An Introduction to Recent Algorithms Behind the DDMF

Bruno Salvy
Inria & ENS de Lyon

SIAM Orthogonal Polynomials, Special Functions and Applications
June 2015

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1. Fast computation at large precision
2. Continued fractions
3. Chebyshev expansions

I. Fast computation at large precision

From large integers to precise numerical values

Fast Fourier Transform (Gauss, Cooley-Tuckey, Schönhage-Strassen). Two integers of n digits can be multiplied with O(n log(n) loglog(n)) bit operations.

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Direct consequence (by Newton iteration):

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O(n log(n) loglog(n)) bit operations.

Direct consequence (by Newton iteration):

inverses, square-roots,… : same cost.

Binary Splitting for linear recurrences (70's and 80's)

- n! by divide-and-conquer:

$$n! := \underbrace{n \times \cdots \times \lfloor n/2 \rfloor}_{\text{size } O(n \log n)} \times \underbrace{(\lfloor n/2 \rfloor + 1) \times \cdots \times 1}_{\text{size } O(n \log n)}$$

Cost: $O(n \log^3 n \operatorname{loglog} n)$ using FFT

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- linear recurrences of order 1 reduce to

$$p!(n) := (p(n) \times \cdots \times p(\lfloor n/2 \rfloor)) \times (p(\lfloor n/2 \rfloor + 1) \times \cdots \times p(1))$$

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- arbitrary order: same idea, same cost (matrix factorial):

ex: $e_n := \sum_{k=0}^{n} \frac{1}{k!}$ satisfies a 2nd order rec, computed via

$$\begin{pmatrix} e_n \\ e_{n-1} \end{pmatrix} = \frac{1}{n} \underbrace{\begin{pmatrix} n+1 & -1 \\ n & 0 \end{pmatrix}}_{A(n)} \begin{pmatrix} e_{n-1} \\ e_{n-2} \end{pmatrix} = \frac{1}{n!} A!(n) \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Code available: NumGfun [Mezzarobba 2010]

Numerical evaluation of solutions of LDEs

Principle: $$f(x) = \underbrace{\sum_{n=0}^{N} a_n x^n}_{\text{fast evaluation}} + \underbrace{\sum_{n=N+1}^{\infty} a_n x^n}_{\text{good bounds}}$$

f solution of a LDE with coeffs in $\mathbb{Q}(x)$ (our data-structure!)

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The technique used for fast evaluation of constants like

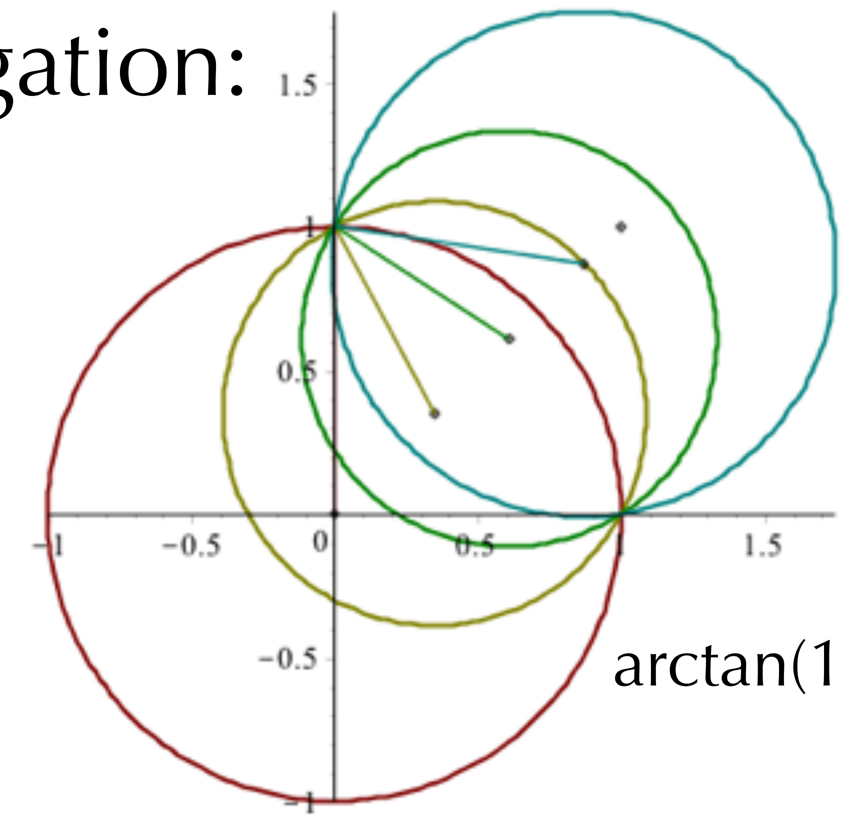
$$\frac{1}{\pi} = \frac{12}{C^{3/2}} \sum_{n=0}^{\infty} \frac{(-1)^n (6n)! (A + nB)}{(3n)! n!^3 C^{3n}}$$

with A=13591409, B=545140134, C=640320.

Code available: NumGfun [Mezzarobba 2010]

Analytic continuation

Compute $f(x), f'(x), \ldots, f^{(d-1)}(x)$ as new initial conditions and handle error propagation:



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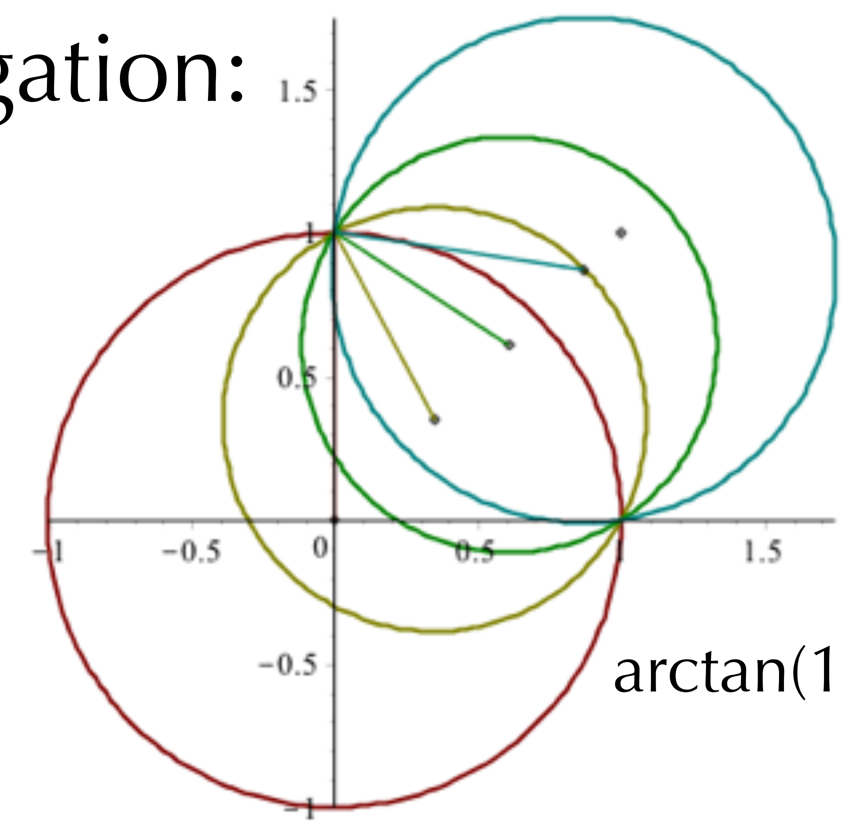


arctan(1+i)

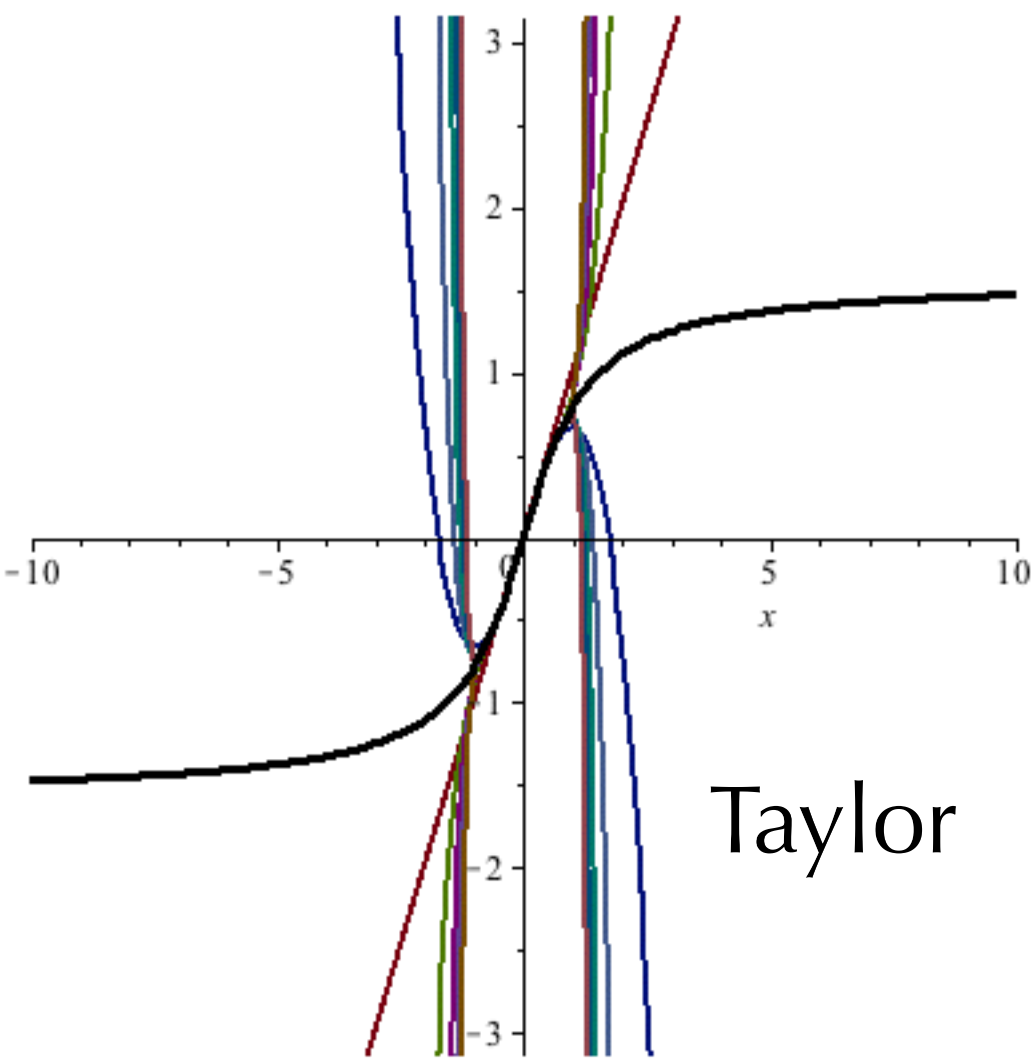
Ex: erf(π) with 15 digits:

$$0 \xrightarrow[\text{200 terms}]{} 3.1416 \xrightarrow[\text{18 terms}]{} 3.1415927 \xrightarrow[\text{6 terms}]{} 3.14159265358979$$

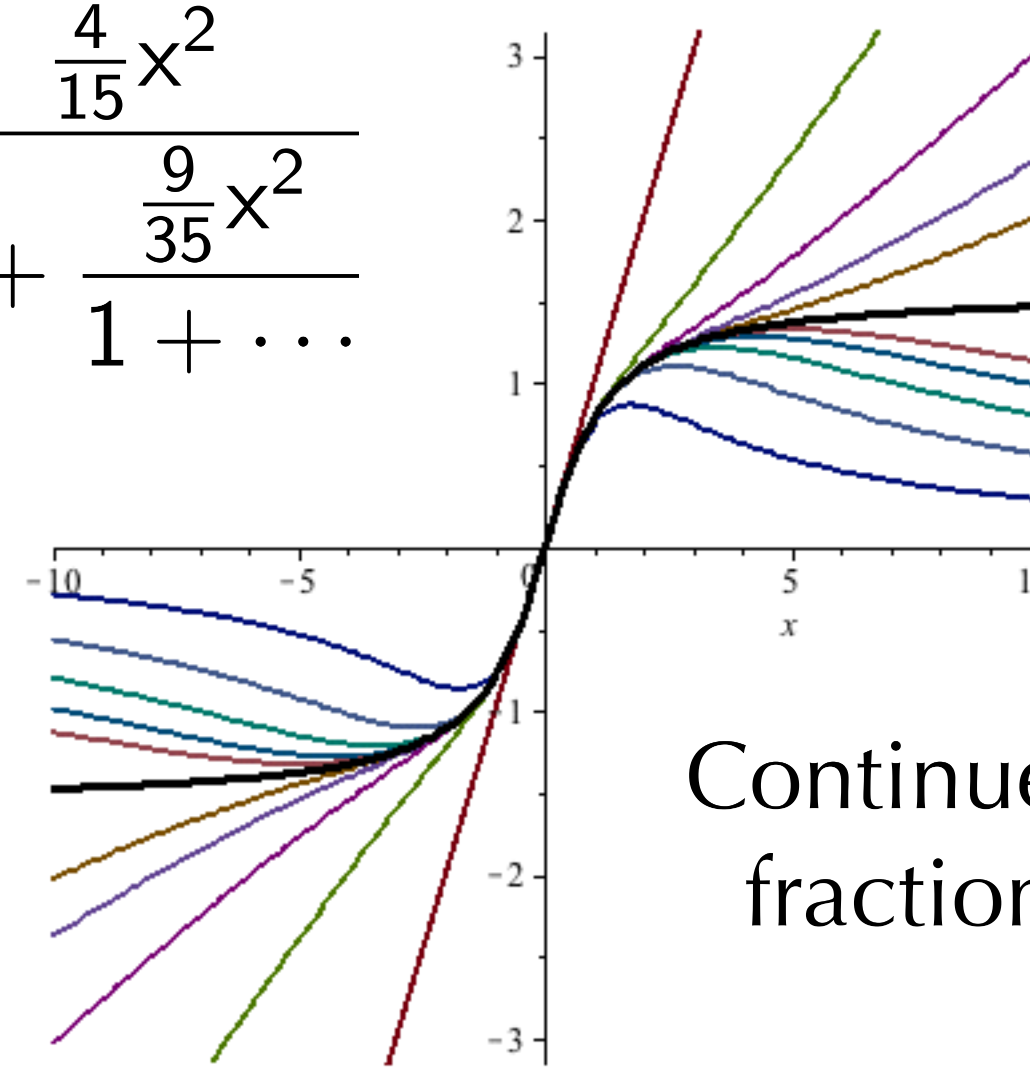
Again: computation on integers. No roundoff errors.

II. Continued Fractions

$$\arctan x = \cfrac{x}{1 + \cfrac{\frac{1}{3}x^2}{1 + \cfrac{\frac{4}{15}x^2}{1 + \cfrac{\frac{9}{35}x^2}{1 + \cdots}}}}$$



Taylor



Continued fraction

A guess & prove approach

(Maulat, S. 2015)

1. Differential equation produces first terms (easy):

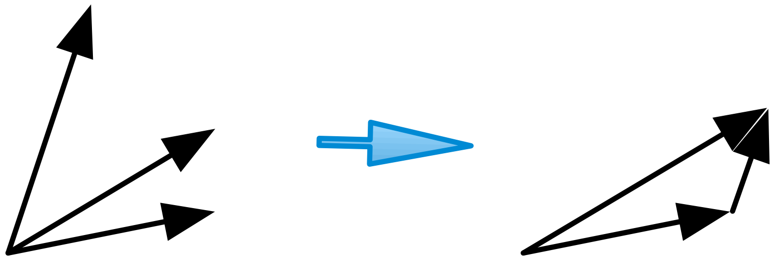
$$\arctan x = \cfrac{x}{1 + \cfrac{\frac{1}{3}x^2}{1 + \cfrac{\frac{4}{15}x^2}{1 + \cfrac{\frac{9}{35}x^2}{1 + \cdots}}}}$$

2. Guess a formula (easy): $a_n = \dfrac{n^2}{4n^2 - 1}$

3. Prove that the CF with these a_n satisfies the differential equation.

No human intervention needed.

Proof technique



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> series(sin(x)^2+cos(x)^2-1,x,4);
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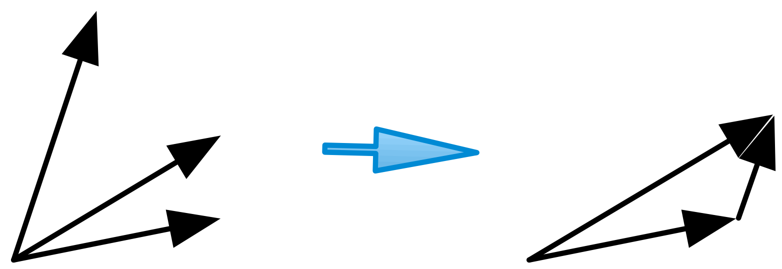
$O(x^4)$

Why is this a proof?

Proof technique



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$\Longleftrightarrow$
f,f',f'',… live in a finite-dim. vector space

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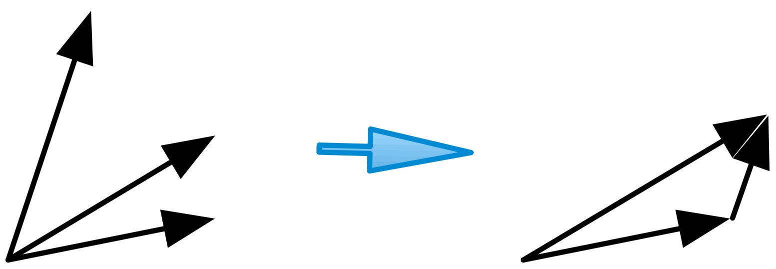

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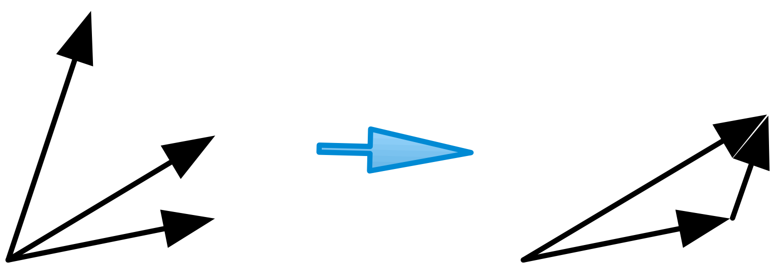

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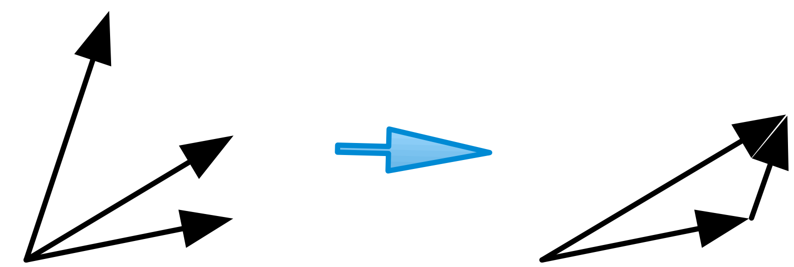

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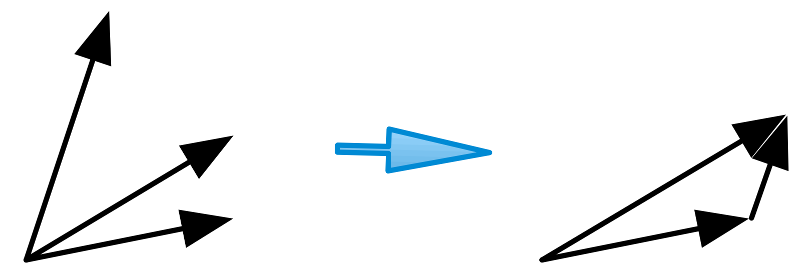

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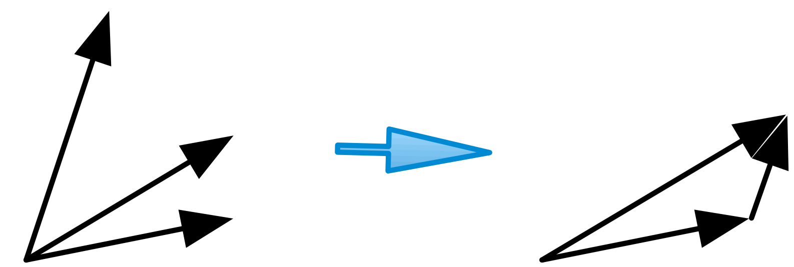

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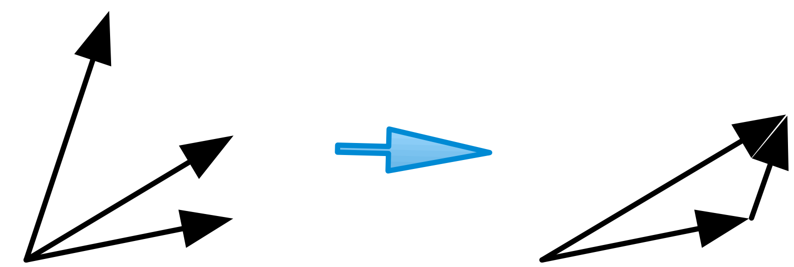

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Proofs of non-linear identities by linear algebra!

Automatic Proof of the guessed CF

$$\arctan x \stackrel{?}{=} \cfrac{x}{1 + \cfrac{\cdots}{1 + \cfrac{\frac{n^2}{4n^2-1}x^2}{1+\cdots}}}$$

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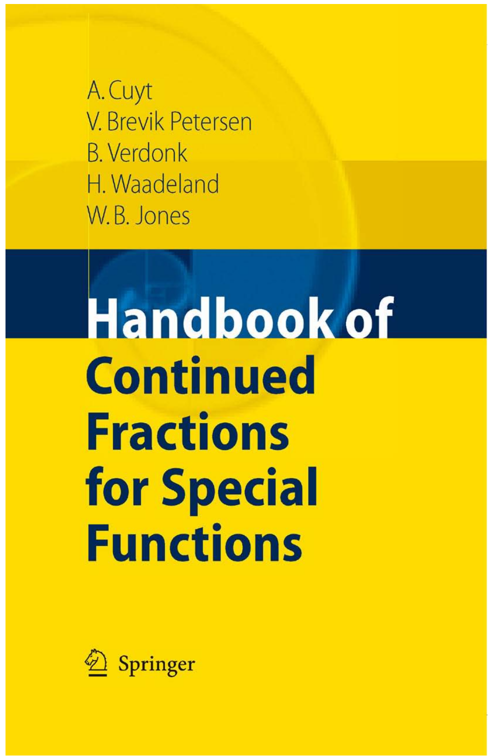


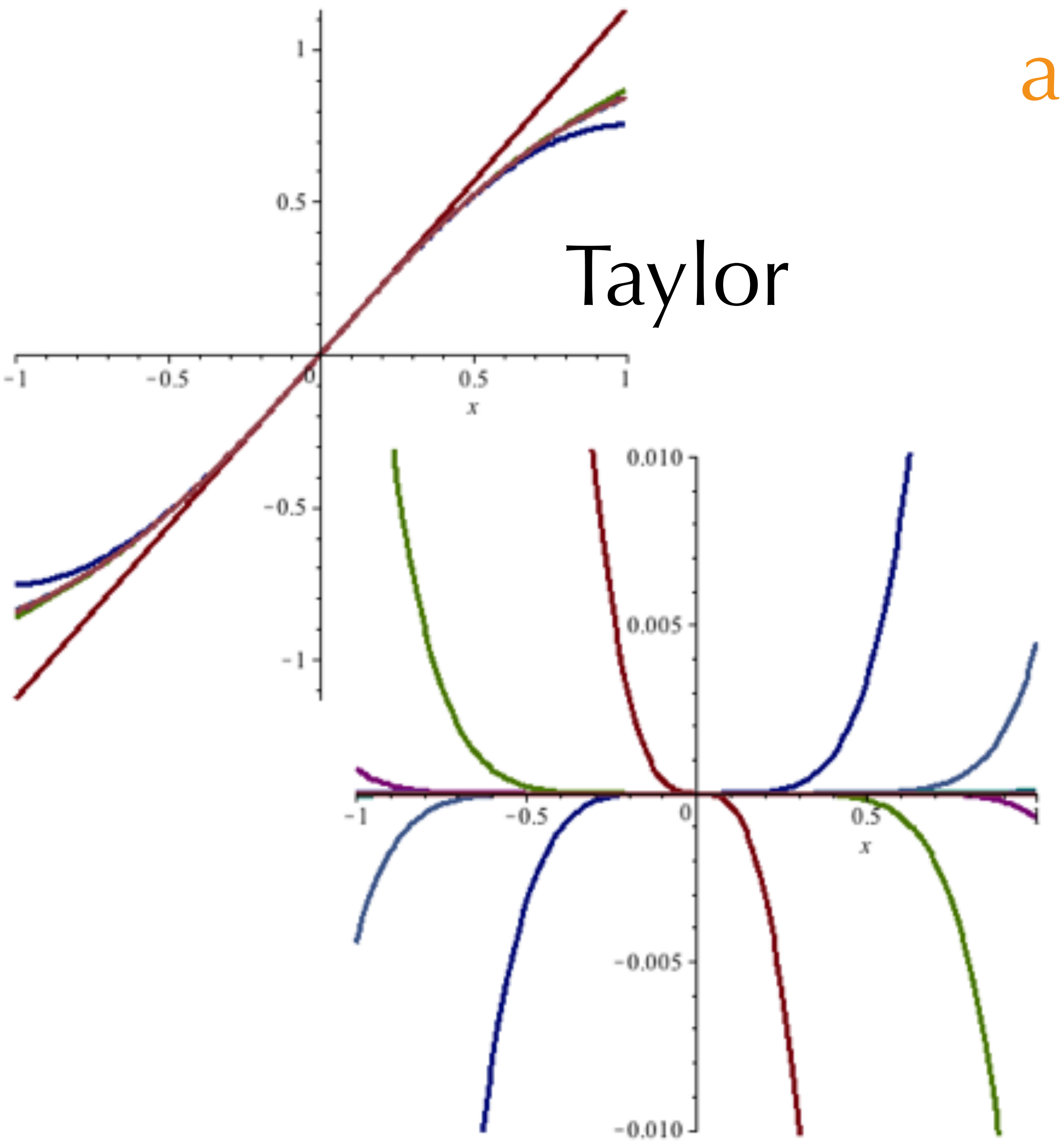
More generally: this guess-and-proof approach applies to CF for solutions of (q-)Ricatti equations

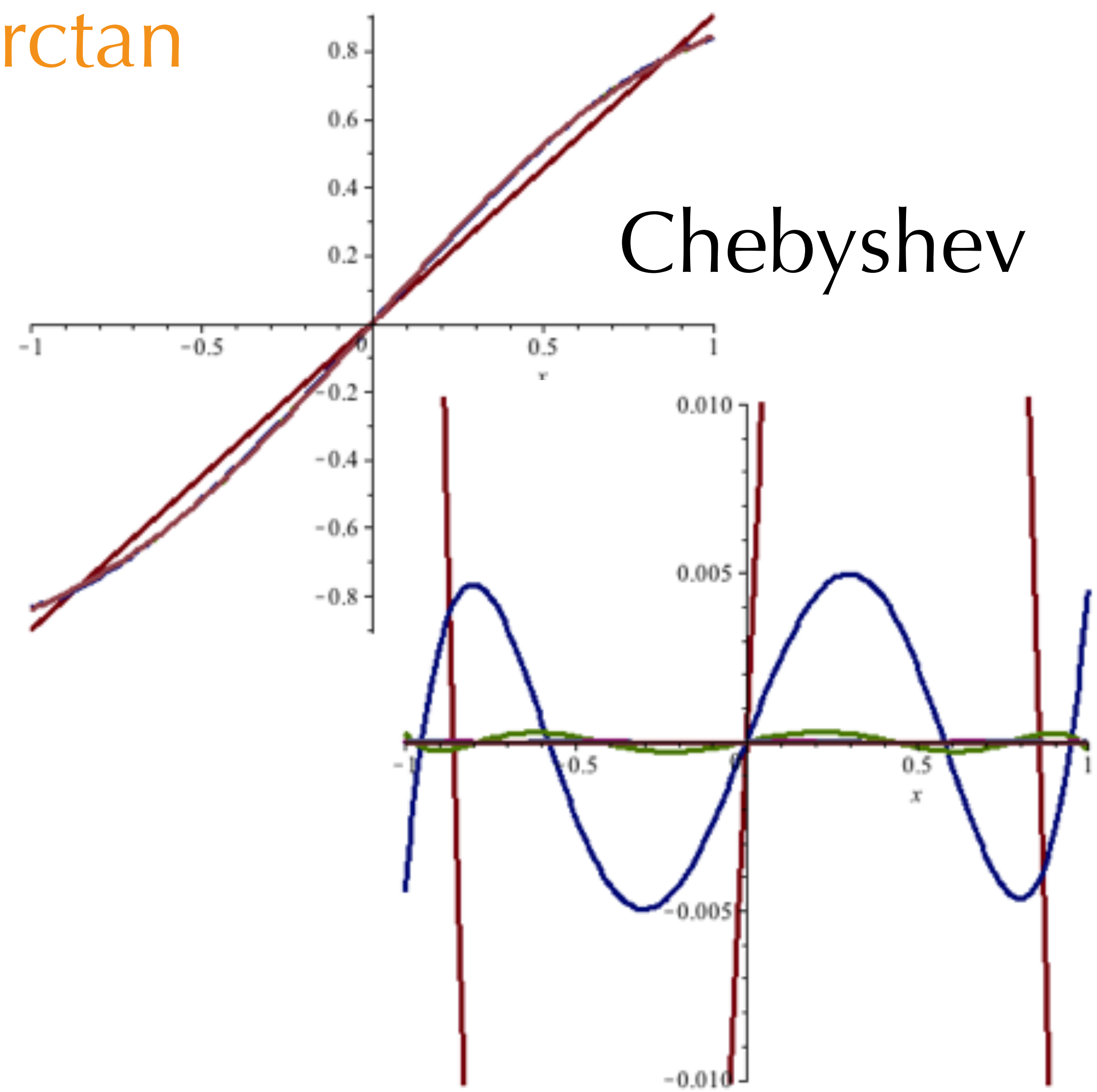
→ **all explicit C-fractions** in Cuyt et alii.

III. Ore polynomials and Chebyshev expansions

Chebyshev expansions

arctan





$$z - \frac{1}{3}z^3 + \frac{1}{5}z^5 + \cdots$$

$$2(\sqrt{2}+1)\left(\frac{T_1(x)}{(2\sqrt{2}+3)} - \frac{T_3(x)}{3(2\sqrt{2}+3)^2} + \frac{T_5(x)}{5(2\sqrt{2}+3)^3} + \cdots\right)$$

From equations to operators

$D \leftrightarrow d/dx$
$x \leftrightarrow$ mult by x
product $\leftrightarrow$ composition
$Dx = xD + 1$

$S \leftrightarrow (n \mapsto n+1)$
$n \leftrightarrow$ mult by n
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$Sn = (n+1)S$

From equations to operators

$D \leftrightarrow d/dx$ $x \leftrightarrow$ mult by x product $\leftrightarrow$ composition $Dx=xD+1$	$S \leftrightarrow (n \mapsto n+1)$ $n \leftrightarrow$ mult by n product $\leftrightarrow$ composition $Sn=(n+1)S$

Taylor morphism: $D \mapsto (n+1)S$; $x \mapsto S^{-1}$
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Ore (1933): general framework for these non-commutative polynomials.
Main property: deg AB=deg A+deg B.
Consequence 1: (non-commutative) Euclidean division
Consequence 2: (non-commutative) Euclidean algorithm
Consequence 3: (non-commutative) fractions

Application: Chebyshev expansions

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Extend Taylor morphism to Chebyshev expansions

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Extend Taylor morphism to Chebyshev expansions

Taylor

$x^{n+1} = x \cdot x^n \leftrightarrow x \mapsto X := S^{-1}$

$(x^n)' = n x^{n-1} \leftrightarrow d/dx \mapsto D := (n+1)S$

Application: Chebyshev expansions

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Chebyshev

$2xT_n(x)=T_{n+1}(x)+T_{n-1}(x)$

$\leftrightarrow x \mapsto X:=(S+S^{-1})/2$

$2(1-x^2)T_n'(x)=-nT_{n+1}(x)+nT_{n-1}(x)$

$\leftrightarrow d/dx \mapsto D:=(1-X^2)^{-1}n(S-S^{-1})/2.$

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erf: $$D^2+2xD \mapsto (2(S^{-1}-S)^{-1}n)^2+2\frac{S+S^{-1}}{2}2(S^{-1}-S)^{-1}n$$

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$$=\mathrm{pol}(n,S)^{-1}(2(n+1)(n+4)S^4-4(n+2)^3S^2+2n(n+3))$$

Prop. [Benoit, S (2009)] If y is a solution of L(x,d/dx), then its Chebyshev coefficients annihilate the numerator of L(X,D).

Application: Chebyshev expansions

Extend Taylor morphism to Chebyshev expansions

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$2(1-x^2)T_n'(x)=-nT_{n+1}(x)+nT_{n-1}(x)$

$\leftrightarrow d/dx \mapsto D:=(1-X^2)^{-1}n(S-S^{-1})/2.$

erf: $$D^2+2xD \mapsto (2(S^{-1}-S)^{-1}n)^2+2\frac{S+S^{-1}}{2}2(S^{-1}-S)^{-1}n$$

$$=\operatorname{pol}(n,S)^{-1}(2(n+1)(n+4)S^4-4(n+2)^3S^2+2n(n+3))$$

Prop. [Benoit, S (2009)] If y is a solution of L(x,d/dx), then its Chebyshev coefficients annihilate the numerator of L(X,D).

See Benoit-Mezzarobba-Joldes for certified numerical approximations on this basis.

Conclusion

Summary

- Linear differential equations and recurrences are a great data-structure;
- Numerous algorithms have been developed in computer algebra;
- Efficient code is available;
- More is true (creative telescoping, diagonals,…);
- More to come in DDMF, including formal proofs.