots \end{array}\right]$$

The nth cone is contracted into the next for n≥20

```
> out:=subs(n=n+1,basis^(-1)).An.basis.Vector([1,c+I*s,c-I*s])
  ;
```

$$out := \quad (4.2.1)$$

$$\left[\begin{array}{c} \cdots \\ \cdots \\ \cdots \end{array}\right]$$

```
> out:=normal(out);
```

$out :=$ **(4.2.2)**

$$\left[\begin{array}{c} \cdots \\ \cdots \\ \cdots \end{array}\right]$$

```
> pp:=normal(out[1]^2-out[2]*out[3]):
> denom(pp);
```

$$1032192\,(7\,n^2-35\,n+48)^2\,(n+5)^2\,(2\,n+5)^2\,(n+4)^4\,n^2 \quad \textbf{(4.2.3)}$$

```
> signum(%) assuming n>0;
```

$$1 \quad \textbf{(4.2.4)}$$

```
> pp:=collect(numer(pp),n,evalc);
```

$$pp := (-202309632\,c^2-202309632\,s^2+202309632)\,n^{14}+(-3034644480\,c^2 \quad \textbf{(4.2.5)}$$
$$-3034644480\,s^2+3843883008)\,n^{13}+(-4025645568\,\sqrt{7}\,c\,s$$
$$+2657270784\,\sqrt{7}\,s-5416411392\,c^2-14800101120\,s^2+4112575488\,c$$
$$+22246834176)\,n^{12}+(-62955099648\,\sqrt{7}\,c\,s+40047221760\,\sqrt{7}\,s$$
$$+121234159872\,c^2-5599754496\,s^2+79021329408\,c-11184832512)\,n^{11}$$
$$+(-270930495528\,\sqrt{7}\,c\,s+141801639168\,\sqrt{7}\,s+455122630620\,c^2$$
$$+212175240228\,s^2+572651073792\,c-536844550144)\,n^{10}$$
$$+(489923366184\,\sqrt{7}\,c\,s-609292879104\,\sqrt{7}\,s-3094820396892\,c^2$$
$$+699692340060\,s^2+1661300438784\,c-1152651845632)\,n^9$$

$$+\left(5608256338638\sqrt{7}\,c\,s-4515731637120\sqrt{7}\,s-19735349438637\,c^{2}\right.$$
$$\left.-1728527285331\,s^{2}-1417091856768\,c+4063885041664\right)n^{8}+\left(\right.$$
$$-1789375480668\sqrt{7}\,c\,s-507872104896\sqrt{7}\,s+8850252144762\,c^{2}$$
$$\left.-17705348366202\,s^{2}-28065364012224\,c+14192912269312\right)n^{7}+\left(\right.$$
$$-94024077039594\sqrt{7}\,c\,s+44228927294976\sqrt{7}\,s+269399919522375\,c^{2}$$
$$\left.-23685271545159\,s^{2}-107552609991168\,c-13403978761216\right)n^{6}+\left(\right.$$
$$-217015290357696\sqrt{7}\,c\,s+52665501630528\sqrt{7}\,s+216967607988768\,c^{2}$$
$$\left.+117611693968608\,s^{2}-172071276409536\,c-54099255289856\right)n^{5}$$
$$+\left(74271280853808\sqrt{7}\,c\,s-169798286754816\sqrt{7}\,s\right.$$
$$-2982628940242824\,c^{2}+273973455777672\,s^{2}+334460244790272\,c$$
$$\left.+98710988684288\right)n^{4}+\left(649607399544384\sqrt{7}\,c\,s\right.$$
$$-217024881652224\sqrt{7}\,s-11031834277480032\,c^{2}-422831698690464\,s^{2}$$
$$\left.+2531971435292160\,c+142581859549184\right)n^{3}+\left(\right.$$
$$-28945727010720\sqrt{7}\,c\,s+452152947185664\sqrt{7}\,s-16772533376360976\,c^{2}$$
$$\left.-1232441449893360\,s^{2}+5886629881116672\,c-635792238944256\right)n^{2}+\left(\right.$$
$$-1556365628468736\sqrt{7}\,c\,s+500488059073536\sqrt{7}\,s$$
$$-11919231142767360\,c^{2}+626528750770944\,s^{2}+6409042045584384\,c$$
$$\left.-1050211584147456\right)n-1156897105422336\sqrt{7}\,c\,s$$
$$-287666361409536\sqrt{7}\,s-3210654871397376\,c^{2}+2076806295954432\,s^{2}$$
$$+2730088510414848\,c-201423249227776$$

```
> pp:=simplify(%,{c^2+s^2=1});
```

$$pp:=-4025645568\left(\left(\left(-\frac{35306}{53487}+c\right)n^{12}+\left(-\frac{532090}{53487}+\frac{278819\,c}{17829}\right)n^{11}+\left(\right.\right.\right. \quad \textbf{(4.2.6)}$$
$$\left.-\frac{26376793}{748818}+\frac{537560507\,c}{7987392}\right)n^{10}+\left(\frac{113335729}{748818}-\frac{972070171\,c}{7987392}\right)n^{9}$$
$$+\left(\frac{3919905935}{3494484}-\frac{311569796591\,c}{223646976}\right)n^{8}+\left(\frac{2645167213}{20966904}\right.$$
$$\left.+\frac{49704874463\,c}{111823488}\right)n^{7}+\left(-\frac{28794874541}{2620863}+\frac{5223559835533\,c}{223646976}\right)n^{6}+\left(\right.$$
$$\left.-\frac{30477720851}{2329656}+\frac{376762656871\,c}{6988968}\right)n^{5}+\left(\frac{110545759606}{2620863}\right.$$
$$\left.-\frac{515772783707\,c}{27955872}\right)n^{4}+\left(\frac{15699137851}{291207}-\frac{1127790624209\,c}{6988968}\right)n^{3}$$
$$+\left(\frac{33501998855\,c}{4659312}-\frac{294370408324}{2620863}\right)n^{2}+\left(\frac{112584319189\,c}{291207}\right.$$
$$\left.-\frac{108612860042}{873621}\right)n+\frac{62427595792}{873621}+\frac{27895859988\,c}{97069}\right)s\sqrt{7}$$

$$+ 809238528\, n^{13} + \left(9383689728\, c^2 + 4112575488\, c + 7446733056\right) n^{12}$$
$$+ \left(126833914368\, c^2 + 79021329408\, c - 16784587008\right) n^{11}$$
$$+ \left(242947390392\, c^2 + 572651073792\, c - 324669309916\right) n^{10} + \big($$
$$-3794512736952\, c^2 + 1661300438784\, c - 452959505572\big)\, n^9 + \big($$
$$-18006822153306\, c^2 - 1417091856768\, c + 2335357756333\big)\, n^8$$
$$+ \left(26555600510964\, c^2 - 28065364012224\, c - 3512436096890\right) n^7$$
$$+ \left(293085191067534\, c^2 - 107552609991168\, c - 37089250306375\right) n^6$$
$$+ \left(99355914020160\, c^2 - 172071276409536\, c + 63512438678752\right) n^5 + \big($$
$$-3256602396020496\, c^2 + 334460244790272\, c + 372684444461960\big)\, n^4 + \big($$
$$-10609002578789568\, c^2 + 2531971435292160\, c - 280249839141280\big)\, n^3 + \big($$
$$-15540091926467616\, c^2 + 5886629881116672\, c - 1868233688837616\big)\, n^2$$
$$+ \left(-12545759893538304\, c^2 + 6409042045584384\, c - 423682833376512\right) n$$
$$- 5287461167351808\, c^2 + 2730088510414848\, c + 1875383046726656$$

```
> pp:=collect(pp,[n,c,s]);
```

$$pp := 809238528\, n^{13} + \left(9383689728\, c^2 + \left(-4025645568\sqrt{7}\, s + 4112575488\right) c\right. \quad \textbf{(4.2.7)}$$
$$\left. + 2657270784\sqrt{7}\, s + 7446733056\right) n^{12} + \big(126833914368\, c^2 + \big($$
$$-62955099648\sqrt{7}\, s + 79021329408\big)\, c + 40047221760\sqrt{7}\, s$$
$$- 16784587008\big)\, n^{11} + \big(242947390392\, c^2 + \big(-270930495528\sqrt{7}\, s$$
$$+ 572651073792\big)\, c + 141801639168\sqrt{7}\, s - 324669309916\big)\, n^{10} + \big($$
$$-3794512736952\, c^2 + \left(489923366184\sqrt{7}\, s + 1661300438784\right) c$$
$$- 609292879104\sqrt{7}\, s - 452959505572\big)\, n^9 + \big(-18006822153306\, c^2$$
$$+ \left(5608256338638\sqrt{7}\, s - 1417091856768\right) c - 4515731637120\sqrt{7}\, s$$
$$+ 2335357756333\big)\, n^8 + \big(26555600510964\, c^2 + \big(-1789375480668\sqrt{7}\, s$$
$$- 28065364012224\big)\, c - 507872104896\sqrt{7}\, s - 3512436096890\big)\, n^7$$
$$+ \big(293085191067534\, c^2 + \left(-94024077039594\sqrt{7}\, s - 107552609991168\right) c$$
$$+ 44228927294976\sqrt{7}\, s - 37089250306375\big)\, n^6 + \big(99355914020160\, c^2 + \big($$
$$-217015290357696\sqrt{7}\, s - 172071276409536\big)\, c + 52665501630528\sqrt{7}\, s$$
$$+ 63512438678752\big)\, n^5 + \big(-3256602396020496\, c^2 + \big(74271280853808\sqrt{7}\, s$$
$$+ 334460244790272\big)\, c - 169798286754816\sqrt{7}\, s + 372684444461960\big)\, n^4$$
$$+ \big(-10609002578789568\, c^2 + \big(649607399544384\sqrt{7}\, s$$
$$+ 2531971435292160\big)\, c - 217024881652224\sqrt{7}\, s - 280249839141280\big)\, n^3$$
$$+ \big(-15540091926467616\, c^2 + \big(-28945727010720\sqrt{7}\, s$$
$$+ 5886629881116672\big)\, c + 452152947185664\sqrt{7}\, s - 1868233688837616\big)\, n^2$$
$$+ \big(-12545759893538304\, c^2 + \big(-1556365628468736\sqrt{7}\, s$$
$$+ 6409042045584384\big)\, c + 500488059073536\sqrt{7}\, s - 423682833376512\big)\, n$$

$- 5287461167351808\, c^2 + \left(-1156897105422336 \sqrt{7}\, s + 2730088510414848\right) c - 287666361409536 \sqrt{7}\, s + 1875383046726656$

```
> lcoeff(pp,n);
```

$$809238528 \tag{4.2.8}$$

Ultimately positive.

```
> eval(pp,[c=RealBox(0,1),s=RealBox(0,1)]);
```

$$\langle \text{RealBox: } 1.87538e+15 \pm 1.18395e+16\rangle + 809238528\, n^{13} + \left(\langle \text{RealBox: } 7.44673e+09 \pm 3.11776e+10\rangle\right) n^{12} + \left(\langle \text{RealBox: -}1.67846e+10 \pm 4.78374e+11\rangle\right) n^{11} + \left(\langle \text{RealBox: -}3.24669e+11 \pm 1.90759e+12\rangle\right) n^{10} + \left(\langle \text{RealBox: -}4.5296e+11 \pm 8.36407e+12\rangle\right) n^{9} + \left(\langle \text{RealBox: } 2.33536e+12 \pm 4.62095e+13\rangle\right) n^{8} + \left(\langle \text{RealBox: -}3.51244e+12 \pm 6.06989e+13\rangle\right) n^{7} + \left(\langle \text{RealBox: -}3.70893e+13 \pm 7.66421e+14\rangle\right) n^{6} + \left(\langle \text{RealBox: } 6.35124e+13 \pm 9.84936e+14\rangle\right) n^{5} + \left(\langle \text{RealBox: } 3.72684e+14 \pm 4.23681e+15\rangle\right) n^{4} + \left(\langle \text{RealBox: -}2.8025e+14 \pm 1.54339e+16\rangle\right) n^{3} + \left(\langle \text{RealBox: -}1.86823e+15 \pm 2.26996e+16\rangle\right) n^{2} + \left(\langle \text{RealBox: -}4.23683e+14 \pm 2.43967e+16\rangle\right) n \tag{4.2.9}$$

```
> lowpol(%,n);
```

$$-9.964122459249664 \times 10^{15} + 809238528\, n^{13} - 2.3730867392 \times 10^{10}\, n^{12} - 4.95158365440 \times 10^{11}\, n^{11} - 2.232254391260 \times 10^{12}\, n^{10} - 8.817025617060 \times 10^{12}\, n^{9} - 4.3874111626323 \times 10^{13}\, n^{8} - 6.4211347144570 \times 10^{13}\, n^{7} - 8.03510133647687 \times 10^{14}\, n^{6} - 9.21423078447904 \times 10^{14}\, n^{5} - 3.864125630518392 \times 10^{15}\, n^{4} - 1.5714117576573344 \times 10^{16}\, n^{3} - 2.4567823311400432 \times 10^{16}\, n^{2} - 2.4820408448967936 \times 10^{16}\, n \tag{4.2.10}$$

```
> fsolve(%,n);
```

$$-9.8562047175071410423,\ -6.7907908966279380910,\ -1.7060595471881057049,\ -0.71203598723939354387,\ 44.576877138296534220 \tag{4.2.11}$$

45 is an upper bound. It can be reduced by inspection:

```
> pp:=subs(sqrt(7)=s7,pp);
```

$$pp := 809238528\, n^{13} + \left(9383689728\, c^2 + \left(-4025645568\, s7\, s + 4112575488\right) c + 2657270784\, s7\, s + 7446733056\right) n^{12} + \left(126833914368\, c^2 + \left(-62955099648\, s7\, s + 79021329408\right) c + 40047221760\, s7\, s - 16784587008\right) n^{11} + \left(242947390392\, c^2 + \left(-270930495528\, s7\, s + 572651073792\right) c + 141801639168\, s7\, s - 324669309916\right) n^{10} + \left(-3794512736952\, c^2 + \left(489923366184\, s7\, s + 1661300438784\right) c - 609292879104\, s7\, s - 452959505572\right) n^{9} + \left(-18006822153306\, c^2 + \left(5608256338638\, s7\, s - 1417091856768\right) c - 4515731637120\, s7\, s + 2335357756333\right) n^{8} + \left(26555600510964\, c^2 + \left(-1789375480668\, s7\, s - 28065364012224\right) c \tag{4.2.12}$$

$- 507872104896\, s7\, s - 3512436096890)\, n^7 + (293085191067534\, c^2 + ($
$-94024077039594\, s7\, s - 107552609991168)\, c + 44228927294976\, s7\, s$
$- 37089250306375)\, n^6 + (99355914020160\, c^2 + (-217015290357696\, s7\, s$
$- 172071276409536)\, c + 52665501630528\, s7\, s + 63512438678752)\, n^5 + ($
$-3256602396020496\, c^2 + (74271280853808\, s7\, s + 334460244790272)\, c$
$- 169798286754816\, s7\, s + 372684444461960)\, n^4 + ($
$-10609002578789568\, c^2 + (649607399544384\, s7\, s + 2531971435292160)\, c$
$- 217024881652224\, s7\, s - 280249839141280)\, n^3 + ($
$-15540091926467616\, c^2 + (-28945727010720\, s7\, s + 5886629881116672)\, c$
$+ 452152947185664\, s7\, s - 1868233688837616)\, n^2 + ($
$-12545759893538304\, c^2 + (-1556365628468736\, s7\, s$
$+ 6409042045584384)\, c + 500488059073536\, s7\, s - 423682833376512)\, n$
$- 5287461167351808\, c^2 + (-1156897105422336\, s7\, s$
$+ 2730088510414848)\, c - 287666361409536\, s7\, s + 1875383046726656$

```
> for i from 40 by -1 while
   SMTLIB[Satisfiable]({eval(pp,n=i)<0,s7^2-7=0,s7>0,c^2+s^2-1=
  0},timelimit=5)=false do print(i) od;
```

40
39
38
37
36
35
34
33
32
31
30
29
28
27
26
25
24
23
22
21
20 **(4.2.13)**

For $n \geq 20$, the cone is contracted

Initial conditions

```
> pp:=gfun:-rectoproc(rec,u(n),remember):
```

It is necessary to check positivity up to 20 and then that it is in the cone for n=20

```
> map(signum,{seq(pp(i),i=0..40)});
```

$$\{1\} \tag{4.3.1}$$

The vector is in the cone since the beginning:

```
> for i to 30 do V:=Vector([pp(i),pp(i+1),pp(i+2)]);W:=eval
  (basis,n=i)^(-1).V;test:=evalc(W[1]^2-W[2]*W[3]); print(i,
  evalb(test>0)) od:
```

1, *true*

2, *true*

3, *false*

4, *false*

5, *true*

6, *true*

7, *true*

8, *true*

9, *true*

10, *true*

11, *true*

12, *true*

13, *true*

14, *true*

15, *true*

16, *true*

17, *true*

18, *true*

19, *true*

20, *true*

21, *true*

22, *true*

23, *true*

24, *true*

25, *true*

26, *true*

27, *true*

28, *true*

29, *true*

30, *true* **(4.3.2)**