Positivity Proofs for Solutions of Linear Recurrences

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Joint work with Alaa Ibrahim

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I. Introduction

Examples

$$s_n = \sum_{k=0}^{n} (-27)^{n-k} 2^{2k-n} \frac{(3k)!}{k!^3} \binom{k}{n-k} > 0$$

[Straub-Zudilin 2015]

$$2(n+2)^2 s_{n+2} = (81n^2 + 243n + 186)s_{n+1} - 81(3n+2)(3n+4)s_n\,.$$

Family of tests

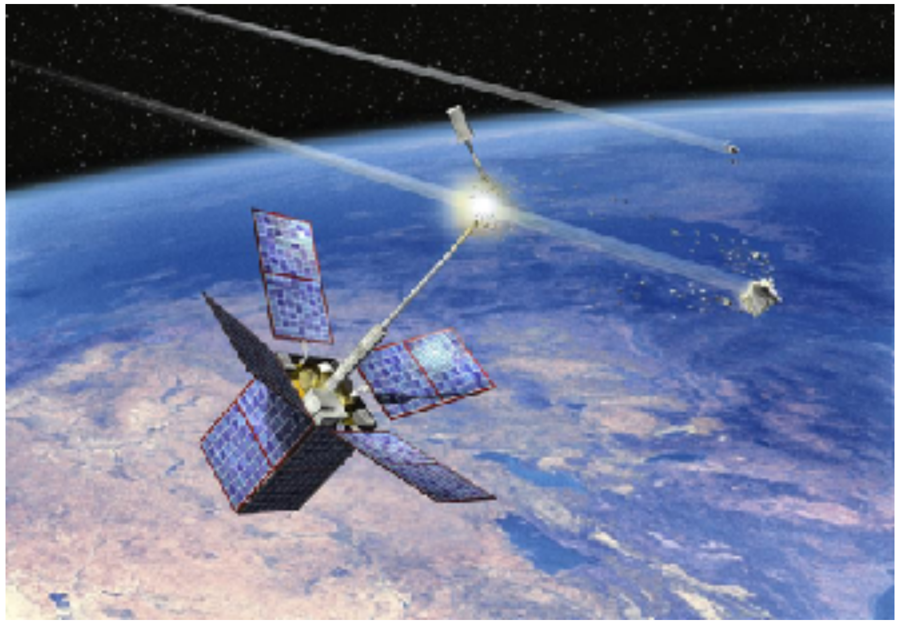
$$u_n^{(k)} = \sum_{j=0}^{n} (-1)^j \frac{(kn-(k-1)j)!\,k!^j}{(n-j)!^k j!} \geq 0 \text{ for } k \geq 4$$

[Yu 2019]

linear rec of order k with coeffs of degree $k(k-1)/2$

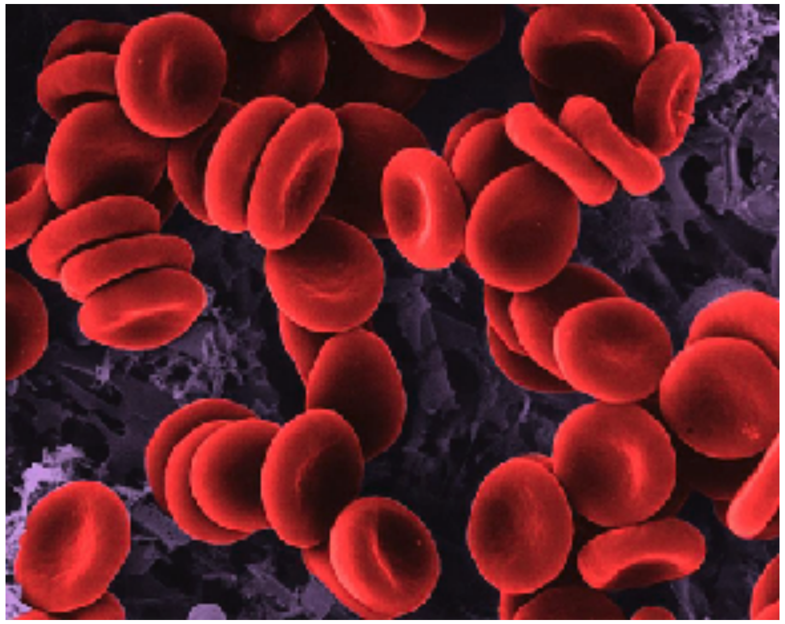
Was conjectured by Gillis-Reznick-Zeilberger (1983)

More Examples



Probability of collision: $\sum_{n\geq 0} u_n$.
Numerical stability ensured by the positivity of (u_n), linear recurrence of order 4, with 2 parameters.

[Serra-Arzelier-Joldes-Lasserre-Rondepierre-S. 2016]



Uniqueness of the Canham model for biomembranes. Reduced to the positivity of a solution of a linear recurrence of order 7, coeffs of degree 7.

[Melczer-Mezzarobba 2022]
[Bostan-Yurkevich 2022]

P-finite Sequences

Def. A sequence (u_n) is P-finite when it is defined by

$$p_d(n)u_{n+d} = p_{d-1}(n)u_{n+d-1} + \cdots + p_0(n)u_n,$$

C-finite when p_i constant

and $u_0, \ldots, u_{d-1}$ with $p_0, \ldots, p_d$ in $\mathbb{Q}[n]$, $u_0, \ldots, u_{d-1}$ in $\mathbb{Q}$.

Our data-structure

Positivity: Is $u_n \geq 0$ for all n?

In this talk, $p_0 \neq 0$ and $0 \notin p_d(\mathbb{N})$

Is this even decidable?

From Positivity to Inequalities

P-finite and C-finite sequences are closed under sum and product.

If $(u_n), (v_n)$ are P-finite, deciding

monotonicity	$u_{n+1} \geq u_n$	
convexity	$u_{n+1} + u_{n-1} \geq 2u_n$	all reduce to positivity.
log-convexity	$u_{n+1}u_{n-1} \geq u_n^2$	
inequality	$u_n \geq v_n$	

Constant Coefficients Already Difficult

$$u_{n+d} = c_{d-1}u_{n+d-1} + \cdots + c_0 u_n, \qquad c_i \in \mathbb{Q}$$

Characteristic polynomial: $x^d - \sum_{i=0}^{d-1} c_i x^i = \prod_{i=1}^{k} (x - \lambda_i)^{m_i}$

Closed form: $u_n = C_1(n)\lambda_1^n + \cdots + C_k(n)\lambda_k^n$ C_i computable from $u_0, \ldots, u_{d-1}$

General case: positivity decidable for $d \leq 5$. [Ouaknine-Worrell 2014]

$d = 6$ related to open pbs in Diophantine approximation.

Skolem problem (decidable for $d \leq 4$) reduces to positivity.

Easy situation: $\boxed{|\lambda_1| > |\lambda_i|, i \neq 1}$ and $C_1 \neq 0$.

Aim: extension to polynomial coefficients

Gerhold-Kauers Method

Quantifier elimination:

$$\exists x, ax^2 + bx + c = 0 \mapsto (a \neq 0 \wedge b^2 - 4ac \geq 0) \vee (a = 0 \vee b \neq 0) \vee (a = 0 \vee b = 0 \vee c = 0)$$

GK: Use quantifier elimination to look for m s.t.

$$\forall n \geq 0, \forall u_n \geq 0, \forall u_{n+1} \geq 0, \ldots, \forall u_{n+d-1} \geq 0,$$
$$u_{n+d} \geq 0 \wedge \cdots \wedge u_{n+m} \geq 0 \Rightarrow u_{n+m+1} \geq 0.$$

Gives a proof when it terminates

[Gerhold-Kauers 2005]

Does not work in general: e.g., $u_{n+2} = 3u_{n+1} - 2u_n$, $u_0 = 1, u_1 = 3$

Variant: add $\exists \mu \geq 0$ and replace $u_{i+1} \geq 0$ by $u_{i+1} \geq \mu u_i$ [Kauers-Pillwein 2010]

Termination (under hypothesis) for $d = 2$ and cases of $d = 3$.

This work: generalize to arbitrary order

Proof by Induction on an Example

$$s_0 = 1,\ s_1 = 12,$$

[Straub-Zudilin 2015]

$$2(n+2)^2 s_{n+2} = (81n^2 + 243n + 186)s_{n+1} - 81(3n+2)(3n+4)s_n\,.$$

$s_n \geq 0, \ldots, s_{n+m-1} \geq 0$ do not imply $s_{n+m} \geq 0$.

Add $s_{n+1} \geq 18 s_n$ to the induction hypothesis, then

$$2\,(n+2)^2\left(s_{n+2} - 18 s_{n+1}\right) =$$

$$\left(45n^2 + 99n + 42\right)\underbrace{\left(s_{n+1} - 18 s_n\right)}_{\geq 0} + \left(81n^2 + 324n + 108\right)\underbrace{s_n}_{\geq 0}$$

implies $s_{n+2} \geq 18 s_{n+1} \geq 0$.

Method: synthesize extra hypotheses

First terms: $s_0 = 1, s_1 = 12, s_2 = 198, s_3 = 3720 > 18 \times s_2\,.$

II. Geometric Viewpoint on the Asymptotic Behaviour

Vector Version of the Recurrence

$$p_d(n)u_{n+d} = p_{d-1}(n)u_{n+d-1} + \cdots + p_0(n)u_n \Leftrightarrow$$

$$\begin{pmatrix} u_{n+1} \\ \vdots \\ u_{n+d} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ \frac{p_0(n)}{p_d(n)} & \cdots & \cdots & \cdots & \frac{p_{d-1}(n)}{p_d(n)} \end{pmatrix}}_{A(n)} \underbrace{\begin{pmatrix} u_n \\ \vdots \\ u_{n+d-1} \end{pmatrix}}_{U_n}$$

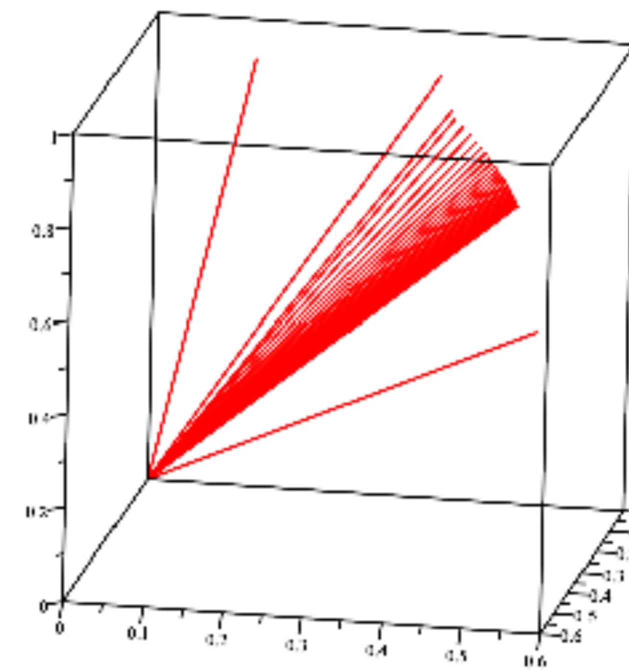
$$\begin{aligned} U_{n+1} &= A(n)U_n \\ &= A(n)A(n-1)\cdots A(0)U_0 \end{aligned}$$

$u_n \geq 0$ for all $n \Leftrightarrow U_n \in \mathbb{R}_+^d$ for all n

Constant Coefficients & Power Method

1. Pick a random U_0
2. For $n = 0,1,\ldots$

$$U_{n+1} := AU_n/\|U_n\|$$



Eigenvalues of A : $\lambda_1, \ldots, \lambda_k$

Hyp.: $\boxed{|\lambda_1| > |\lambda_i|, i \neq 1}$ and $\boxed{\lambda_1 \text{ simple.}}$

Principle: polynomial coefficients as a perturbation

Then $A^n/\|A^n\| \to VW^\top \quad (AV = \lambda_1 V, A^\top W = \lambda_1 W)$

Convergence to a rank 1 matrix

Generic $U_0 : W^\top U_0 \neq 0$

$$\Rightarrow U_n/\|U_n\| \to \pm V/\|V\|$$

Convergence to a dominant eigenvector

Generalized Power Method

Recall that
$U_{n+1} = A(n)\cdots A(0)U_0$

Thm. $A(n) \in \mathrm{GL}_d(\mathbb{R})$ with limit A s.t. $\lambda_1 > |\lambda_i|, i \neq 1,$ λ_1 simple, then

$$\frac{A(n)A(n-1)\cdots A(0)}{\|A(n)A(n-1)\cdots A(0)\|} \to VW^\top,$$

with $AV = \lambda_1 V$. Initial conditions generic when $W^\top U_0 \neq 0$.

[Friedland 2006]

Ex. Apéry recurrence

$$(n+2)^3 u_{n+2} = (2n+3)(17n^2+51n+39)u_{n+1} - (n+1)^3 u_n$$

$$V = \begin{pmatrix} 1 \\ (3+2\sqrt{2})^2 \end{pmatrix}, \quad W = \begin{pmatrix} 1 \\ 6/\zeta(3) - 5 \end{pmatrix}.$$

algebraic

difficult to control in general

Recurrences of Poincaré Type

$$U_{n+1} = A(n)U_n$$

Def. Poincaré type: $A := \lim_{n\to\infty} A(n)$ finite.

Polynomial coefficients as a perturbation

Ex. $2(n+2)^2 s_{n+2} = (81n^2 + 243n + 186)s_{n+1} - 81(3n+2)(3n+4)s_n$. ✔

$$A(n) = \begin{pmatrix} 0 & 1 \\ -\frac{81(3n+2)(3n+4)}{2(n+2)^2} & \frac{3(27n^2+81n+62)}{2(n+2)^2} \end{pmatrix} \to A = \begin{pmatrix} 0 & 1 \\ -\frac{729}{2} & \frac{81}{2} \end{pmatrix}$$

Ex. $u_{n+3} + u_{n+2} + nu_{n+1} + (n+1)u_n = 0$ ✘

Positivity-preserving reduction

$$v_n := \psi_n u_n$$
$$(n+2)\psi_{n+2} = \psi_n \quad \longrightarrow \quad (n+6)(n+4)v_{n+6} + 2(n+4)(n+1)v_{n+4}$$
$$\psi_0 = \psi_1 = 1 \quad \sim 1/\sqrt{n!} \qquad +(n^2-n-5)v_{n+2} - (n+1)v_n = 0$$

✔

[Mezzarobba-S. 2010]

III. Cones

Cones in $\mathbb{R}^d$

Def. $K \subset \mathbb{R}^d$ is a cone when

. $\mathbb{R}_+ K \subset K$;

. $K + K = K$;

. $K \cap (-K) = \{0\}$.

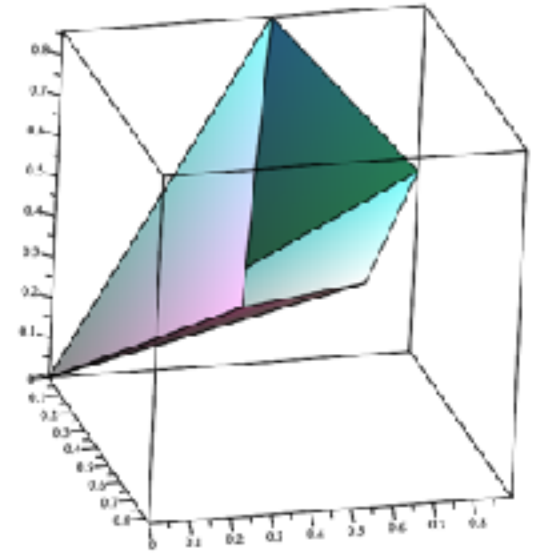
Def. $F \subset K$ is a face of K when

. F is a cone;

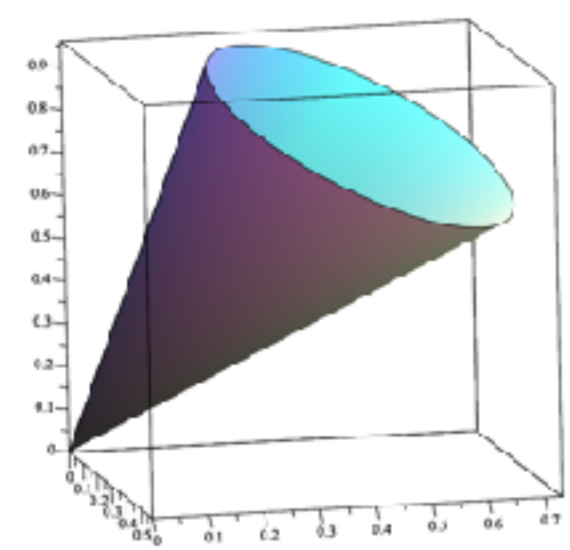
. u, v in K, $u + v \in F \Rightarrow u, v$ in F.

Def. $x \in K$ is an extremal vector of K when

$\mathbb{R}_+ x = \cap \{F \text{ face of } K \mid x \in F\}$



polyhedral: finite number of extremal vectors



In this talk, all cones are closed and solid ($\mathring{K} \neq \emptyset$).

Perron-Frobenius for Cones

(Classical case: $K = \mathbb{R}^d_+$)

$A \in \mathbb{R}^{d\times d}$, eigenvalues λ_i

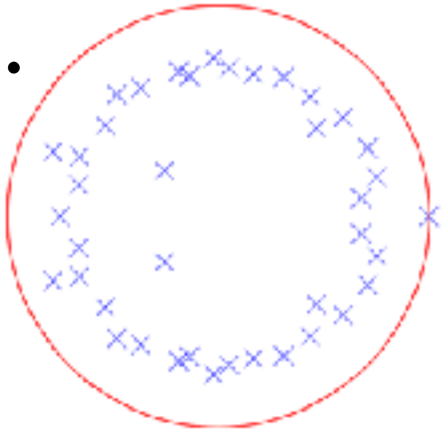
[Perron 1907, Frobenius 1912]

K-positive A: $A(K\backslash\{0\}) \subset \mathring{K}$.

[Birkhoff 1967]

Then K contains an eigenvector for $\lambda_1 = \rho(A)$.

Also, $\lambda_1 > |\lambda_i|, i \neq 1$, λ_1 simple.



[Vandergraft 1968]

A with this property is K-positive for some K.

K-irreducible & K-primitive also defined, with spectral characterizations.

Positivity by Contracted Cones

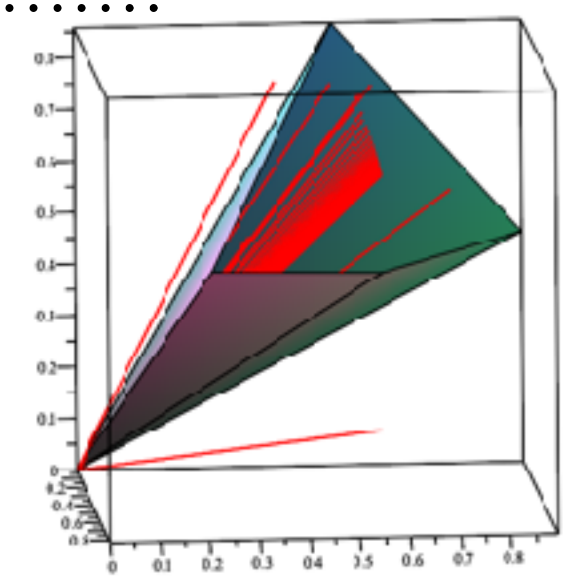
$A := \lim_{n\to\infty} A(n)$

Depends on $A(n)$ only

0. Check $\lambda_1 > |\lambda_i|$, $i \neq 1$;
1. Construct a cone $K \subset \mathbb{R}_+^d$ s.t. $A(K\backslash\{0\}) \subset \mathring{K}$;
2. Compute m s.t. for all $n \geq m$, $A(n)K \subset K$;

Depends also on U_0

3. For $i = 1,2,\ldots$ do
   1. $U_i = A(i)U_{i-1}$;
   2. If $U_i \notin \mathbb{R}_+^d$ return false
   3. If $i \geq m$ and $U_i \in K$ return true



Constructs a formula $\forall n \geq m, U_n \in K \subset \mathbb{R}_+^d$ and proves it by induction

Thm. If $\lambda_1 > |\lambda_i|, i \neq 1$, λ_1 simple, then for arbitrary order d, the algorithm terminates for generic U_0.

[Ibrahim-S. 2024]

Next: algorithms for the computation of K and m.

It works in practice!

```
> rec:=(16*n+1)*u(n+3)-(32*n-2)*u(n+2)+(20*n-4)*u(n+1)-(5*n-3)*u(n):
> ini:=u(0)=5,u(1)=5,u(2)=1:
> Positivity({rec,ini},u,n);
```

$$\mathit{true},\ \begin{bmatrix} \frac{44}{79} & 2 & 0 \\ \frac{41}{64} & \frac{213}{250} & \frac{301}{500} \\ \frac{81}{110} & \frac{909}{5000} & \frac{513}{1000} \end{bmatrix},\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix},\ 5$$

Certificate that can be checked separately

The sequence is positive

M, V s.t. $M \cdot V$ are the extremal vectors of a contracted cone in $\mathbb{R}^3_{>0}$

Starting at $n = 5$, U_n is in that cone

IV. Cones from Eigenvectors

Example. Part 1. Contracted Cone

1. Construct a cone $K \subset \mathbb{R}_+^d$ s.t. $A(K \backslash \{0\}) \subset \mathring{K}$;

$$\left(16n+1\right) u_{n+3} = \left(5n-3\right) u_n + (20n-4)\, u_{n+1} + (32n-2)\, u_{n+2}$$

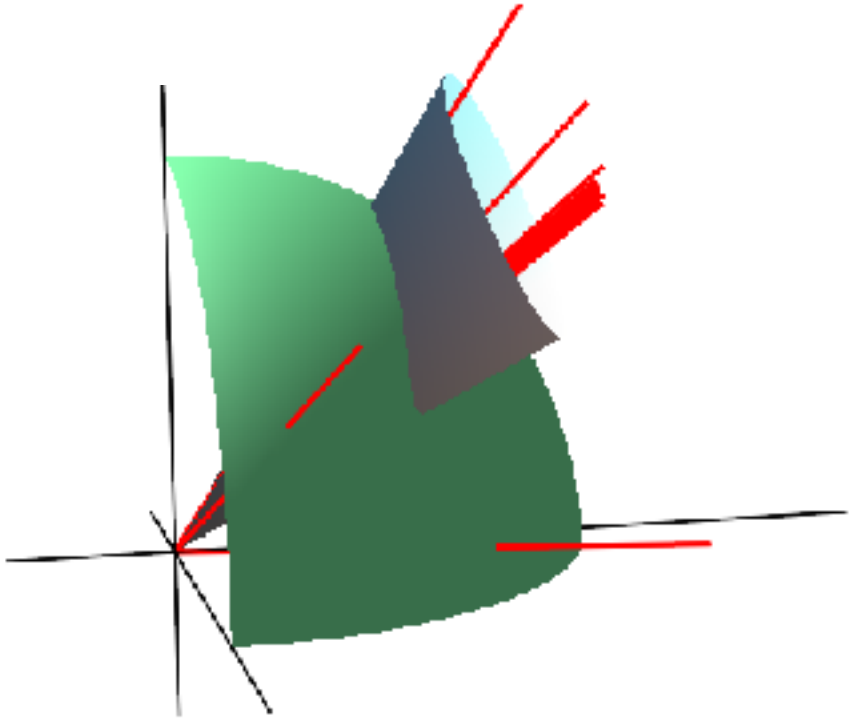
$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{5}{16} & -\frac{5}{4} & 2 \end{pmatrix}$$

Eigenvectors: $V_1, V_2, \overline{V_2}$

$$V_i = (1, \lambda_i, \lambda_i^2)^\mathsf{T}$$

$$\lambda_1 \approx 1.15 > |\lambda_2|,$$

$$\lambda_2 \approx 0.43 + 0.30i\,.$$



$$K_\mu := \{a\mu V_1 + bV_2 + \bar{b}\overline{V_2} \mid |b| \le a\} \text{ (real)}$$

$$\forall \mu > 0,\ AK_\mu \subset \mathring{K}_\mu$$

$$\exists \mu > 0,\ K_\mu \subset \mathbb{R}_{>0}^d$$

General Case: Vandergraft's Construction

1. Construct a cone $K \subset \mathbb{R}^d_+$ s.t. $A(K \backslash \{0\}) \subset \mathring{K}$

K-positive A: $A(K \backslash \{0\}) \subset \mathring{K}$.

[Vandergraft 1968]

Then $\lambda_1 > |\lambda_i|, i \neq 1$, λ_1 simple.

A with these properties is K-positive for some K.

From slide 13

Take a basis $V_1, \ldots, V_d$ where A has the form

$$A = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & J_2 & \ddots & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & J_k \end{pmatrix} \quad J_i = \begin{pmatrix} \lambda_i & \varepsilon & 0 & \cdots & 0 \\ 0 & \lambda_i & \varepsilon & \cdots & 0 \\ 0 & 0 & \lambda_i & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \varepsilon \\ 0 & 0 & 0 & \cdots & \lambda_i \end{pmatrix}$$

with $0 < \varepsilon < \lambda_1 - |\lambda_i|$ and $V_j = \overline{V_i}$ when $\lambda_j = \overline{\lambda_i}$.

$$K := \{a_1 V_1 + \cdots + a_d V_d \mid |a_i| \leq a_1 \text{ and } a_j = \overline{a_i} \text{ when } V_j = \overline{V_i}\}$$

satisfies $AK \subset \mathring{K}$.

Example. Part 2. Contraction Index

2. Compute m s.t. for all $n \geq m$, $A(n)K \subset K$;

$$K = \{aV_1 + bV_2 + \overline{b}\overline{V_2} \mid |b| \leq a\}$$

Extremal vectors: $V_1 + e^{it}V_2 + e^{-it}\overline{V_2}$

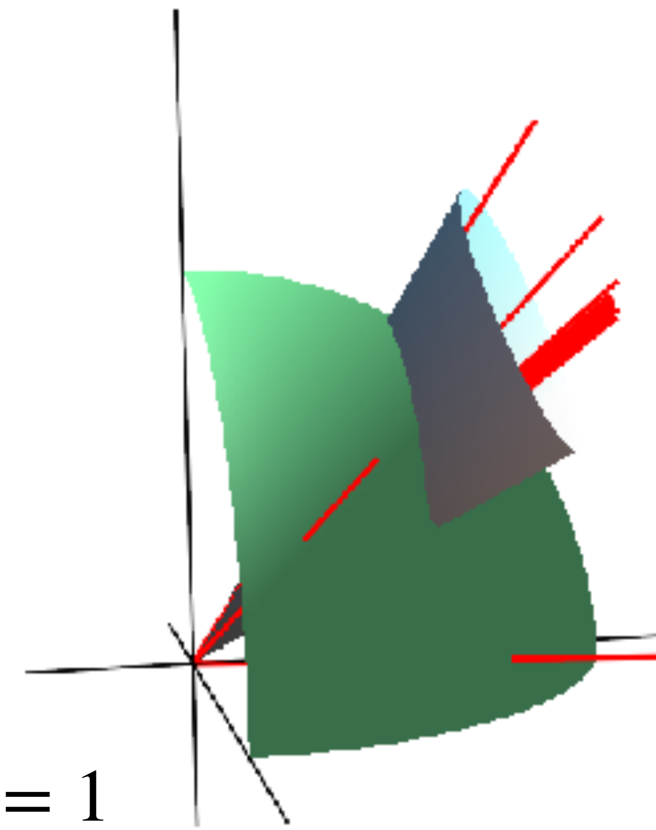


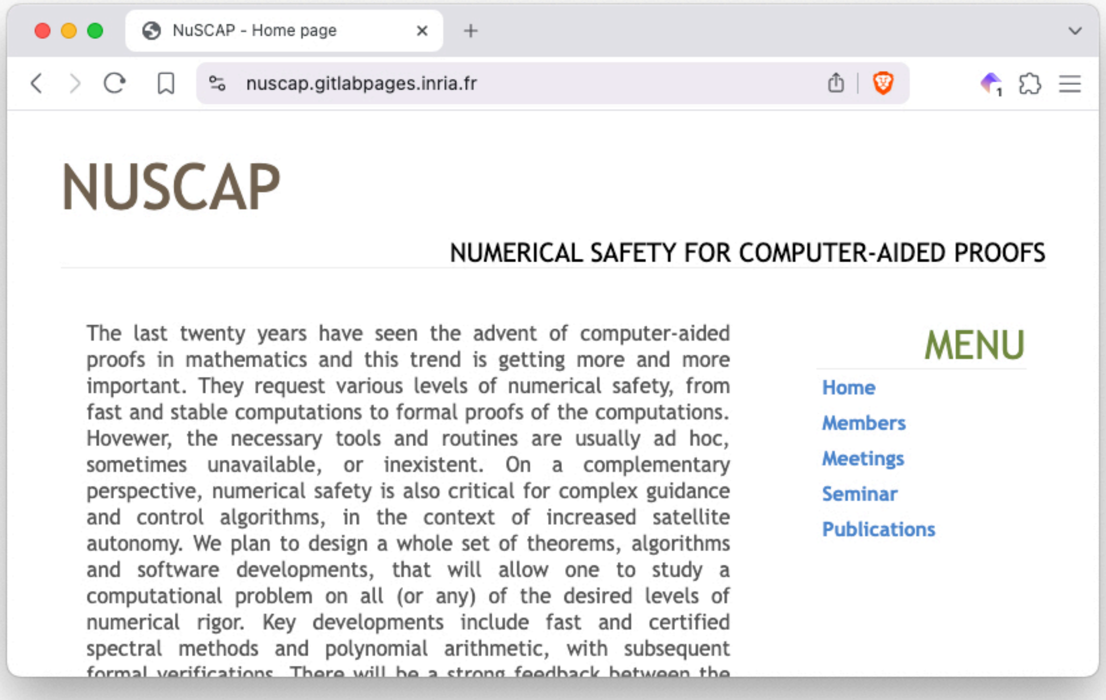
Image of extremal vectors

$$\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} = V^{-1}A(n)V\begin{pmatrix} 1 \\ c+is \\ c-is \end{pmatrix} \qquad c^2+s^2=1$$

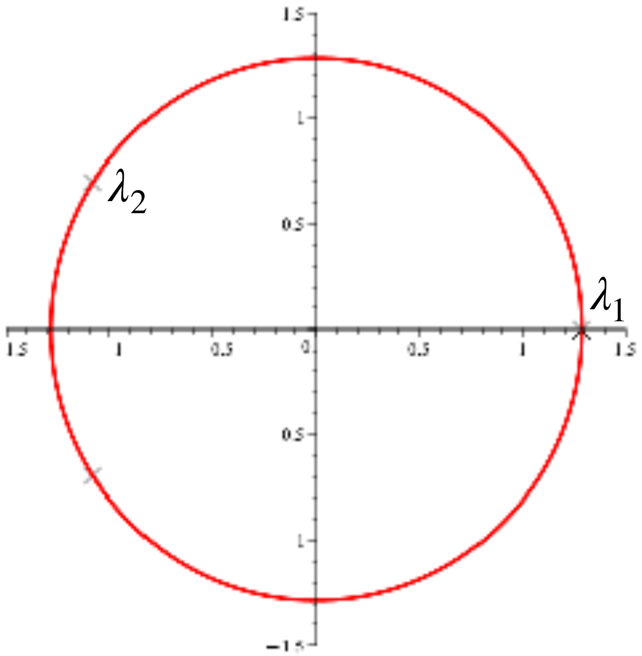
Wanted: m s.t. $\underbrace{a_n^2 - b_n c_n}_{\in \mathbb{Q}(n,\lambda_1,\lambda_2,\overline{\lambda_2})[c,s]/(c^2+s^2-1)} \geq 0$ for all c, s and $n \geq m$.

m computable by quantifier elimination (in theory)

IV. Symbolic-Numeric Aspects



Step 0: Test $\lambda_1 > |\lambda_i|,\ i \neq 1$



$8x^3 + 7x^2 - 9x - 17$
$\lambda_1 - |\lambda_2| \simeq 2.10^{-5}$

$$P = a_0 + \cdots + a_d X^d \in \mathbb{Z}[X], \quad |a_i| \leq H,$$

Absolute Separation:

[Bugeaud-Dujella-Pejkovic-S-Wang 2022]

$$\min_{\substack{P(\alpha)=P(\beta)=0,\\ |\alpha| \neq |\beta|}} \big| |\alpha| - |\beta| \big| > \kappa(d) H^{-e(d)}$$

probably not tight

explicit function of d

$O(d^3)$ in general, $O(d^2)$ if α real

Approximate roots with precision larger than the bound sufficient to decide equality of absolute values

Disks of radius ε for all roots can be computed in time $\tilde{O}(d^3 + d^2 \log H - d \log \varepsilon)$.

[Mehlhorn-Sagraloff 2016]

Approximate Cones

1. Contracted cone

$$\tilde{K} := \{a\tilde{V}_1 + b\tilde{V}_2 + \overline{b}\overline{\tilde{V}_2} \mid |b| \leq a\},$$

$\tilde{V}_1, \tilde{V}_2$ rational approximations of V_1, V_2

$$\tilde{\lambda}_1 = \frac{7}{6}, \qquad \tilde{\lambda}_2 = \frac{2}{5} + \frac{i}{3}$$

Check that $A\tilde{K} \subset \mathring{\tilde{K}} \subset \mathbb{R}_+^d$

otherwise, increase precision

2. Contraction index

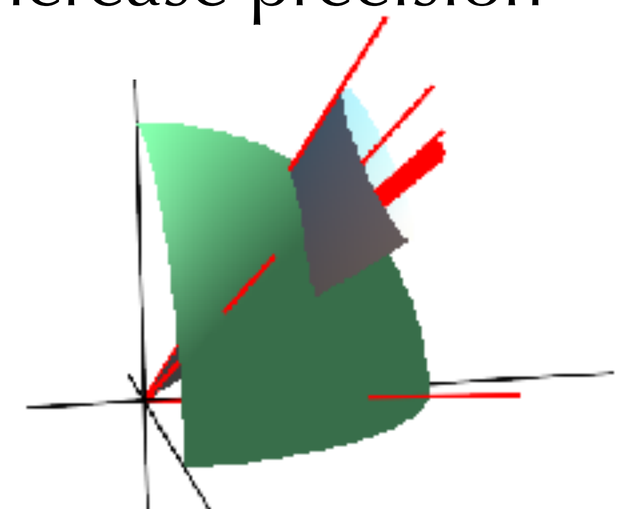


Image of extremal vectors $\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} = \tilde{V}^{-1}A(n)\tilde{V}\begin{pmatrix} 1 \\ c+is \\ c-is \end{pmatrix}$

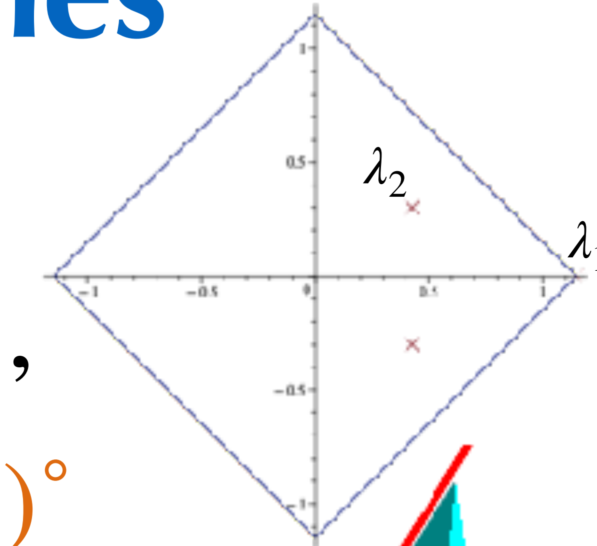
Wanted: m s.t. $\underbrace{a_n^2 - b_n c_n}_{p_2(s,c)n^2 + p_1(s,c)n + p_0(s,c)} \geq 0$ for all c, s and $n \geq m$.

now in $\mathbb{Q}[n,s,c]/(s^2+c^2-1)$
instead of $\mathbb{Q}(n,\lambda_1,\lambda_2,\overline{\lambda_2})[c,s]/(c^2+s^2-1)$

Interval analysis

$\geq 204.7n^2 - 408.4n - 324.0 > 0$ for $n \geq 3$

Polyhedral Contracted Cones



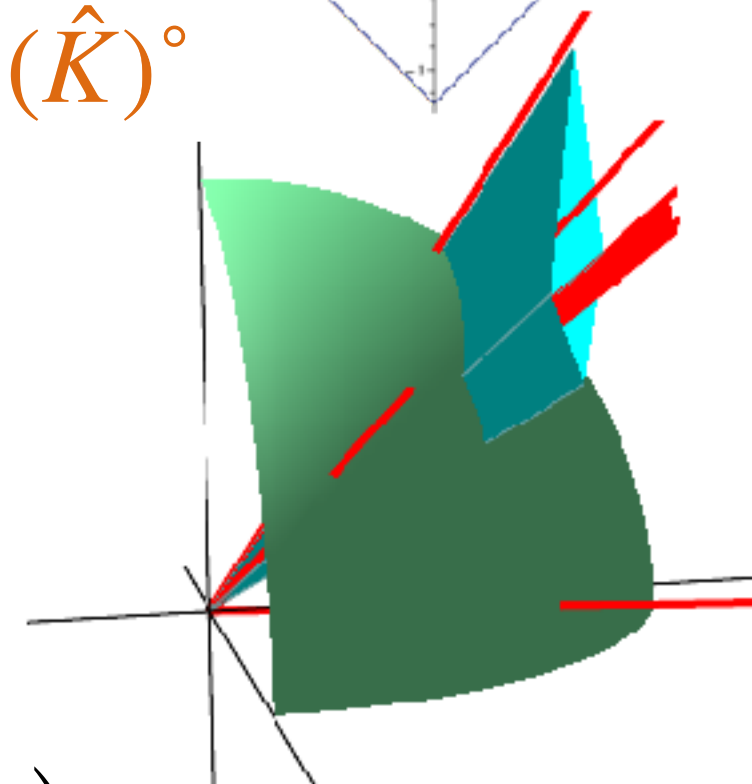
1. Contracted cone

$|\Re(b)| + |\Im(b)|$

$$\hat{K} := \{aV_1 + bV_2 + \overline{b}\overline{V_2} \mid \|b\|_1 \le a\},$$

$$\|uv\|_1 \le \|u\|_1\|v\|_1, \ \|\lambda_2\|_1 < \lambda_1 \Rightarrow A\hat{K} \subset (\hat{K})^\circ$$

+ approximate version as before



2. Contraction index

Image of 4 extremal vectors

$$\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} = \tilde{V}^{-1}A(n)\tilde{V}\begin{pmatrix} 1 \\ c+is \\ c-is \end{pmatrix}$$

Wanted: m s.t. $a_n \ge \|b_n\|_1$ for $\binom{c}{s} \in \{\begin{pmatrix} \pm 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \pm 1 \end{pmatrix}\}$ and $n \ge m$.

$\binom{0}{1} \to \frac{18811}{3774}n - \frac{452275}{30192} > 0$ for $n \ge 4$, $\quad \binom{1}{0} \to \frac{138361}{18870}n + \frac{576563}{150960} > 0$ for $n \ge 5$,

$\binom{0}{-1} \to \frac{139657}{18870}n + \frac{2215391}{150960} > 0$ for $n \ge 3$, $\quad \binom{-1}{0} \to \frac{681019}{94350}n - \frac{3927583}{754800} > 0$ for $n \ge 1$

$\tilde{\hat{K}}$ contracted by A_n for $n \ge 5$

simple computations in $\mathbb{Q}[n]$

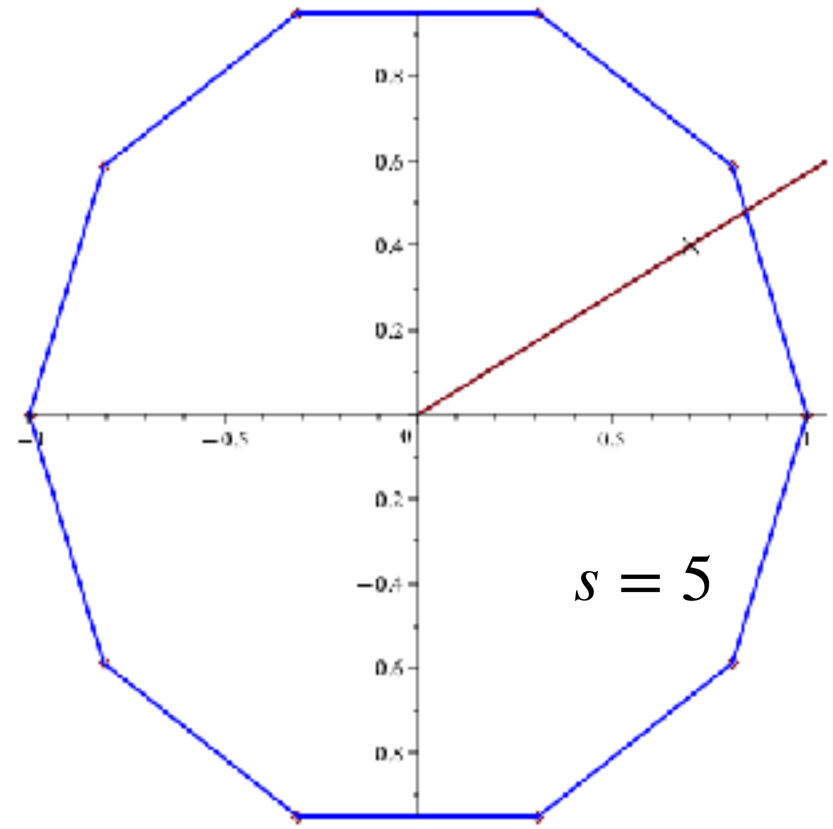
General Case: Minkowski Functional

For $\mathscr{P} \subset \mathbb{C}$,

$$\|z\|_{\mathscr{P}} = \inf\{r > 0 \,|\, z \in r\mathscr{P}\}$$

$\mathscr{P}_s :=$ polygon with vertices at $e^{ik\pi/s}$

$\|\cdot\|_{\mathscr{P}_s}$ is a sub-multiplicative norm



$$K := \{a_1V_1 + \cdots + a_dV_d \mid \|a_i\|_{\mathscr{P}_{s_i}} \leq a_1 \text{ and } a_j = \overline{a_i} \text{ when } V_j = \overline{V_i}\}$$

with s_i minimal s.t. $\lambda_i \in \mathscr{P}_{s_i}$

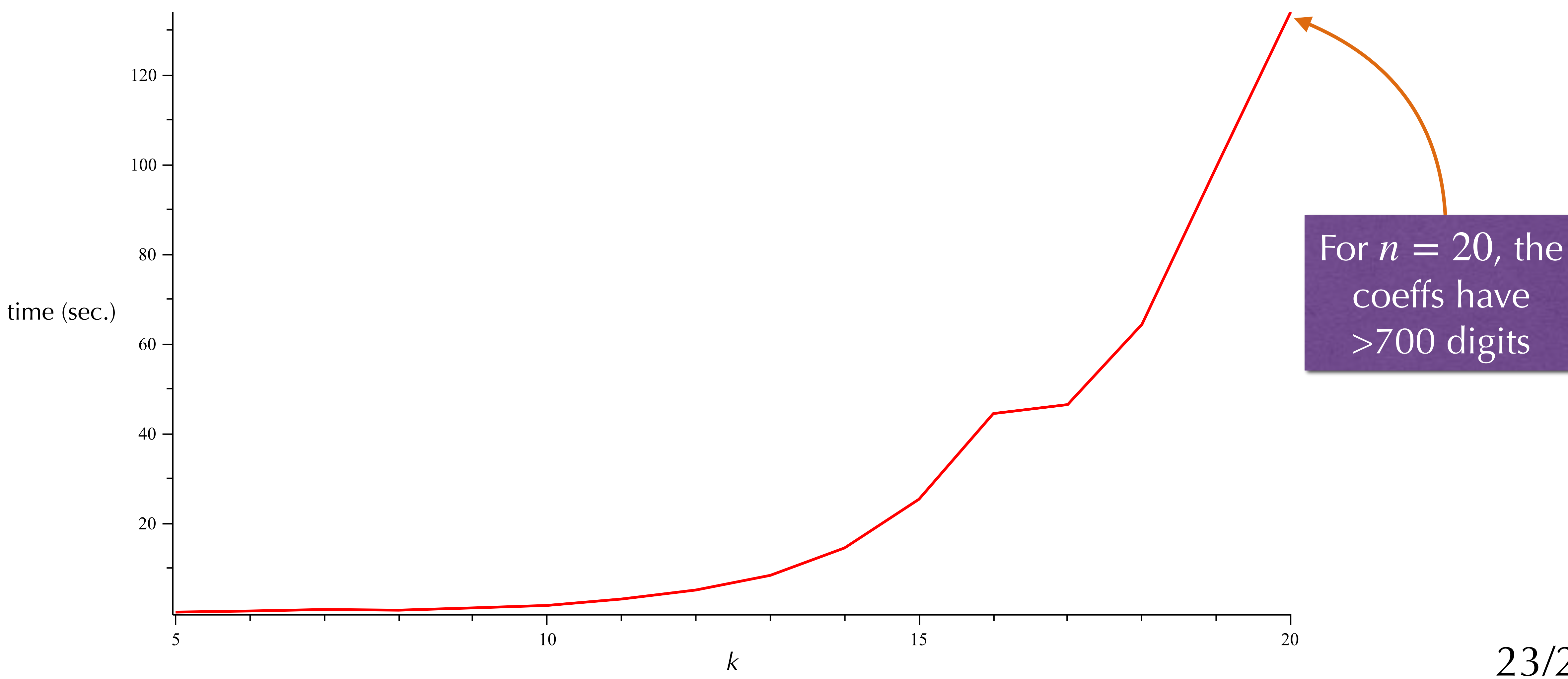
is a polyhedral cone that satisfies $AK \subset \mathring{K}$.

Timings

$$u_n^{(k)} = \sum_{j=0}^{n} (-1)^j \frac{(kn-(k-1)j)!k!^j}{(n-j)!^k j!} \geq 0 \text{ for } k \geq 4$$

Gillis-Reznick-Zeilberger

linear rec of order k with coeffs of degree $k(k-1)/2$



Conclusions

Contracted cones give an access to positivity proofs for many sequences;

the cone, plus contraction index, give a certificate $\equiv$ a property that can proved by induction;

certified numerical computations save the day.

In progress:

positivity proofs without simple dominant eigenvalue [Ibrahim 2025]

recurrences with parameters

Thank You!

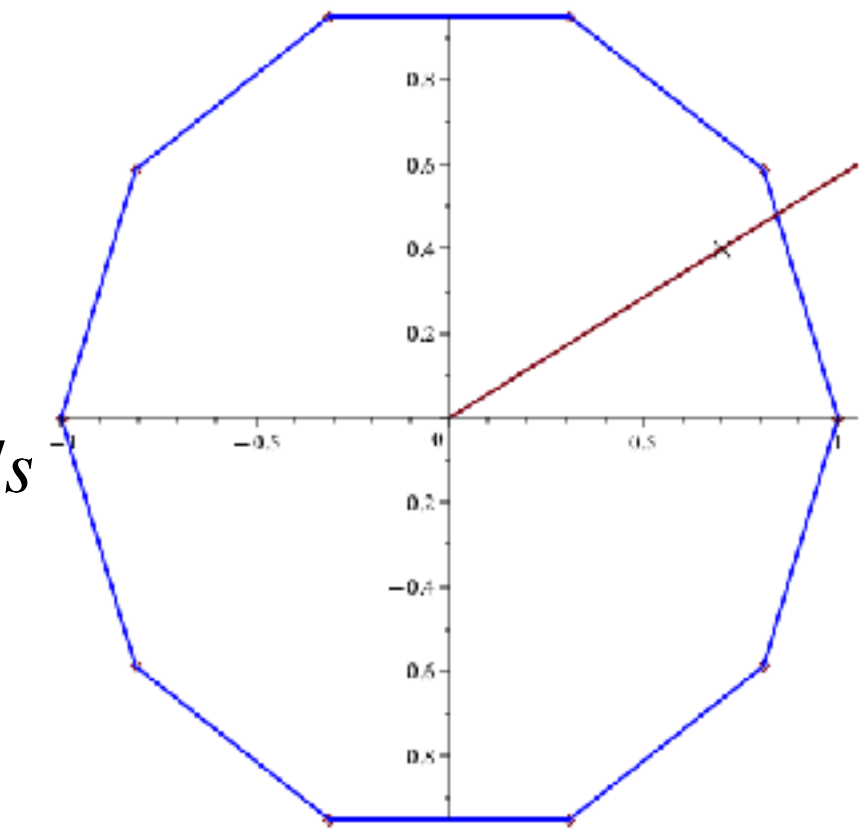
Computation of the Minkowski Functional

For $\mathscr{P} \subset \mathbb{C}$,

$$\|z\|_{\mathscr{P}} = \inf\{r > 0 \,|\, z \in r\mathscr{P}\}$$

$\mathscr{P}_s :=$ polygon with vertices at $\omega_s = e^{ik\pi/s}$

$\|\cdot\|_{\mathscr{P}_s}$ is a sub-multiplicative norm



$$\|z\|_{\mathscr{P}_s} = \Re\left(\frac{1+\omega_s^{-1}}{1+\cos(\frac{\pi}{s})}\frac{z}{\omega_s^m}\right), \qquad \frac{m\pi}{s} \le \arg z \le \frac{(m+1)\pi}{s}.$$