Newton's Iteration for Combinatorial Systems and Applications

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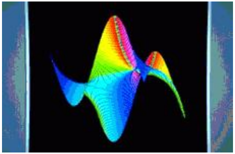
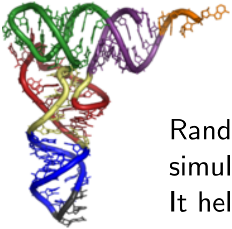


Joint work with Carine Pivoteau and Michèle Soria

Vancouver, May 17, 2011

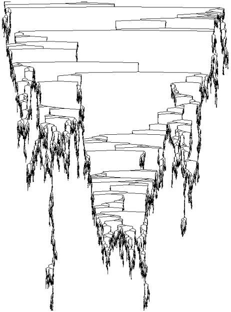
I Introduction

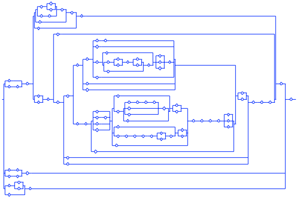
Motivation: Random Generation

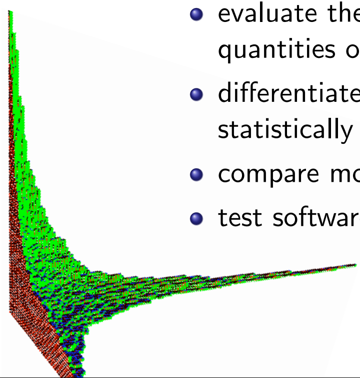


Random generation of large objects = simulation in the discrete world.
It helps

- evaluate the order of magnitude of quantities of interest;
- differentiate exceptional values from statistically expected ones;
- compare models;
- test software.







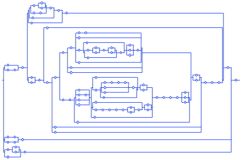
Framework: Constructible Species

A small set of species

$1, \mathcal{Z}, \times, +, \textsc{Seq}, \textsc{Set}, \textsc{Cyc},$
cardinality constraints that are finite unions of intervals,
used recursively.

Examples:

- Regular languages
- Unambiguous context-free languages
- Trees ($\mathcal{B} = \mathcal{Z} + \mathcal{Z} \times \mathcal{B}^2$, $\mathcal{T} = \mathcal{Z} \times \textsc{Set}(\mathcal{T})$)
- Mappings, ...



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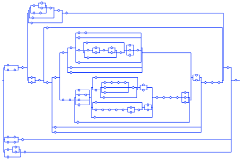

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- Mappings, . . .

Two related problems:

1. Enumeration: number of objects of size n for $n = 0, 1, 2, \ldots$.
2. Random generation: all objects of size n with the same proba.

Two contexts: labelled/unlabelled.

Boltzmann Samplers

Principle (Duchon, Flajolet, Louchard, Schaeffer 2004)

Generate each $t \in \mathcal{T}$ with probability $x^{|t|}/T(x)$, where: $x > 0$ fixed; $T(z) := \sum_{t\in\mathcal{T}} z^{|t|}$ = generating series of $\mathcal{T}$; $|t|$ = size.

Same size, same probability

Expected size $xT'(x)/T(x)$ increases with x.

Complexity linear in $|t|$ when the values $T(x)$ are available.

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Cartesian Product $\mathcal{C} = \mathcal{A} \times \mathcal{B}$

- Generate $a \in \mathcal{A}$; $b \in \mathcal{B}$;
- Return (a, b).

Proof. $C(x) = \sum_{(a,b)} x^{|a|+|b|} = A(x)B(x)$; $\frac{x^{|a|+|b|}}{C(x)} = \frac{x^{|a|}}{A(x)} \frac{x^{|b|}}{B(x)}$.

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- Draw $b = \text{Bernoulli}(A(x)/C(x))$;
- If $b = 1$ then generate $a \in \mathcal{A}$ else generate $b \in \mathcal{B}$.

Proof. $\frac{x^{|a|}}{C(x)} = \frac{x^{|a|}}{A(x)} \frac{A(x)}{C(x)}$.

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Use recursively (e.g., binary trees $\mathcal{B} = \mathcal{Z} \cup \mathcal{Z} \times \mathcal{B} \times \mathcal{B}$)

Also: sets, cycles,...;

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Also: sets, cycles,...; labelled case

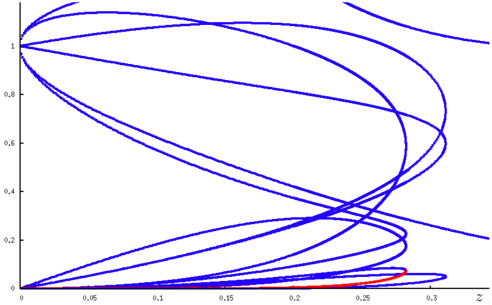
Complexity linear in $|t|$ when the values $T(x)$ are available.

Oracle: Large Systems that are Interesting to Solve

The generating series are given by systems of equations.

We need:

- only one solution;
- the right one;
- only numerically.



In the worst case, these requirements would make no difference.
But these systems inherit structure from combinatorics.

Results (1/2): Fast Enumeration

Theorem (Enumeration in Quasi-Optimal Complexity)

First N coefficients of gfs of constructible species in

1. *arithmetic complexity:*
   - $O(N \log N)$ *(both ogf and egf);*
2. *binary complexity:*
   - $O(N^2 \log^2 N \log\log N)$ *(ogf);*
   - $O(N^2 \log^3 N \log\log N)$ *(egf).*

Results (2/2): Oracle

1. The egfs and the ogfs of constructible species are convergent in the neighborhood of 0;
2. A numerical iteration converging to $\mathbf{Y}(\alpha)$ in the labelled case (inside the disk);
3. A numerical iteration converging to the sequence $\mathbf{Y}(\alpha), \mathbf{Y}(\alpha^2), \mathbf{Y}(\alpha^3), \dots$ for $\|\cdot\|_\infty$ in the unlabelled case (inside the disk).

Examples (I): Polynomial Systems

Random generation following given XML grammars

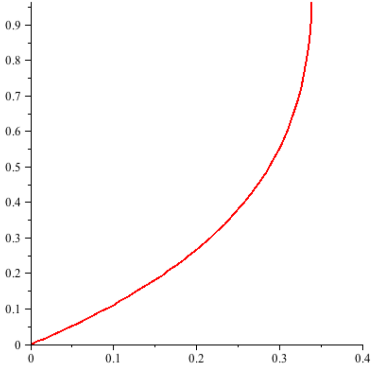
Grammar	nb eqs	max deg	nb sols	oracle (s.)	FGb (s.)
rss	10	5	2	0.02	0.03
PNML	22	4	4	0.05	0.1
xslt	40	3	10	0.4	1.5
relaxng	34	4	32	0.4	3.3
xhtml-basic	53	3	13	1.2	18
mathml2	182	2	18	3.7	882
xhtml	93	6	56	3.4	1124
xhtml-strict	80	6	32	3.0	1590
xmlschema	59	10	24	0.5	6592
SVG	117	10		5.8	$>$1.5Go
docbook	407	11		67.7	$>$1.5Go
OpenDoc	500			3.9	

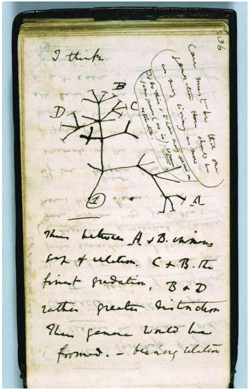
[Darrasse 2008]

Example (II): A Non-Polynomial "System"

Unlabelled rooted trees:

$$f(x) = x\exp\left(f(x) + \frac{1}{2}f(x^2) + \frac{1}{3}f(x^3) + \cdots\right)$$

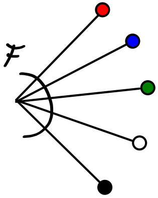




II Combinatorics

Mini-Introduction to Species

- Species $\mathcal{F}$:



Examples:

- 0, 1, $\mathcal{Z}$;
- Set;
- Seq, Cyc.

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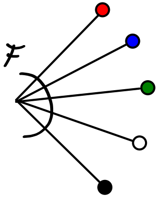
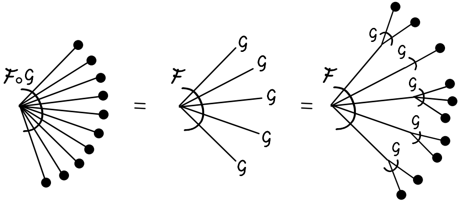


- Composition $\mathcal{F} \circ \mathcal{G}$:



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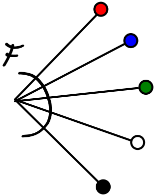

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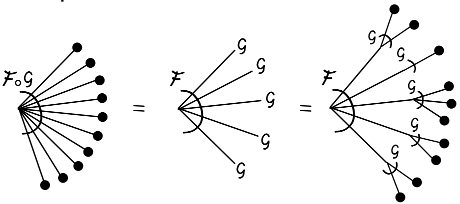

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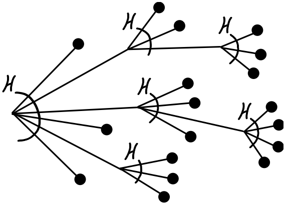
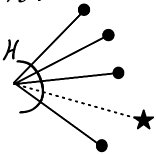


Derivative

$\mathcal{H}'$:



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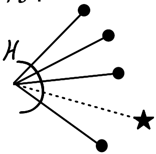
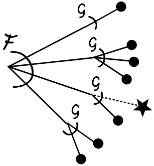


$(\mathcal{F} \circ \mathcal{G})'$:



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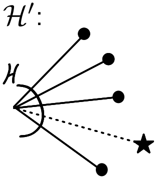

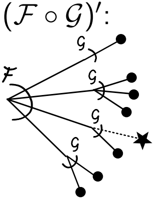

species	derivative
$\mathcal{A}+\mathcal{B}$	$\mathcal{A}'+\mathcal{B}'$
$\mathcal{A}\cdot\mathcal{B}$	$\mathcal{A}'\cdot\mathcal{B}+\mathcal{A}\cdot\mathcal{B}'$
$\textsc{Seq}(\mathcal{B})$	$\textsc{Seq}(\mathcal{B})\cdot\mathcal{B}'\cdot\textsc{Seq}(\mathcal{B})$
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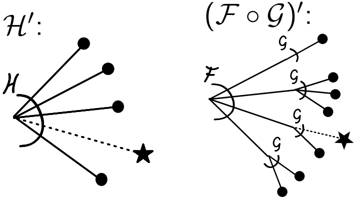

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Example:

$$\boldsymbol{\mathcal{H}}(\mathcal{G},\mathcal{S},\mathcal{P}) := (\mathcal{S}+\mathcal{P}, \mathsf{Seq}(\mathcal{Z}+\mathcal{P}), \mathsf{Set}(\mathcal{Z}+\mathcal{S}))\,.$$

$$\frac{\partial\boldsymbol{\mathcal{H}}}{\partial\boldsymbol{\mathcal{Y}}} = \begin{pmatrix} \varnothing & \mathcal{E} & \mathcal{E} \\ \varnothing & \varnothing & \mathsf{Seq}(\mathcal{Z}+\mathcal{P})\cdot\mathcal{E}\cdot\mathsf{Seq}(\mathcal{Z}+\mathcal{P}) \\ \varnothing & \mathsf{Set}(\mathcal{Z}+\mathcal{S})\cdot\mathcal{E} & \varnothing \end{pmatrix}$$

Joyal's Implicit Species Theorem

Theorem

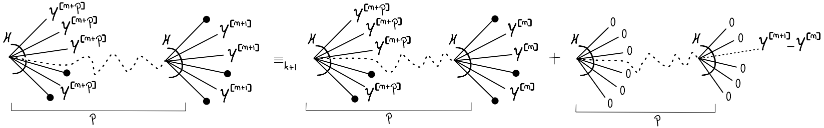
If $\mathcal{H}(0,0) = 0$ *and* $\partial\mathcal{H}/\partial\mathcal{Y}(0,0)$ *is nilpotent, then* $\mathcal{Y} = \mathcal{H}(\mathcal{Z},\mathcal{Y})$ *has a unique solution, limit of*

$$\mathcal{Y}^{[0]} = 0, \qquad \mathcal{Y}^{[n+1]} = \mathcal{H}(\mathcal{Z},\mathcal{Y}^{[n]}) \quad (n \geq 0).$$

Def. $\mathcal{A} =_k \mathcal{B}$ if they coincide up to size k (*contact* k).

Key Lemma

If $\mathcal{Y}^{[n+1]} =_k \mathcal{Y}^{[n]}$, then $\mathcal{Y}^{[n+p+1]} =_{k+1} \mathcal{Y}^{[n+p]}$, ($p =$ dimension).



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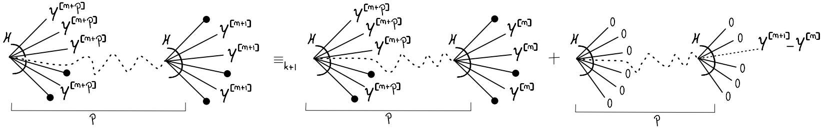

Newton Iteration for Binary Trees

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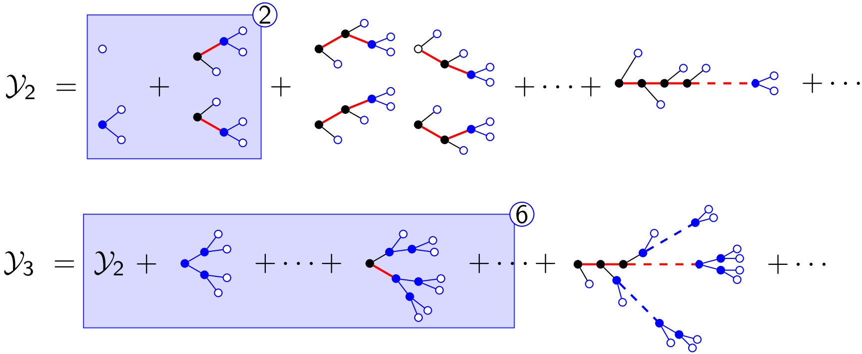

[Décoste, Labelle, Leroux 1982]

Newton Iteration for Binary Trees

$$\mathcal{Y} = \mathcal{E} \cup \mathcal{Z} \times \mathcal{Y}^2$$

$$\mathcal{Y}_{n+1} = \mathcal{Y}_n \cup \operatorname{SEQ}(\mathcal{Z} \times \mathcal{Y}_n \times \square \cup \mathcal{Z} \times \square \times \mathcal{Y}_n) \times (\mathcal{E} \cup \mathcal{Z} \times \mathcal{Y}_n^2 \setminus \mathcal{Y}_n).$$

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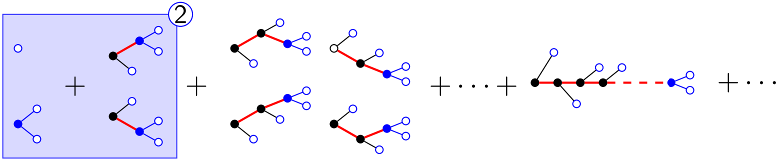
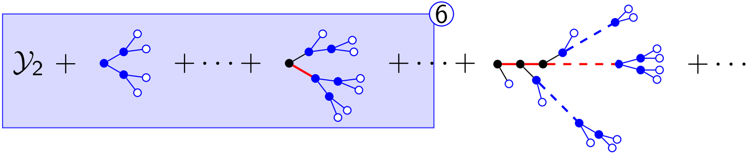
$\mathcal{Y}_2 =$



$\mathcal{Y}_3 =$



[Décoste, Labelle, Leroux 1982]

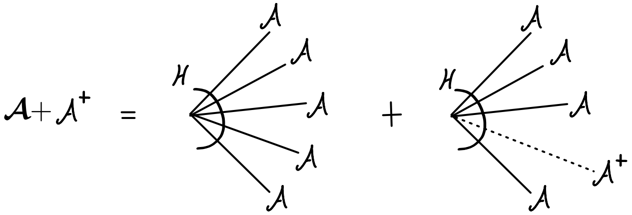
Combinatorial Newton Iteration

Theorem (essentially Labelle)

For any well-founded system $\mathcal{Y} = \mathcal{H}(\mathcal{Z}, \mathcal{Y})$, *if* $\mathcal{A}$ *has contact* k *with the solution and* $\mathcal{A} \subset \mathcal{H}(\mathcal{Z}, \mathcal{A})$, *then*

$$\mathcal{A} + \sum_{i \geq 0} \left(\frac{\partial \mathcal{H}}{\partial \mathcal{Y}}(\mathcal{Z}, \mathcal{A}) \right)^i \cdot (\mathcal{H}(\mathcal{Z}, \mathcal{A}) - \mathcal{A})$$

has contact $2k+1$ *with it.*



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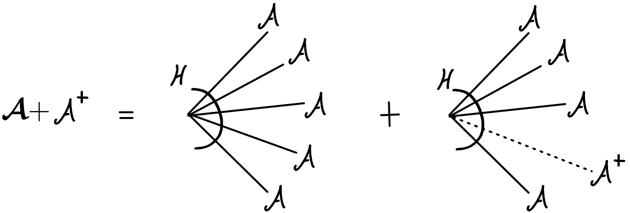

Generation by increasing Strahler numbers.

III Newton Iteration for Power Series

Newton Did It in 1671!

$y^3 + a^2y - 2a^3 + axy - x^3 = 0.\quad y = a - \frac{x}{4} + \frac{x^2}{64a} + \frac{131x^3}{512a^2} + \frac{509x^4}{16384a^3}$ &c.

$+a+p=y.$	$+y^3$ $+axy$ $+a^2y$ $-x^3$ $-2a^3$	$+a^3 + 3a^2p + 3ap^2 + p^3$ $+a^2x + axp$ $+a^3 + a^2p$ $-x^3$ $-2a^3$
$-\frac{1}{4}x+q=p.$	$+p^3$ $+3ap^2$ $+axp$ $+4a^2p$ $+a^2x$ $-x^3$	$-\frac{1}{64}x^3 + \frac{3}{16}x^2q - \frac{3}{4}xq^2 + q^3$ $+\frac{3}{16}ax^2 - \frac{3}{2}axq + 3aq^2$ $-\frac{1}{4}ax^2 + axq$ $-a^2x + 4a^2q$ $+a^2x$ $-x^3$
$+\frac{x^2}{64a}+r=q.$	$+q^3$ $-\frac{3}{4}xq^2$ $+3aq^2$ $+\frac{3}{16}x^2q$ $-\frac{3}{2}axq$ $+4a^2q$ $-\frac{65}{64}x^3$ $-\frac{1}{16}ax^2$	* * $+\frac{3x^4}{4096a}$ * $-\frac{3}{32}x^2r + 3ar^2$ $+\frac{3x^4}{1024a}$ * $+\frac{3}{16}x^2r$ $-\frac{3}{128}x^3$ $-\frac{3}{2}axr$ $+\frac{1}{16}ax^2$ $+4a^2r$ $-\frac{65}{64}x^3$ $-\frac{1}{16}ax^2$

$+4a^2 - \frac{3}{2}ax + \frac{9}{32}x^2) + \frac{131}{128}x^3 - \frac{15x^4}{4096a}\left(+\frac{131x^3}{512a^2} + \frac{509x^4}{16384a^3}\right.$

Generating Series: a Simple Dictionary

$$\text{ogf} := \sum_{t\in\mathcal{T}} z^{|t|}, \quad \text{egf} := \sum_{t\in\mathcal{T}} \frac{z^{|t|}}{|t|!}.$$

Language and Gen. Fcns (labelled)

$\mathcal{A}\cup\mathcal{B}$	$A(z)+B(z)$
$\mathcal{A}\times\mathcal{B}$	$A(z)\times B(z)$
$\textsc{Seq}(\mathcal{C})$	$\frac{1}{1-C(z)}$
$\mathcal{A}'$	$A'(z)$
$\textsc{Cyc}(\mathcal{C})$	$\log\frac{1}{1-C(z)}$
$\textsc{Set}(\mathcal{C})$	$\exp(C(z))$

Consequences:

1. Newton iteration for EGFs easy;

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$\textsc{Seq}(\mathcal{C})$	$\frac{1}{1-C(z)}$	$\frac{1}{1-C(z)}$
$\mathcal{A}'$	$A'(z)$	—
$\textsc{Cyc}(\mathcal{C})$	$\log\frac{1}{1-C(z)}$	$\sum_{k\geq 1}\frac{\phi(k)}{k}\log\frac{1}{1-C(z^k)}$
$\textsc{Set}(\mathcal{C})$	$\exp(C(z))$	$\exp(\sum C(z^i)/i)$

Consequences:

1. Newton iteration for EGFs easy;
2. Pólya operators for ogfs.

Newton Iteration for Power Series has Good Complexity

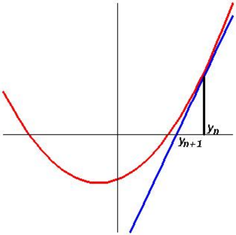
To solve $\phi(y) = 0$, iterate

$$y^{[n+1]} = y^{[n]} - u^{[n+1]}, \quad \phi'(y^{[n]})u^{[n+1]} = \phi(y^{[n]}).$$

Quadratic convergence

$\Updownarrow$

Divide-and-Conquer



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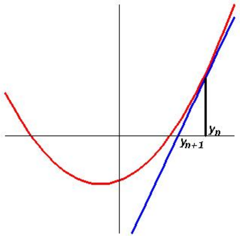

Quadratic convergence
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To solve at precision N

1. Solve at precision $N/2$;
2. Compute ϕ and ϕ' there;
3. Solve for $u^{[n+1]}$.

$$\mathrm{Cost}(y^{[n]}) = \mathrm{constant} \times \mathrm{Cost}(\text{last step}).$$

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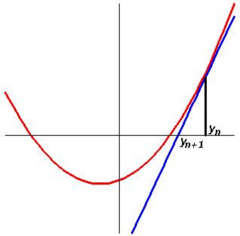

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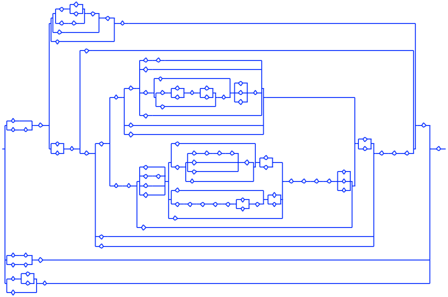
Useful in conjunction with fast multiplication (quasi-linear):

- power series at order N: $O(N \log N)$ ops on the coefficients;
- N-bit integers: $O(N \log N \log\log N)$ bit ops.

Example: Series-Parallel Graphs

$$\begin{cases} \mathcal{G} & = \mathcal{S} + \mathcal{P}, \\ \mathcal{S} & = \mathrm{Seq}(\mathcal{Z} + \mathcal{P}), \\ \mathcal{P} & = \mathrm{Set}_{>0}(\mathcal{Z} + \mathcal{S}). \end{cases} \qquad \frac{\partial \boldsymbol{\mathcal{H}}}{\partial \boldsymbol{\mathcal{Y}}} = \begin{pmatrix} \varnothing & \mathcal{E} & \mathcal{E} \\ \varnothing & \varnothing & \mathrm{Seq}^2(\mathcal{Z} + \mathcal{P}) \\ \varnothing & \mathrm{Set}(\mathcal{Z} + \mathcal{S}) & \varnothing \end{pmatrix}$$



Example: Series-Parallel Graphs (labelled case)

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Newton iteration: $\mathbf{Y}^{[n]} := \begin{pmatrix} G^{[n]} \\ S^{[n]} \\ P^{[n]} \end{pmatrix}$,

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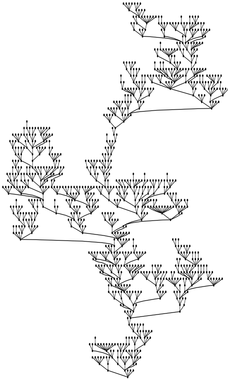
$\Rightarrow$ Wanted: efficient matrix inverse, efficient exp.

Example: Unlabelled Rooted Trees

1. Well-founded system: $\mathcal{Y} = \mathcal{Z} \cdot \textsc{Set}(\mathcal{Y}) =: \mathcal{H}(\mathcal{Z}, \mathcal{Y})$;



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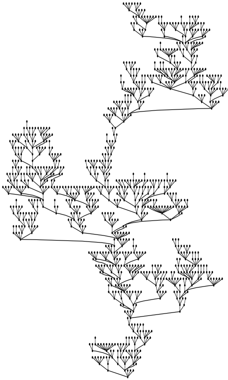

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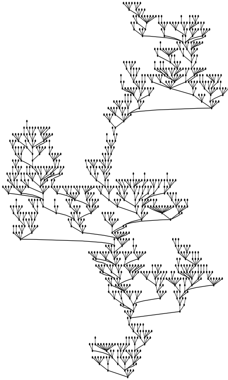

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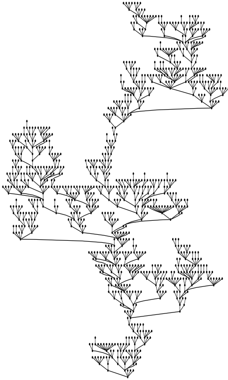

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4. Newton for OGF:

$$\tilde{Y}^{[n+1]} = \tilde{Y}^{[n]} + \frac{H(z, \tilde{Y}^{[n]}) - \tilde{Y}^{[n]}}{1 - H(z, \tilde{Y}^{[n]})}$$

0,

$z + z^2 + z^3 + z^4 + \cdots$,

$z + z^2 + 2z^3 + 4z^4 + 9z^5 + 20z^6 + \cdots$

Newton Iteration for Inverses

$$\phi(y) = a - 1/y \Rightarrow 1/\phi'(y) = y^2 \Rightarrow \boxed{y^{[n+1]} = y^{[n]} - y^{[n]}(ay^{[n]} - 1).}$$

Cost: a small number of multiplications

Works for:

1. Numerical inversion;
2. Reciprocal of power series;
3. Inversion of matrices.

Applications:

- Seq
- $(I - \frac{\partial H}{\partial Y})^{-1}$

[Schulz 1933; Cook 1966; Sieveking 1972; Kung 1974]

Inverses for Series-Parallel Graphs

$$\begin{cases} G &= S + P, \\ S &= (1 - z - P)^{-1}, \\ P &= \exp(z + S) - 1. \end{cases} \qquad \frac{\partial \mathbf{H}}{\partial \mathbf{Y}} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & (1 - z - P)^{-2} \\ 0 & \exp(z + S) & 0 \end{pmatrix}$$

Newton iteration:

$$\begin{cases} U^{[n+1]} &= U^{[n]} + U^{[n]} \cdot \left(\frac{\partial \mathbf{H}}{\partial \mathbf{Y}}(\mathbf{Y}^{[n]}) \cdot U^{[n]} + \mathrm{Id} - U^{[n]} \right) \bmod z^{2^n}, \\ \mathbf{Y}^{[n+1]} &= \mathbf{Y}^{[n]} + U^{[n+1]} \cdot \left(\mathbf{H}(\mathbf{Y}^{[n]}) - \mathbf{Y}^{[n]} \right) \bmod z^{2^{n+1}}. \end{cases}$$

Can be lifted combinatorially.
Also a numerical iteration!

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Can be lifted combinatorially.
Also a numerical iteration!

$\Rightarrow$ Wanted: efficient exp.

From the Inverse to the Exponential

1. Logarithm of power series: $\log f = \int (f'/f)$;
2. exponential of power series: $\phi(y) = a - \log y$.

$$e^{[n+1]} = e^{[n]} + \frac{a - \log e^{[n]}}{1/e^{[n]}} \bmod z^{2^{n+1}},$$
$$= e^{[n]} + e^{[n]} \left(a - \int e^{[n]'}/e^{[n]} \right) \bmod z^{2^{n+1}}.$$

And $1/e^{[n]}$ is computed by Newton iteration too!

[Brent 1975]

Application: Power Sums

$$F = t^N + a_{N-1}t^{N-1} + \cdots + a_0 \leftrightarrow S_i = \sum_{F(\alpha)=0} \alpha^i, \quad i = 0, \ldots, N.$$

Application: Power Sums

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Fast conversion using the generating series:

$$\frac{\operatorname{rev}(F)'}{\operatorname{rev}(F)} = -\sum_{i\geq 0} S_{i+1}t^i \leftrightarrow \operatorname{rev}(F) = \exp\left(-\sum \frac{S_i}{i}t^i\right).$$

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Application: composed product and sums

$$(F, G) \mapsto \prod_{F(\alpha)=0, G(\beta)=0} (t - \alpha\beta) \quad \text{or} \quad \prod_{F(\alpha)=0, G(\beta)=0} (t - (\alpha + \beta)).$$

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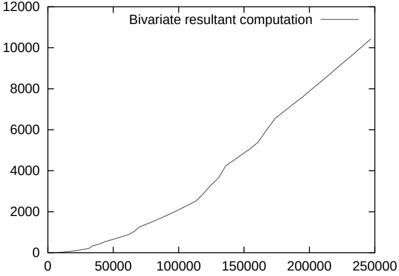
Easy in Newton representation: $\sum \alpha^s \sum \beta^s = \sum (\alpha\beta)^s$ and

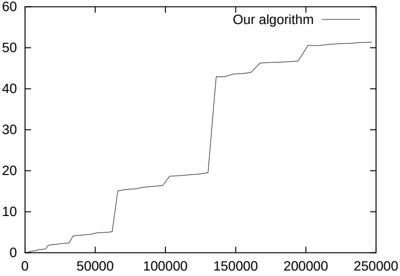
$$\sum \frac{\sum(\alpha+\beta)^s}{s!}t^s = \left(\sum \frac{\sum \alpha^s}{s!}t^s\right)\left(\sum \frac{\sum \beta^s}{s!}t^s\right).$$

[Schönhage 1982; Bostan, Flajolet, Salvy, Schost 2006]

Timings

Applications (crypto): over finite fields, degree > 200000 expected.





Timings in seconds vs. output degree N, over $\mathbb{F}_p$, 26 bits prime p

Exponential for Series-Parallel Graphs

$$\mathcal{G} = \mathcal{S} + \mathcal{P}, \quad \mathcal{S} = \mathrm{SEQ}(\mathcal{Z} + \mathcal{P}), \quad \mathcal{P} = \mathrm{SET}_{>0}(\mathcal{Z} + \mathcal{S})$$

compiles into the Newton iteration:

$$\begin{cases} i^{[n+1]} = i^{[n]} - i^{[n]}(e^{[n]} i^{[n]} - 1), \\ e^{[n+1]} = e^{[n]} - e^{[n]} \left(1 + \frac{d}{dz} S^{[n]} - \int (\frac{d}{dz} e^{[n]}) i^{[n]}\right), \\ v^{[n+1]} = v^{[n]} - v^{[n]}((1 - z - P^{[n]}) v^{[n]} - 1), \\ U^{[n+1]} = U^{[n]} + U^{[n]} \cdot \left(\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & {v^{[n+1]}}^2 \\ 0 & e^{[n+1]} & 0 \end{pmatrix} \cdot U^{[n]} + \mathrm{Id} - U^{[n]} \right), \\ \begin{pmatrix} G^{[n+1]} \\ S^{[n+1]} \\ P^{[n+1]} \end{pmatrix} = \begin{pmatrix} G^{[n]} \\ S^{[n]} \\ P^{[n]} \end{pmatrix} + U^{[n+1]} \cdot \begin{pmatrix} S^{[n]} + P^{[n]} - G^{[n]} \\ v^{[n+1]} - S^{[n]} \\ e^{[n+1]} - P^{[n]} \end{pmatrix} \bmod z^{2^{n+1}}. \end{cases}$$

Computation reduced to products and linear ops.

Linear Differential Equations of Arbitrary Order

Given a linear differential equation with power series coefficients,

$$a_r(t)y^{(r)}(t) + \cdots + a_0(t)y(t) = 0,$$

compute the first N terms of a basis of power series solutions.

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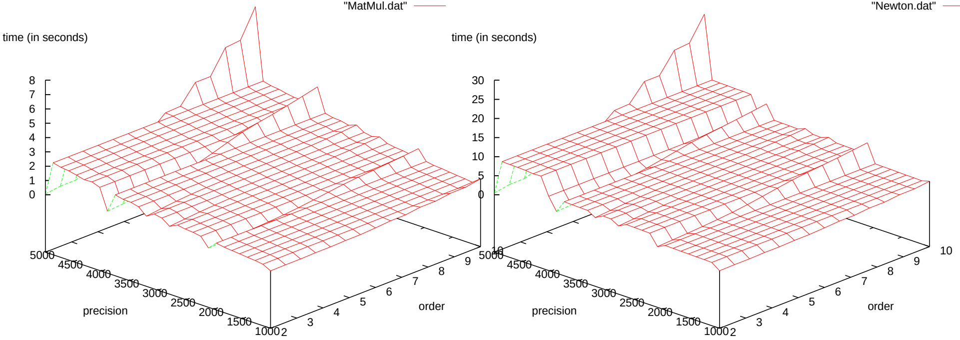
Algorithm

1. Convert into a system $\Phi : Y \mapsto Y' - A(t)Y$ $(D\Phi = \Phi)$;
2. $D\Phi|_Y(U) = \Phi(Y)$ rewrites $U' - AU = Y' - AY$;
3. Variation of constants: $U = Y \int Y^{-1}(Y' - AY)$;
4. Y^{-1} by Newton iteration too.

Special case: recover good exponential.

[Bostan, Chyzak, Ollivier, Salvy, Schost, Sedoglavic 2007]

Timings

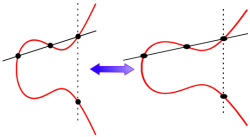


Polynomial matrix multiplication vs. solving $Y' = AY$.

Non-Linear Differential Equations

Example from cryptography:

$$\phi : y \mapsto (x^3 + Ax + B)y'^2 - (y^3 + \tilde{A}y + \tilde{B}).$$



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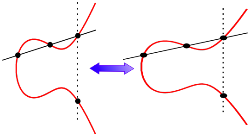


Differential:

$$D\phi|_y : u \mapsto 2(x^3 + Ax + B)y'u' - (3y^2 + \tilde{A})u.$$

Solve the linear differential equation

$$D\phi|_y\, u = \phi(y)$$

at each iteration.

Again, quasi-linear complexity.

[Bostan, Morain, Salvy, Schost 2008]

IV Oracle

Exponential Generating Series

Arbitrary Combinatorial Specification

$\Downarrow$

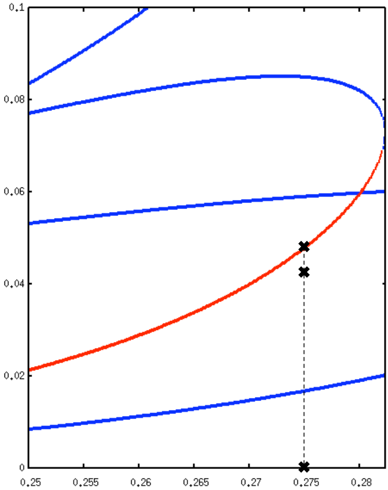
Combinatorial Newton iteration for $\mathcal{Y}$

$\Downarrow$

Newton iteration for the gf $Y(z)$
($(y_0, \ldots, y_N)$ fast)

$\Downarrow$

Numerical Newton iteration starting from 0 converges to the value of $Y(x)$.



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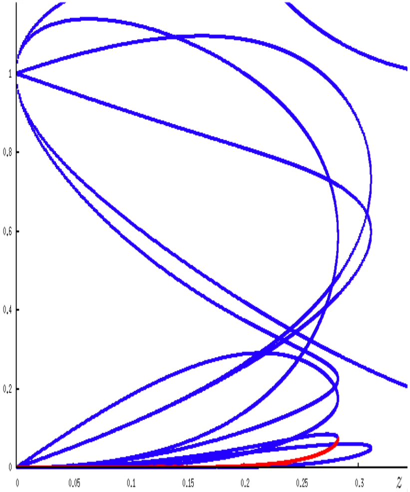

Ordinary Generating Function on an Example

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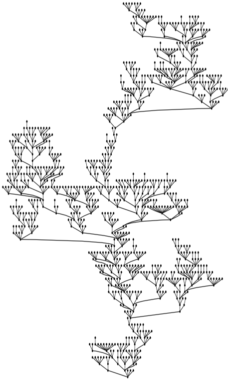
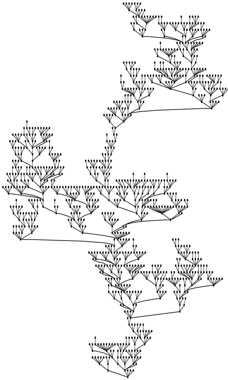


Ordinary Generating Function on an Example

1. Well-founded system: $\mathcal{Y} = \mathcal{Z} \cdot \mathrm{SET}(\mathcal{Y}) =: \mathcal{H}(\mathcal{Z}, \mathcal{Y})$;
2. Combinatorial Newton iteration:
$\mathcal{Y}^{[n+1]} = \mathcal{Y}^{[n]} + \mathrm{SEQ}(\mathcal{H}(\mathcal{Y}^{[n]})) \cdot (\mathcal{H}(\mathcal{Y}^{[n]}) \setminus \mathcal{Y}^{[n]})$



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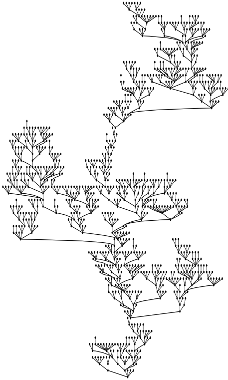

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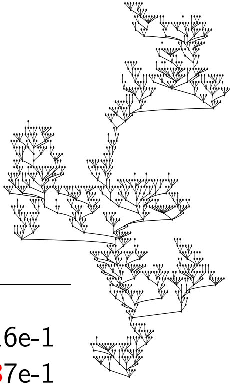
n	$\tilde{Y}^{[n]}(0.3)$	$\tilde{Y}^{[n]}(0.3^2)$	$\tilde{Y}^{[n]}(0.3^3)$
0	0	0	0
1	.43021322639	0.99370806338e-1	0.27759817516e-1
2	.54875612912	0.99887132154e-1	0.27770629187e-1
3	.55709557053	0.99887147197e-1	0.27770629189e-1
4	.55713907945	0.99887147198e-1	0.27770629189e-1
5	.55713908064	0.99887147198e-1	0.27770629189e-1



V Conclusion

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THE END