Positivity of Linear Recurrent Sequences via Invariant Cones

Bruno Salvy

AriC, Inria at ENS de Lyon

Joint and on-going work with Alaa Ibrahim

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I. Positivity Proofs

Positivity Problem

Input: $p_0, \ldots, p_d$ in $\mathbb{Q}[n]$, $u_0, \ldots, u_{d-1}$ in $\mathbb{Q}$. $p_0 p_d \neq 0$ and $0 \notin p_d(\mathbb{N})$.

A sequence (u_n) is defined by

$$p_d(n) u_{n+d} = p_{d-1}(n) u_{n+d-1} + \cdots + p_0(n) u_n,$$

Def: P-finite sequence. C-finite when p_i constant.

and $u_0, \ldots, u_{d-1}$. Is $u_n \geq 0$ for all n?

Is this even decidable?

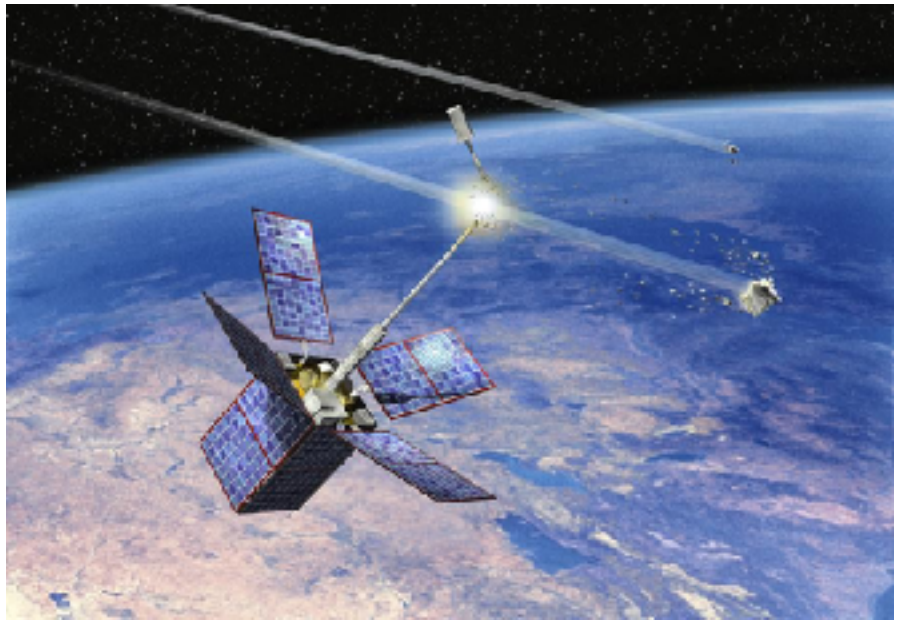
From Positivity to Inequalities

P-finite and C-finite sequences are closed under sum and product.

If $(u_n), (v_n)$ are P-finite, deciding

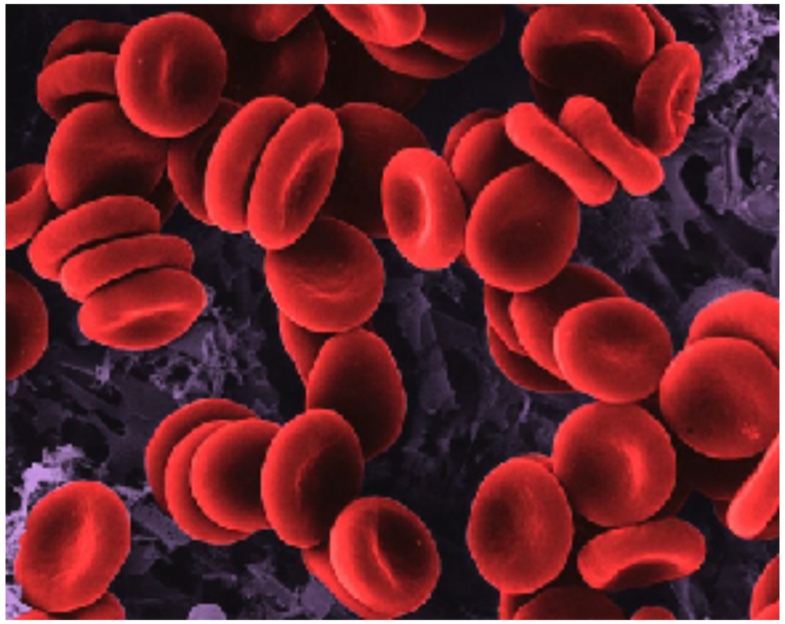
monotonicity	$u_{n+1} \geq u_n$	
convexity	$u_{n+1} + u_{n-1} \geq 2u_n$	all reduce to positivity.
log-convexity	$u_{n+1}u_{n-1} \geq u_n^2$	
inequality	$u_n \geq v_n$	

Examples



Probability of collision: $\sum_{n \geq 0} u_n$.
Numerical stability ensured by the positivity of (u_n), P-finite of order 4, with 2 parameters.

[Serra-Arzelier-Joldes-Lasserre-Rondepierre-S. 2016]



Uniqueness of the Canham model for biomembranes. Reduced to the positivity of a P-finite sequence of order 7, coeffs of degree 7.

[Melczer-Mezzarobba 2022]
[Bostan-Yurkevich 2022]

Examples from Diagonals

$$\operatorname{diag} \frac{1}{1-2(x+y+z)+3(xy+xz+yz)} =: \sum_{n\geq 0} s_n t^n$$

$$s_n = \sum_{k=0}^{n} (-27)^{n-k} 2^{2k-n} \frac{(3k)!}{k!^3} \binom{k}{n-k} > 0$$

[Straub-Zudilin 2015]

$$2(n+2)^2 s_{n+2} = (81n^2 + 243n + 186)s_{n+1} - 81(3n+2)(3n+4)s_n\,.$$

A family of tests

$$\operatorname{diag} \frac{1}{1-(x_1+\cdots+x_k)+k!x_1\cdots x_k} =: \sum_{n\geq 0} u_n^{(k)} t^n$$

$$u_n^{(k)} = \sum_{j=0}^{n} (-1)^j \frac{(kn-(k-1)j)!k!^j}{(n-j)!^k j!} \geq 0 \text{ for } k \geq 4$$

[Yu 2019]

Was conjectured by Gillis-Reznick-Zeilberger (1983)

linear rec of order k with coeffs of degree $k(k-1)/2$

Constant Coefficients Already Difficult

$$u_{n+d} = c_{d-1}u_{n+d-1} + \cdots + c_0 u_n, \qquad c_i \in \mathbb{Q}$$

Characteristic polynomial: $x^d - \sum_{i=0}^{d-1} c_i x^i = \prod_{i=1}^{k} (x - \lambda_i)^{m_i}$

Closed form: $u_n = C_1(n)\lambda_1^n + \cdots + C_k(n)\lambda_k^n$ — C_i computable from $u_0, \ldots, u_{d-1}$

Aim: extension to polynomial coefficients

Easy situation: $|\lambda_1| > |\lambda_i|, i \neq 1$ and $C_1 \neq 0$.

General case: positivity decidable for $d \leq 5$. [Ouaknine-Worrell 2014]

$d = 6$ related to open pbs in Diophantine approximation.

Skolem problem (decidable for $d \leq 4$) reduces to positivity:

$m :=$ lcm denominators c_i and initial conditions

$$u_n \neq 0 \Leftrightarrow v_n := (m^n u_n)^2 - 1 \geq 0$$

Gerhold-Kauers Method

Use quantifier elimination to look for m s.t.

$$\forall n \geq 0, \forall u_n \geq 0, \forall u_{n+1} \geq 0, \ldots, \forall u_{n+d-1} \geq 0,$$
$$u_{n+d} \geq 0 \wedge \cdots \wedge u_{n+m} \geq 0 \Rightarrow u_{n+m+1} \geq 0.$$

No guarantee.
Termination unclear.

[Gerhold-Kauers 2005]

Variant: add $\exists \mu \geq 0$ and replace $u_{i+1} \geq 0$ by $u_{i+1} \geq \mu u_i$ [Kauers-Pillwein 2010]

Termination (under hypothesis) for $d = 2$ and cases of $d = 3$.

Aim: a larger class

II. Recurrences on Vectors

Vector Version of the Recurrence

$$p_d(n)u_{n+d} = p_{d-1}(n)u_{n+d-1} + \cdots + p_0(n)u_n \Leftrightarrow$$

$$\begin{pmatrix} u_{n+1} \\ \vdots \\ u_{n+d} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ \frac{p_0(n)}{p_d(n)} & \cdots & \cdots & \cdots & \frac{p_{d-1}(n)}{p_d(n)} \end{pmatrix} \begin{pmatrix} u_n \\ \vdots \\ u_{n+d-1} \end{pmatrix}$$

U_n

$A(n)$

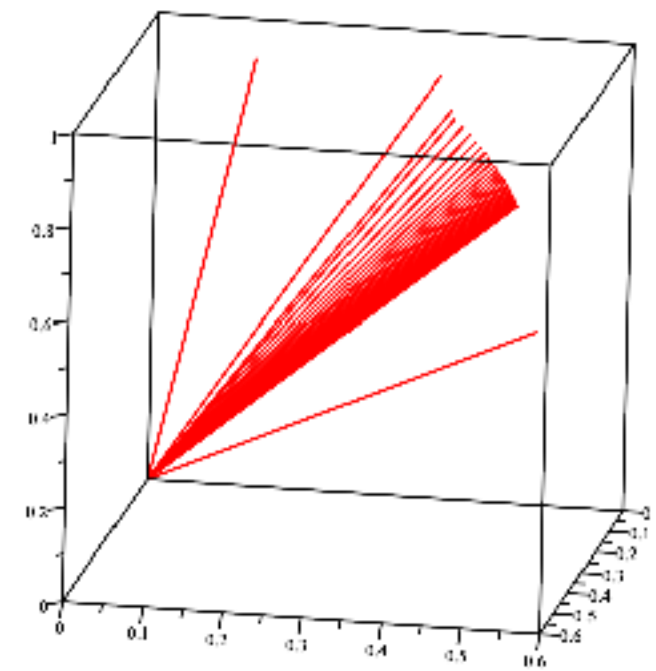
$$U_{n+1} = A(n)U_n$$
$$= A(n)A(n-1)\cdots A(0)U_0$$

$u_n \geq 0$ for all $n \Leftrightarrow U_n \in \mathbb{R}_+^d$ for all n

Constant Coefficients & Power Method

1. Pick a random U_0
2. For $n = 0,1,\ldots$

$$U_{n+1} := AU_n / \|U_n\|$$



Eigenvalues of A : $\lambda_1, \ldots, \lambda_k$

Hyp.: $|\lambda_1| > |\lambda_i|, i \neq 1$ and λ_1 simple.

Principle: polynomial coefficients as a perturbation

Then $A^n / \|A^n\| \to VW^\top \quad (AV = \lambda_1 V, A^\top W = \lambda_1 W)$

Convergence to a rank 1 matrix

Generic $U_0 : W^\top U_0 \neq 0$

$$\Rightarrow U_n / \|U_n\| \to \pm V / \|V\|$$

Convergence to a dominant eigenvector

Recurrences of Poincaré Type

$$U_{n+1} = A(n)U_n$$

Def. Poincaré type: $A := \lim_{n\to\infty} A(n)$ finite.

Ex. $2(n+2)^2 s_{n+2} = (81n^2 + 243n + 186)s_{n+1} - 81(3n+2)(3n+4)s_n$. ✔

$$A(n) = \begin{pmatrix} 0 & 1 \\ -\frac{81(3n+2)(3n+4)}{2(n+2)^2} & \frac{3(27n^2+81n+62)}{2(n+2)^2} \end{pmatrix} \to A = \begin{pmatrix} 0 & 1 \\ -\frac{729}{2} & \frac{81}{2} \end{pmatrix}$$

Ex. $u_{n+3} + u_{n+2} + nu_{n+1} + (n+1)u_n = 0$ ✘

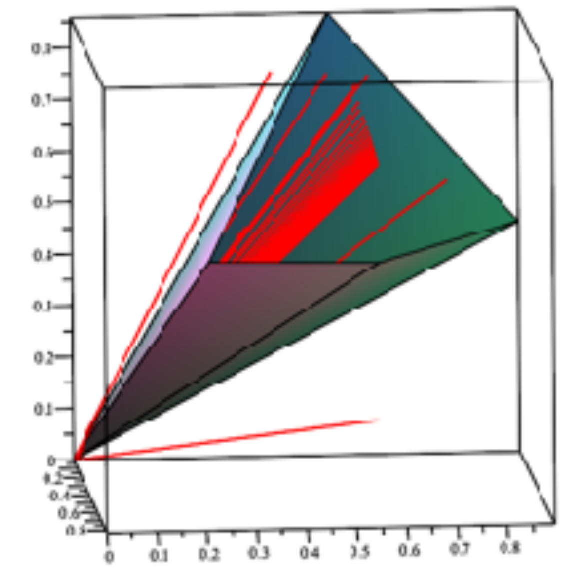
Positivity-preserving reduction

$$v_n := \psi_n u_n$$
$$(n+2)\psi_{n+2} = \psi_n$$
$$\psi_0 = \psi_1 = 1 \quad \sim 1/\sqrt{n!}$$

$\longrightarrow$

$$(n+6)(n+4)v_{n+6} + 2(n+4)(n+1)v_{n+4} + (n^2 - n - 5)v_{n+2} - (n+1)v_n = 0$$ ✔

Positivity by Contracted Cones



Outline of the algorithm:

1. Construct a cone $K \subset \mathbb{R}_+^d$ s.t. $A(K \backslash \{0\}) \subset \mathring{K}$;
2. Compute m s.t. for all $n \geq m$, $A(n)K \subset K$;
3. Compute $n_0 \geq m-1$ s.t. $U_{n_0} \in K$;
4. Check that $U_0, \ldots, U_{n_0-1}$ are positive.

Constructs a formula
$\forall n \geq m, U_n \in K \subset \mathbb{R}_+^d$
and proves it by induction

Thm. If $\lambda_1 > |\lambda_i|, i \neq 1, \lambda_1$ simple, then for arbitrary order d, positivity is decidable for generic U_0.

[Ibrahim-S. 2024]

Next: proof of existence of K, n_0 (m by compactness), + algorithms for their computation.

Cones in $\mathbb{R}^d$

Def. $K \subset \mathbb{R}^d$ is a cone when

. $\mathbb{R}_+ K \subset K$;

. $K + K = K$;

. $K \cap (-K) = \{0\}$.

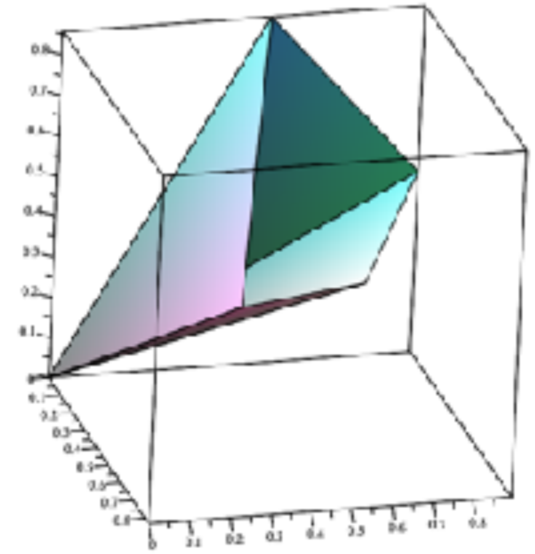
Def. $F \subset K$ is a face of K when

. F is a cone;

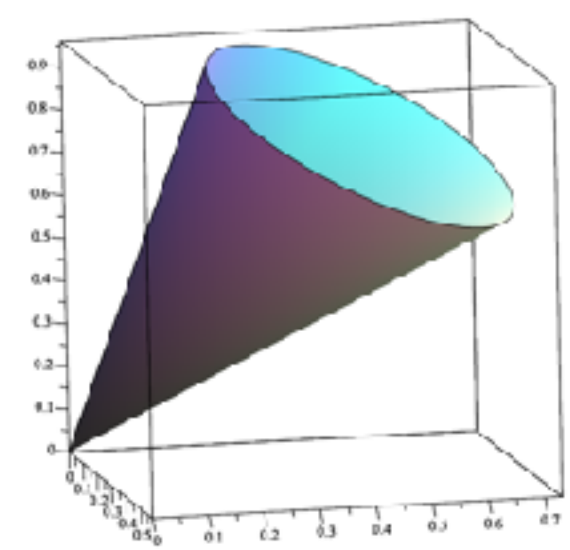
. u, v in $K, u + v \in F \Rightarrow u, v$ in F.

Def. $x \in K$ is an extremal vector of K when

$\mathbb{R}_+ x = \cap \{F \text{ face of } K \mid x \in F\}$



polyhedral: finite number of extremal vectors



In this talk, all cones are closed and solid ($\mathring{K} \neq \emptyset$).

Perron-Frobenius for Cones

(Classical case: $K = \mathbb{R}^d_+$)

$$A \in \mathbb{R}^{d \times d}$$

[Perron 1907, Frobenius 1912]

K-nonnegative A: $AK \subset K$. [Birkhoff 1967]

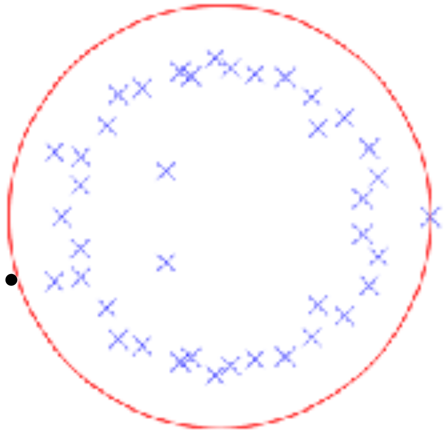
Then K contains an eigenvector for $\lambda_1 = \rho(A)$.

K-positive A: $A(K \backslash \{0\}) \subset \mathring{K}$. [Vandergraft 1968]

Gives K for the algorithm

Then $\lambda_1 > |\lambda_i|, i \neq 1$, λ_1 simple.

A with this property is K-positive for some K.



K-irreducible & *K*-primitive also defined,
with spectral characterizations.

Generalized Power Method

3. Compute $n_0 \geq m - 1$ s.t. $U_{n_0} \in K$;

Thm. $A(n) \in \mathrm{GL}_d(\mathbb{R})$ with limit A s.t. $\lambda_1 > |\lambda_i|, i \neq 1$, λ_1 simple, then

$$\frac{A(n)A(n-1)\cdots A(0)}{\|A(n)A(n-1)\cdots A(0)\|} \to VW^\top,$$

with $AV = \lambda_1 V$. Initial conditions generic when $W^\top U_0 \neq 0$.

[Friedland 2006]

Gives n_0 for the algorithm

Ex. Apéry recurrence

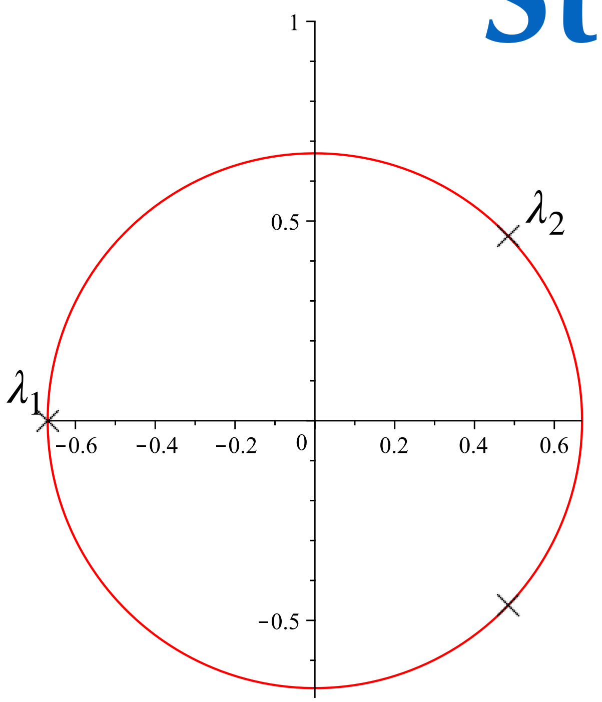
$$(n+2)^3 u_{n+2} = (2n+3)(17n^2+51n+39)u_{n+1} - (n+1)^3 u_n$$

$$V = \begin{pmatrix} 1 \\ (3+2\sqrt{2})^2 \end{pmatrix}, \quad W = \begin{pmatrix} 1 \\ 6/\zeta(3) - 5 \end{pmatrix}.$$

algebraic

difficult to control in general

Step 0: Test $\lambda_1 > |\lambda_i|,\ i \neq 1$



$$P = a_0 + \cdots + a_d X^d \in \mathbb{Z}[X], \quad |a_i| \leq H,$$

Absolute Separation:

[Bugeaud-Dujella-Pejkovic-S-Wang 2022]

$$\min_{\substack{P(\alpha)=P(\beta)=0,\\ |\alpha| \neq |\beta|}} \Big| |\alpha| - |\beta| \Big| > \kappa(d) H^{-e(d)}$$

$\kappa(d)$: explicit function of d

$e(d)$: $O(d^3)$, probably not tight

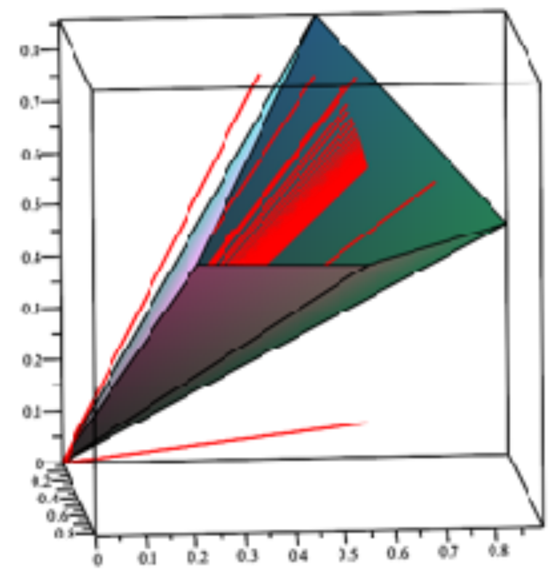
Approximate roots with precision larger than the bound sufficient to test equality of absolute values

Disks of radius ε for all roots can be computed in time $\tilde{O}(d^3 + d^2 \log H - d \log \varepsilon)$.

[Mehlhorn-Sagraloff 2016]

Positivity by Contracted Cones

0. Check $\lambda_1 > |\lambda_i|$, $i \neq 1$;
1. Construct a cone $K \subset \mathbb{R}_+^d$ s.t. $A(K \backslash \{0\}) \subset \mathring{K}$;
2. Compute m s.t. for all $n \geq m$, $A(n)K \subset K$;
3. For $i = 1,2,\ldots$ do
   1. $U_i = A(i)U_{i-1}$;
   2. If $U_i \notin \mathbb{R}_+^d$ return false
   3. If $i \geq m$ and $U_i \in K$ return true



Thm. If $\lambda_1 > |\lambda_i|, i \neq 1, \lambda_1$ simple, then for arbitrary order d, the algorithm terminates for generic U_0.

[Ibrahim-S. 2024]

Next: algorithms for the computation of K and m.

III. Contraction for Hilbert's Metric

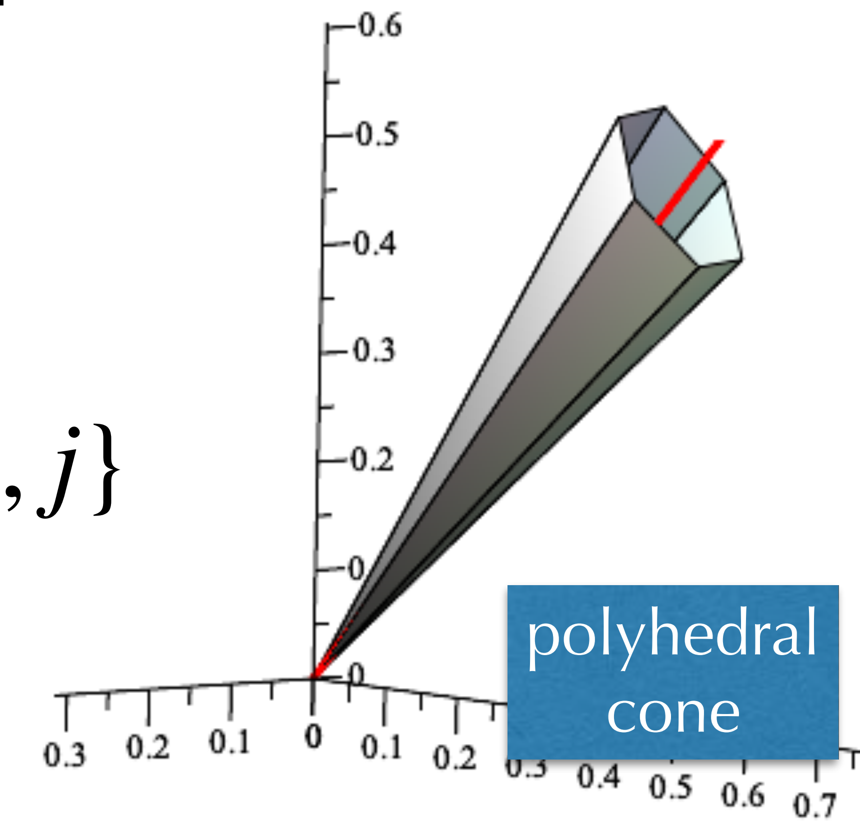
Algorithm 1

Hilbert's Metric in $\mathbb{PR}^d_{>0}$

$$d_H(x, y) := \log \frac{\max_i(x_i/y_i)}{\min_i(x_i/y_i)}$$

Ball of radius $\log r$:

$$B_r(v) = \{x \in \mathbb{R}^d_{>0} \mid x_i v_j \leq r x_j v_i \text{ for all } i, j\}$$



Positive matrices are contractions

$$A > 0 \Rightarrow \sup_{x \neq \alpha y} \frac{d_H(Ax, Ay)}{d_H(x, y)} < 1$$

[Birkhoff 1957]

Implies Perron's theorem

General to Positive

Input: A with $\lambda_1 > |\lambda_i|, i \neq 1, \lambda_1$ simple

There exists $T \in \mathrm{GL}_d(\mathbb{Q})$ s.t. TAT^{-1} has positive right and left eigenvectors v, w for λ_1

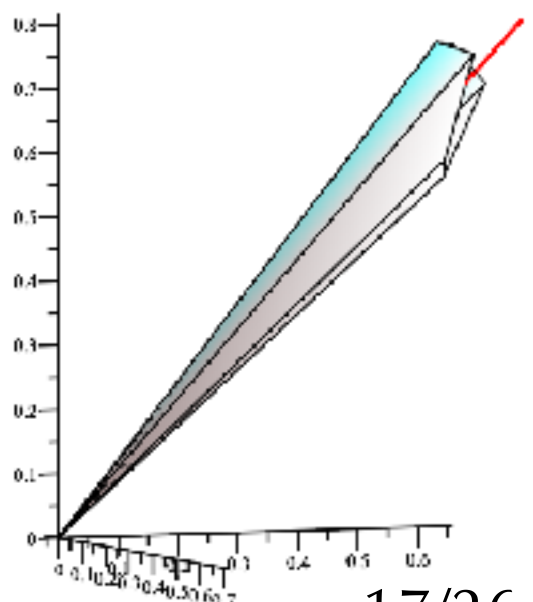
[Friedland 2006]

$\Rightarrow TA^mT^{-1}/\lambda_1^m \to vw^\top > 0, \quad m \to \infty$

$\Rightarrow \exists m, TA^mT^{-1} > 0$ and thus contracts $B_r(v)$

Gives K for the algorithm

1. Take $r > 1$ s.t. $K := T^{-1}B_r(v) > 0$
   (then $A^mK = T^{-1}TA^mT^{-1}B_r(v) \subset \mathring{K}$)
2. Replace $A(n)$ by $A(n+m-1)\cdots A(n)$



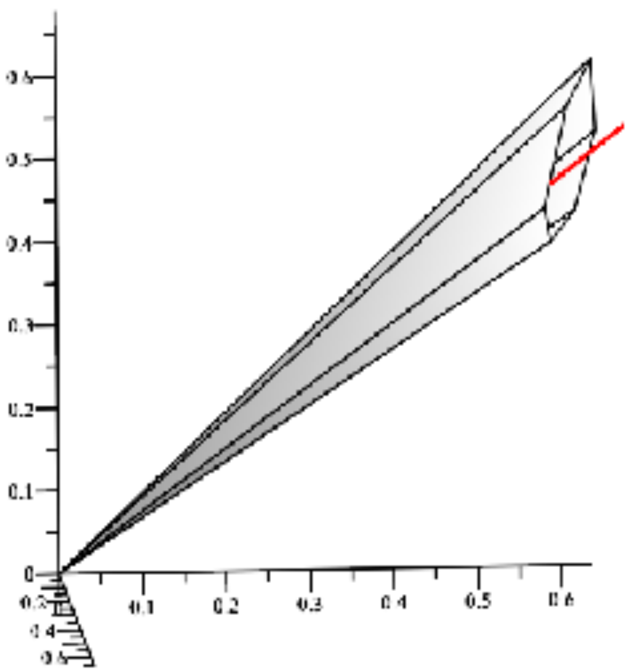
Computation of the Index of Contraction

2. Compute m s.t. for all $n \geq m$, $A(n)K \subset K$

Ex. $A(n) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{5n-7}{60(n+1)} & \frac{-(2n-1)}{3(n+1)} & \frac{19n+55}{12(n+1)} \end{pmatrix}$ $\quad T = \begin{pmatrix} 1 & -4 & 4 \\ 0 & -2 & 3 \\ 0 & 0 & 1 \end{pmatrix}$ s.t. $TAT^{-1} > 0$

$K := T^{-1}B_{6/5}((1,1,1))$ characterized by $V_K \in \mathbb{Q}^{d \times (2^d - 2)}$ extremal vectors

$v \in K \Leftrightarrow M_K \cdot v \geq 0, \ \ M_K \in \mathbb{Q}^{d(d-1) \times d}$ faces



$$A(n)K \subset K \Leftrightarrow \underbrace{M_K \cdot A(n) \cdot V_K}_{} \geq 0$$

$$\begin{pmatrix} \frac{20n-211}{25(n+1)} & \cdots & \frac{35n-152}{25(n+1)} \\ \vdots & & \vdots \\ \frac{85n+8863}{100(n+1)} & \cdots & \frac{135n+7241}{100(n+1)} \end{pmatrix} \Leftrightarrow n \geq 68$$

Examples

$$2(n+2)^2 s_{n+2} = (81n^2+243n+186)s_{n+1} - 81(3n+2)(3n+4)s_n,\ s_0=1,\ s_1=12$$

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> PositivityProof(rec,s,n,27);
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[Straub-Zudilin 2015]

$$true,\ \left[T = \begin{pmatrix} -28 & \frac{29}{27} \\ 26 & -\frac{25}{27} \end{pmatrix}, m=1, r=\infty, n_0=30\right]$$

Certificate can be used to check the proof

$$(n+1)u_{n+3} = \left(\frac{77}{30}n+2\right)u_{n+2} - \left(\frac{13}{6}n-3\right)u_{n+1} + \left(\frac{3}{5}n+2\right)u_n,\ u_0=1,\ u_1=\frac{15}{14},\ u_2=\frac{8}{7}$$

```
> PositivityProof(rec,u,n,1);
```

$$true,\ \left[T = \begin{pmatrix} 54 & -142 & 89 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, m=30, r=\frac{98}{61}, n_0=11209\right]$$

Works, but gives large values

IV. Cones from Eigenvectors

Algorithm 2

Example

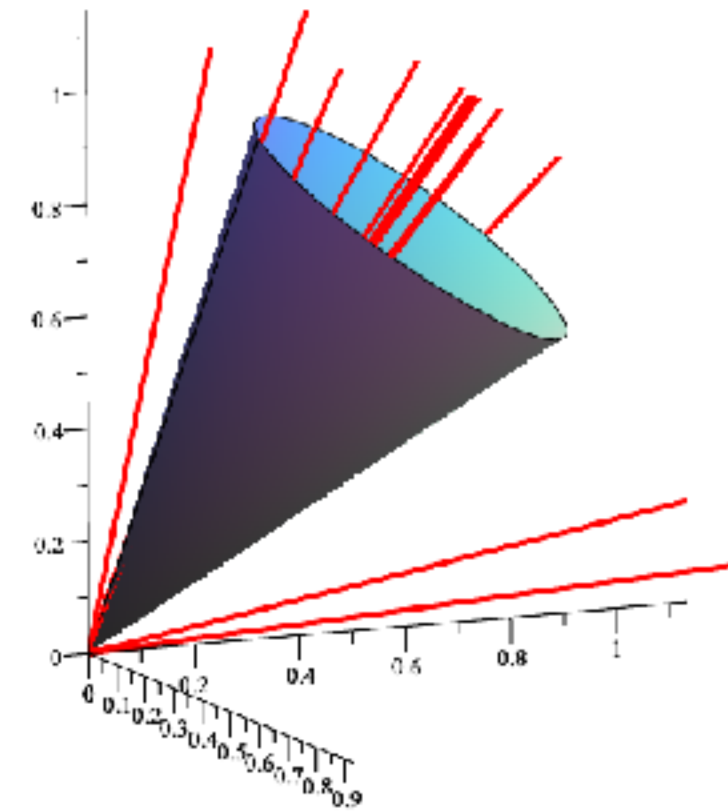
$$(30n-47)\,u_{n+3} = \left(59n-96\right)u_n + \left(15n+72\right)u_{n+1} + (27n-87)\,u_{n+2}$$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{59}{30} & \frac{1}{2} & \frac{9}{10} \end{pmatrix}$$

Eigenvectors: $V_1, V_2, \overline{V_2}$

$\lambda_1 \approx 1.79 > |\lambda_2|,$

$\lambda_2 \approx -0.44 + 0.95i\,.$



$$K := \{aV_1 + bV_2 + \overline{b}\,\overline{V_2} \mid |b| \le a\}$$

$$AK \subset \mathring{K}$$

General Case: Vandergraft's Construction

K-positive A: $A(K\backslash\{0\}) \subset \mathring{K}$.

[Vandergraft 1968]

Then $\lambda_1 > |\lambda_i|, i \neq 1$, λ_1 simple.

A with these properties is K-positive for some K.

See slide 9

Take a basis $V_1, \ldots, V_d$ where A has the form

$$A = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & J_2 & \ddots & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & J_k \end{pmatrix} \quad J_i = \begin{pmatrix} \lambda_i & \varepsilon & 0 & \cdots & 0 \\ 0 & \lambda_i & \varepsilon & \cdots & 0 \\ 0 & 0 & \lambda_i & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \varepsilon \\ 0 & 0 & 0 & \cdots & \lambda_i \end{pmatrix}$$

with $0 < \varepsilon < \lambda_1 - |\lambda_2|$ and $V_j = \overline{V_i}$ when $\lambda_j = \overline{\lambda_i}$.

$$K := \{a_1 V_1 + \cdots + a_d V_d \mid |a_i| \leq a_1 \text{ and } a_j = \overline{a_i} \text{ when } V_j = \overline{V_i}\}$$

satisfies $AK \subset \mathring{K}$.

Computation of the Index of Contraction

$$K = \{aV_1 + bV_2 + \overline{b}\overline{V_2} \mid |b| \leq a\}$$

Boundary vectors: $V_1 + e^{it}V_2 + e^{-it}\overline{V_2}$

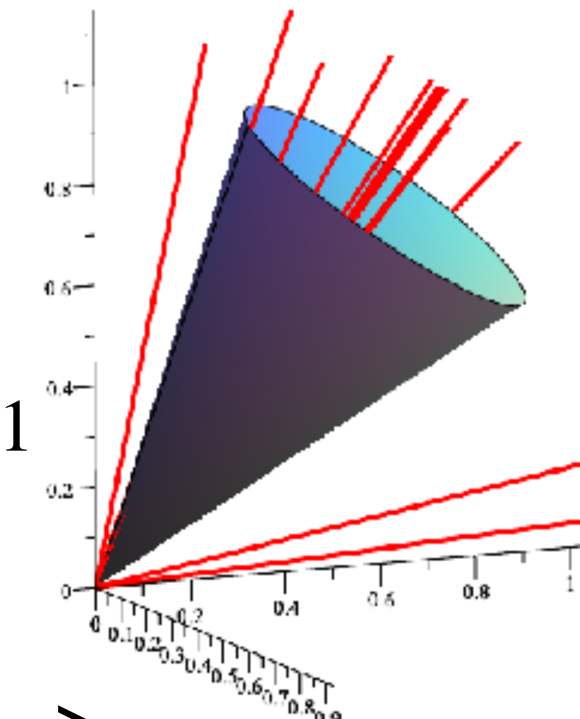


Image: $\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} = V^{-1}A(n)V\begin{pmatrix} 1 \\ c+is \\ c-is \end{pmatrix} \quad c^2+s^2=1$

Wanted: m s.t. $\underbrace{a_n^2 - b_n c_n}_{\in \mathbb{Q}[n,\lambda_1,\lambda_2,\overline{\lambda_2},c,s]/(c^2+s^2-1)} \geq 0$ for all c, s and $n \geq m$.

m computable by quantifier elimination

Approximate Cone

$\tilde{\lambda}_1 = \frac{9}{5}$, $\tilde{\lambda}_2 = \frac{-9+19i}{20}$

$$\tilde{K} := \{a\tilde{V}_1 + b\tilde{V}_2 + \overline{b}\overline{\tilde{V}_2} \mid |b| \le a\},$$

$\tilde{V}_1, \tilde{V}_2$ rational approximations of V_1, V_2

Check that $A\tilde{K} \subset \mathring{\tilde{K}} \subset \mathbb{R}^d_+$

otherwise, increase precision

Image: $$\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} = \tilde{V}^{-1} A(n) \tilde{V} \begin{pmatrix} 1 \\ c+is \\ c-is \end{pmatrix}$$

Wanted: m s.t. $\underbrace{a_n^2 - b_n c_n}_{p_2(s,c)n^2 + p_1(s,c)n + p_0(s,c)} \ge 0$ for all c, s and $n \ge m$.

interval analysis

$$\ge 1972n^2 - 10752n - 9728 \quad \to \quad m \ge 7$$

It Works!

$$2(n+2)^2 s_{n+2} = (81n^2 + 243n + 186)s_{n+1} - 81(3n+2)(3n+4)s_n, \; s_0 = 1, \; s_1 = 12$$

[Straub-Zudilin 2015]

Algorithm 1: $n_0 = 30$
Algorithm 2: $n_0 = 1$

$$(n+1)u_{n+3} = \left(\frac{77}{30}n + 2\right)u_{n+2} - \left(\frac{13}{6}n - 3\right)u_{n+1} + \left(\frac{3}{5}n + 2\right)u_n, \; u_0 = 1, \; u_1 = \frac{15}{14}, \; u_2 = \frac{8}{7}$$

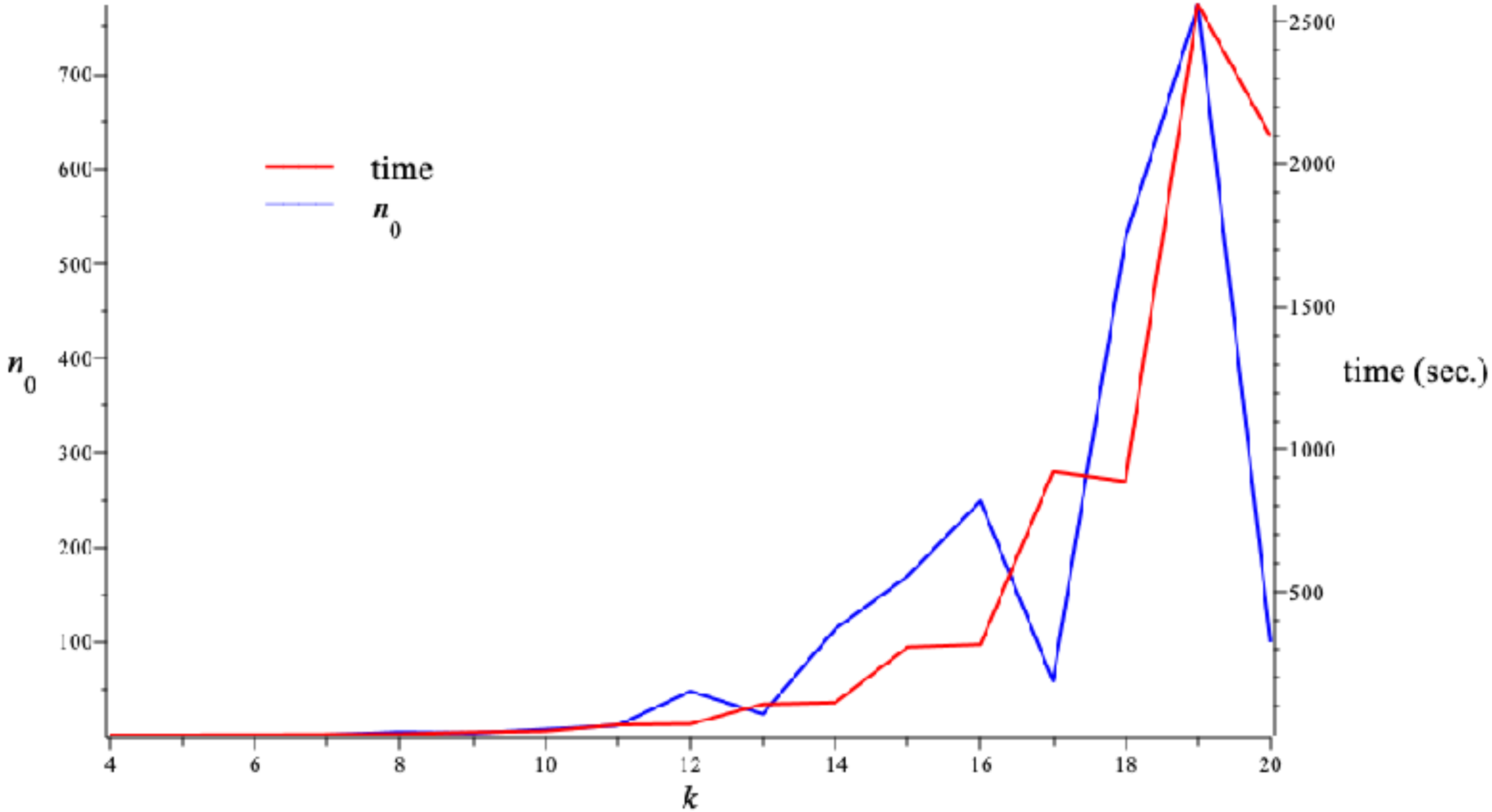
Algorithm 1: $n_0 = 11209$
Algorithm 2: $n_0 = 1200$

demo

Timings

$$u_n^{(k)} = \sum_{j=0}^{n} (-1)^j \frac{(kn-(k-1)j)!k!^j}{(n-j)!^k j!} \geq 0 \text{ for } k \geq 4$$

Gillis-
Reznick-
Zeilberger



Conclusions

Contracted cones give an access to positivity proofs for many sequences;
the cone, plus contraction index, give a certificate
$\equiv$ a property that can proved by induction.

In progress: positivity proofs without simple dominant eigenvalue

$$\text{Perron-Schaefer condition} \Rightarrow \exists K, AK \subset K.$$

A satisfies PS and $\exists m$, s.t. $A(n)K \subset K$ for $m \geq n$;
A does not satisfy PS but $A(n)$ does and $A(n)K_n \subset K_{n+1} \subset \mathbb{R}_+^d$.

Thank You!