Introduction to D-finiteness and creative telescoping

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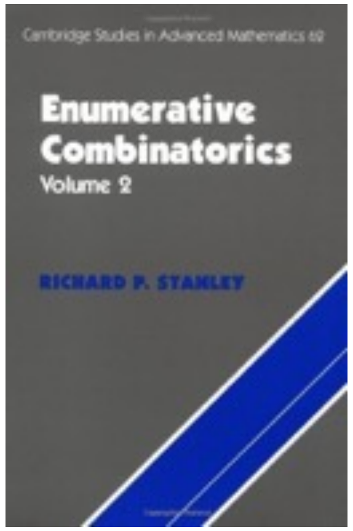
The objects of study

Def. A univariate power series is called D-finite when it is the solution of a linear differential equation with polynomial coefficients.

Exs: sin, cos, exp, log, arcsin, arccos, arctan, arcsinh, hypergeometric series, Bessel functions,…

Def. A sequence is P-recursive when it is the solution of a linear recurrence with polynomial coefficients.

Reference:



Prop. $f = \sum_{n=0}^{\infty} f_n z^n$ D-finite $\Longleftrightarrow$ f_n P-recursive.

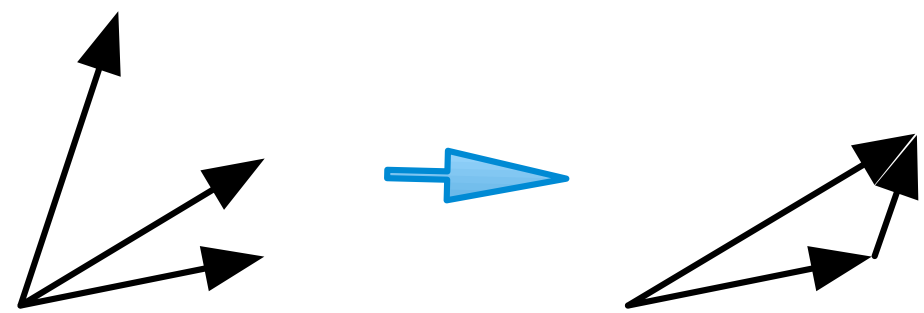
Classes of power series

on the blackboard

I. Closure properties

Bounds give proofs of identities

Confinement



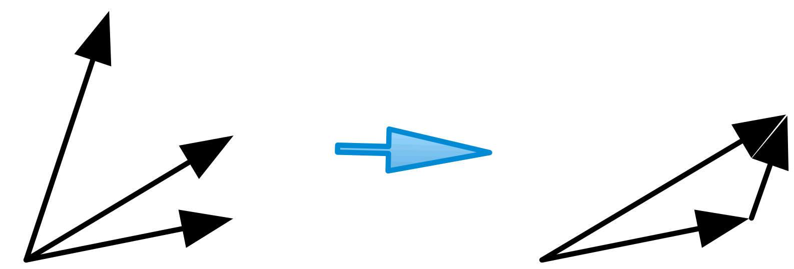
k+1 vectors in dimension k → an identity

LDE ⟺ the function and all its derivatives are confined in a finite dimensional vector space

⇒ the sum and product of solutions of LDEs satisfy LDEs

⇒ same property for P-recursive sequences

Proof technique



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> series(sin(x)^2+cos(x)^2-1,x,4);
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$O(x^4)$

f satisfies a LDE
$\Longleftrightarrow$
f,f',f'',… live in a finite-dim. vector space

Why is this a proof?

1. sin and cos satisfy a 2nd order LDE: $y''+y=0$;
2. their squares and their sum satisfy a 3rd order LDE;
3. the constant -1 satisfies $y'=0$;
4. thus $\sin^2+\cos^2-1$ satisfies a LDE of order at most 4;
5. Cauchy's theorem concludes.

Proofs of non-linear identities by linear algebra!

Example: Mehler's identity for Hermite polynomials

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\frac{u^n}{n!} = \frac{\exp\left(\frac{4u(xy-u(x^2+y^2))}{1-4u^2}\right)}{\sqrt{1-4u^2}}$$

1. Definition of Hermite polynomials:
   recurrence of order 2;
2. Product by linear algebra: $H_{n+k}(x)H_{n+k}(y)/(n+k)!$, $k \in \mathbb{N}$
   generated over $\mathbb{Q}(x,n)$ by
   $$\frac{H_n(x)H_n(y)}{n!}, \frac{H_{n+1}(x)H_n(y)}{n!}, \frac{H_n(x)H_{n+1}(y)}{n!}, \frac{H_{n+1}(x)H_{n+1}(y)}{n!}$$
   → recurrence of order at most 4;
3. Translate into differential equation.

II. Algebraic series

Algebraic series are D-finite

Thm. [Abel, Cockle] If the power series S(X) is a zero of the irreducible polynomial P(X,Y)∈$\mathbb{K}$[X,Y] of degree D in Y and char $\mathbb{K}$=0, then S(X) is solution of a linear differential equation of order at most D with coefficients in $\mathbb{K}$[X].

Proof (= Algorithm)

1. Invert P_Y mod P in $\mathbb{K}(X)[Y]$;
2. $S' = P_Y^{-1}(S)P_X(S) = Q_1(S)$ with $\deg_Y Q_1 < D$;
3. obtain $S^{(i)} = Q_i(S)$ for $i = 2, \ldots, D$ with $\deg_Y Q_i < D$;
4. linear algebra to eliminate $1, S^2, \ldots, S^{D-1}$.

A variant gives: F D-finite, S algebraic ⇒ F∘S D-finite.

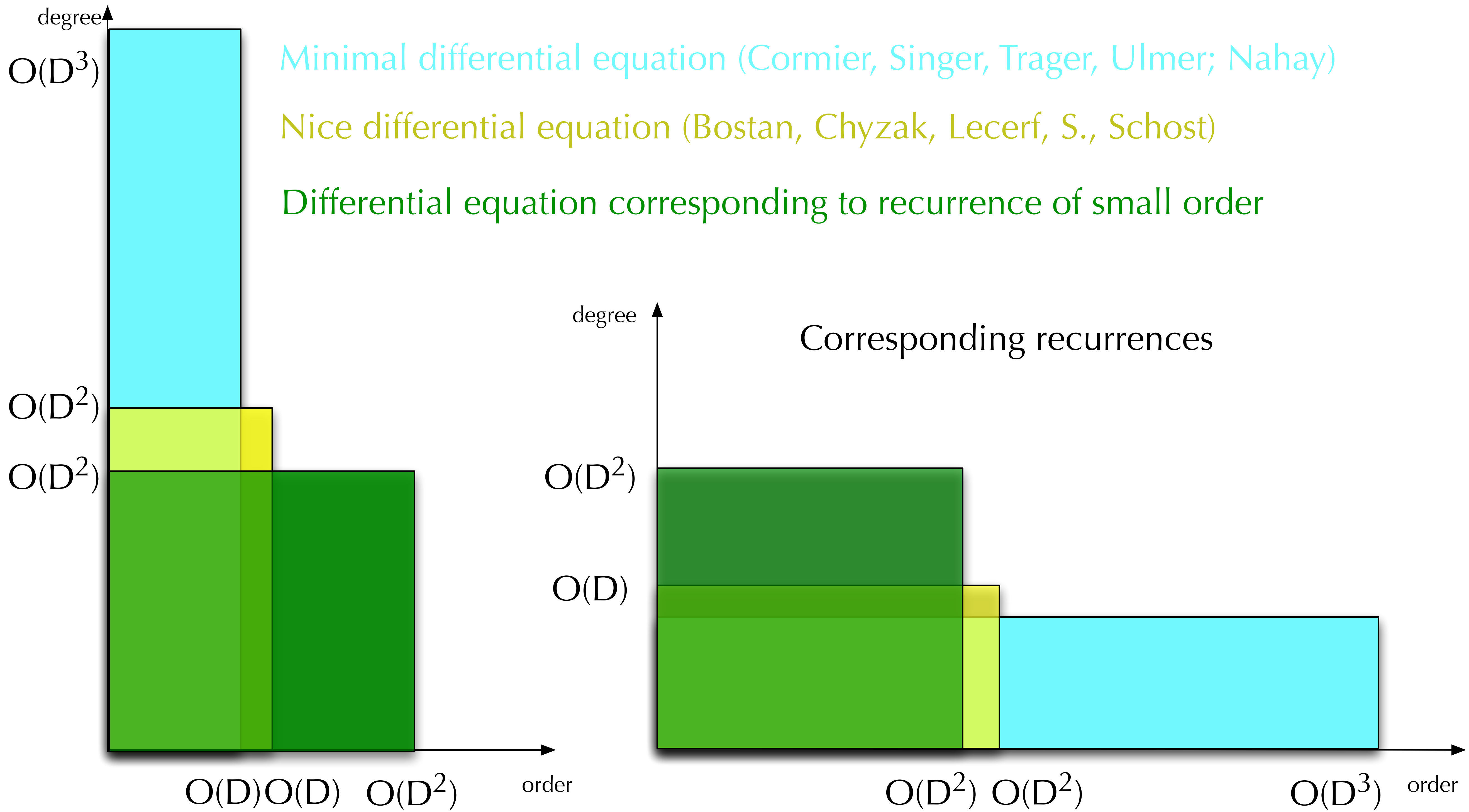
Minimality has a cost

(here P has *total* degree D)

Minimal differential equation (Cormier, Singer, Trager, Ulmer; Nahay)

Nice differential equation (Bostan, Chyzak, Lecerf, S., Schost)

Differential equation corresponding to recurrence of small order



proof by creative telescoping

A useful approximation result

$S(x) \in \mathbb{K}[[x]]$ zero of $P(x,y)$, irreducible of degrees d_x and d_y;
$L(x,\partial_x)$ a linear differential operator, δ_x and δ_∂ its degrees;
If $L(x,\partial_x) \cdot S(x) = O(x^\sigma)$, with $\sigma \geq 4 d_x d_y \delta_\partial + \delta_x d_y - 2 d_x \delta_\partial$, then

$$L(x,\partial_x) \cdot S(x) = 0.$$

III. Diagonals

$$\operatorname{Diag} \sum_{i_1,\ldots,i_m} a_{i_1,\ldots,i_m} x_1^{i_1} \cdots x_m^{i_m} := \sum_i a_{i,\ldots,i} x^i$$

Algebraicity/D-Finiteness

Diag F is **algebraic** (and conversely):

when $\mathsf{F} \in \mathbb{C}(\mathsf{X}, \mathsf{Y})$ [Pólya,Furstenberg]

$$\mathrm{Diag}\,\mathsf{F}(\mathsf{t}) = \frac{1}{2\pi\mathsf{i}} \oint_{|\mathsf{y}|=\epsilon} \mathsf{F}(\mathsf{t}/\mathsf{y}, \mathsf{y}) \frac{\mathsf{dy}}{\mathsf{y}}$$

= sum of residues

when $\mathsf{F} \in \mathbb{K}(\mathsf{X}_1, \ldots, \mathsf{X}_\mathsf{m})$ and $\mathsf{char}\mathbb{K} > 0$ [Furstenberg]

Diag F is **D-finite** when $\mathsf{F} \in \mathbb{K}(\mathsf{X}_1, \ldots, \mathsf{X}_\mathsf{m})$, arbitrary $\mathbb{K}$ [Lipshitz]

Equations for Diag F

$$F = \frac{A}{B} \in \mathbb{C}(X, Y)$$

$$d_x := \deg_x B > \deg_x A, \qquad d_y := \deg_y B > \deg_y A$$

Polynomial P(X,Y) s.t. P(X,Diag F)=0, with $\deg_y P = \binom{d_x + d_y}{d_x}$

(generically minimal) $\deg_x P$ similar

exponential size

Linear differential equation with

(order,degree)≤(O(d_x+d_y),O(d_x(d_x+d_y))).

polynomial size

direct computation by creative telescoping

IV. Creative Telescoping

Creative telescoping

$$I(x) = \int f(x,t)\,dt = ? \quad \text{or} \quad S(n) = \sum_k u(n,k) = ?$$

Input: equations (differential for *f* or recurrence for *u*).

Output: equations for the sum or the integral.

Method: integration (summation) by parts and differentiation (difference) under the integral (sum) sign

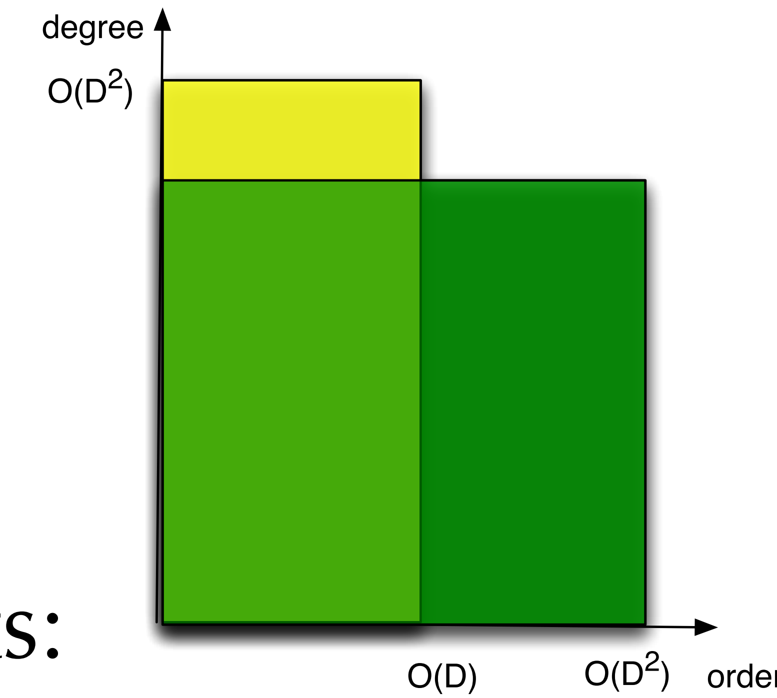
Example (with Pascal's triangle):

$$u(n,k) = \binom{n}{k} \text{ def. by } \left\{ \binom{n+1}{k} = \frac{n+1}{n+1-k}\binom{n}{k}, \binom{n}{k+1} = \frac{n-k}{k+1}\binom{n}{k} \right\}$$

$$S(n+1) = \sum_k \binom{n+1}{k} = \sum_k \underbrace{\binom{n+1}{k} - \binom{n+1}{k+1}}_{\text{telesc.}} + \underbrace{\binom{n}{k+1} - \binom{n}{k}}_{\text{telesc.}} + 2\binom{n}{k} = 2S(n).$$

Example: size of LDE for algebraic F

$$F(z) = \frac{1}{2\pi i} \oint \underbrace{\frac{yP_y(z,y)}{P(z,y)}}_{U(z,y)} \, dy$$



Differentiation under $\int$ and integration by parts:

find $\Lambda = A(z,\partial_z) + \partial_y B(z,\partial_z,y,\partial_y)$ s.t. $\Lambda \cdot U = 0$ and return A.

certificate

Bounds by counting dimensions

$$z^i \partial_z^j \partial_y^k \cdot U = \frac{Q}{P^{j+k+1}}, \qquad \deg Q \leq i + (j+k+1)D.$$

Taking $i \leq N_z,\ j+k \leq N_\partial,\ N_z = 4D^2, N_\partial = 4D,$

$$\dim(\text{lhs}) = (N_z+1)\binom{N_\partial+2}{2} > \dim(\text{rhs}) = \binom{(N_\partial+1)D+N_z+2}{2}.$$

Telescoping Ideal

$$\mathsf{T_t(f)} := \Big(\mathrm{Ann}\,\mathsf{f} + \underbrace{\partial_t\mathbb{Q}(\boldsymbol{x},t)\langle\boldsymbol{\partial_x},\partial_t\rangle}_{\text{int. by parts}}\Big) \cap \underbrace{\mathbb{Q}(\boldsymbol{x})\langle\boldsymbol{\partial_x}\rangle}_{\text{diff. under }\int}\,.$$

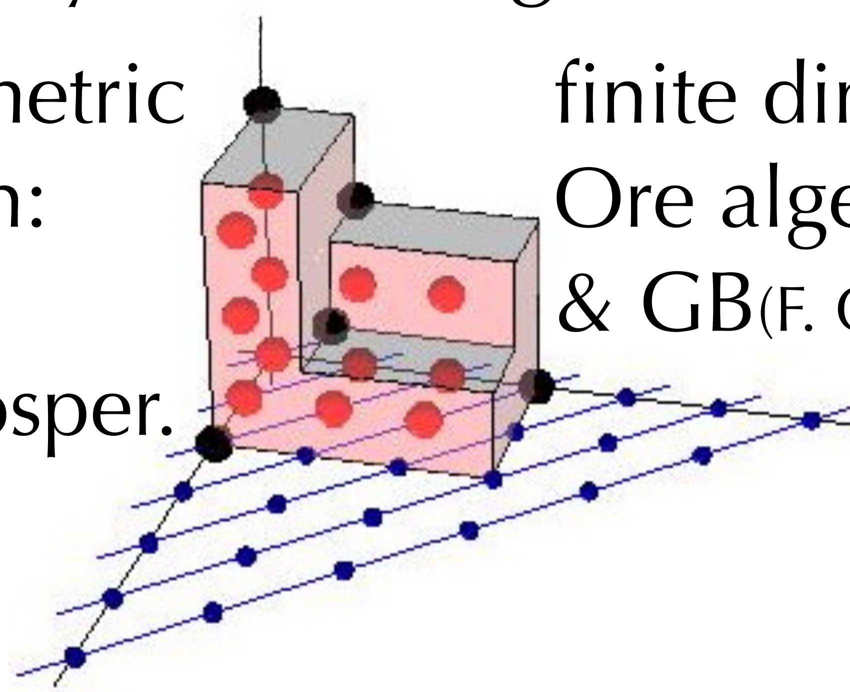
Approximated by:

1. Reducing the search space

   restrict int. by parts to $\mathbb{Q}(\boldsymbol{x})\langle\boldsymbol{\partial_x},\partial_{\mathsf{t}}\rangle$ and use Gröbner bases. (The « holonomic » approach) [Wilf-Zeilberger, also Sister Celine].

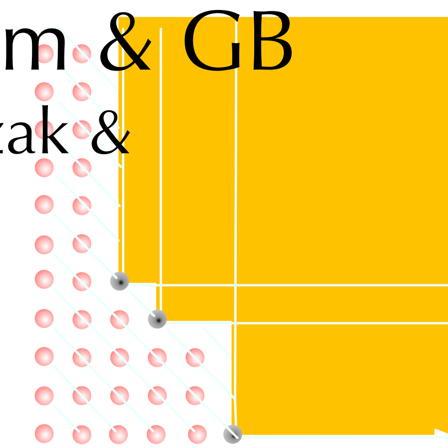
2. Proceeding by increasing slices (and indeterminate coeffs)

hypergeometric summation: dim=1 + param. Gosper. [Zeilberger]



finite dim, Ore algebras & GB(F. Chyzak)

infinite dim & GB (with F. Chyzak & M. Kauers)



Certificates are big

$$C_n := \sum_{r,s} \underbrace{(-1)^{n+r+s}\binom{n}{r}\binom{n}{s}\binom{n+s}{s}\binom{n+r}{r}\binom{2n-r-s}{n}}_{f_{n,r,s}}$$

$$(n+2)^3C_{n+2} - 2(2n+3)(3n^2+9n+7)C_{n+1} - (4n+3)(4n+4)(4n+5)C_n = 180\text{ kB} \simeq 2\text{ pages}$$

$$I(z) = \oint \frac{(1+t_3)^2 dt_1 dt_2 dt_3}{t_1t_2t_3(1+t_3(1+t_1))(1+t_3(1+t_2)) + z(1+t_1)(1+t_2)(1+t_3)^4}$$

$$z^2(4z+1)(16z-1)I'''(z) + 3z(128z^2+18z-1)I''(z) + (444z^2+40z-1)I'(z) + 2(30z+1)I(z) = 1\,080\text{ kB}$$

$$\simeq 12\text{ pages}$$

Next, in $T_t(f) := \Big(\operatorname{Ann} f + \underbrace{\partial_t \mathbb{Q}(\boldsymbol{x},t)\langle \boldsymbol{\partial_x}, \partial_t\rangle}_{\text{int. by parts}}\Big) \cap \underbrace{\mathbb{Q}(\boldsymbol{x})\langle \boldsymbol{\partial_x}\rangle}_{\text{diff. under } \int}$.

we restrict to integrals of rational f and $\partial_t \mathbb{Q}(\boldsymbol{x})[t, 1/\operatorname{den} f]\langle \boldsymbol{\partial_x}, \partial_t\rangle$

Bivariate integrals by Hermite reduction

[BostanChenChyzakLi10]

$$I(t) = \oint \frac{P(t,x)}{Q^m(t,x)}\,dx$$

Q square-free

Int. over a cycle where $Q \neq 0$.

If $m=1$, Euclidean division: $P=aQ+r$, $\deg_x r < \deg_x Q$

$$\frac{P}{Q} = \frac{r}{Q} + \partial_x \int a$$

Def. Reduced form: $\left[\frac{P}{Q}\right] := \frac{r}{Q}$

If $m>1$, Bézout identity and integration by parts

$$P = uQ + v\partial_x Q \quad \rightarrow \quad \frac{P}{Q^m} = \underbrace{\frac{u + \frac{\partial_x v}{m-1}}{Q^{m-1}}}_{A_{m-1}} + \partial_x \frac{v/(1-m)}{Q^{m-1}}$$

$$\left[\frac{P}{Q^m}\right] := [A_{m-1}]$$

Algorithm: $R_0 := [P/Q^m]$
for $i=1,2,\ldots$ do $R_i := [\partial_t R_{i-1}]$
when there is a relation $c_0(t)R_0 + \ldots + c_i(t)R_i = 0$
return $c_0 + \ldots + c_i\partial_t^i$

More variables: Griffiths-Dwork reduction

$$I(t) = \oint \frac{P(t,\underline{x})}{Q^m(t,\underline{x})}\,d\underline{x}$$

Q square-free
Int. over a cycle
where Q≠0.

1. Control degrees by homogenizing $(x_1,\dots,x_n)\mapsto(x_0,\dots,x_n)$
2. If $m=1$, $[P/Q]:=P/Q$
3. If $m>1$, reduce modulo Jacobian ideal $J := \langle \partial_0 Q, \dots, \partial_n Q\rangle$

$$P = r + v_0\partial_0 Q + \cdots + v_n \partial_n Q$$

$$\frac{P}{Q^m} = \frac{r}{Q^m} - \frac{1}{m-1}\left(\partial_0 \frac{v_0}{Q^{m-1}} + \cdots + \partial_n \frac{v_n}{Q^{m-1}}\right) + \underbrace{\frac{1}{m-1}\,\frac{\partial_0 v_0 + \cdots + \partial_n v_n}{Q^{m-1}}}_{A_{m-1}}$$

$$\left[\frac{P}{Q^m}\right] := \frac{r}{Q^m} + [A_{m-1}]$$

Thm. [Griffiths] In the regular case ($\mathbb{Q}(t)[\underline{x}]/J$ finite dim), if $R=P/Q^m$ hom of degree -n-1, $[R]=0 \Leftrightarrow \oint R\,d\underline{x} = 0$.

→ SAME ALGORITHM.

Size and complexity

$$I(t) = \oint \underbrace{\frac{P(t,\underline{x})}{Q^m(t,\underline{x})}}_{\in\mathbb{Q}(t,\underline{x})} \, d\underline{x}$$

no regularity assumed

$$N := \deg_{\underline{x}} Q, \quad d_t := \max(\deg_t Q, \deg_t P)$$

$\deg_x P$ not too big

Thm. [Bostan-Lairez-S. 2013] A linear differential equation for $I(t)$ can be computed in $O(e^{3n}N^{8n}d_t)$ operations in $\mathbb{Q}$. It has order $\leq N^n$ and degree $O(e^n N^{3n} d_t)$.

tight

Note: generically, the certificate has at least $N^{n^2/2}$ monomials.

Non-regular case by deformation, better way in Pierre Lairez's work.

Conclusion

Perhaps (tight) approximation theorems would be useful to circumvent the certificates?

Is any of this useful in approximation?

Other ideas?

Questions?