Demo Session for the STACS2018 Tutorial on Recursive Combinatorial Structures

To be read jointly with the slides available at

http://perso.ens-lyon.fr/bruno.salvy/talks/STACS2018.pdf

0. Load a few packages

In this session, we are using Maple's combstruct package (developed in the Algorithms group at Inria) and the more recent NewtonGF that you can download from

http://perso.ens-lyon.fr/bruno.salvy/software/the-newtongf-package/

```
> with(NewtonGF);
```

$$[BoltzmannExpectedSize, BoltzmannParameter, GFSeries, NumericalNewtonIteration, Radius, SeriesNewtonIteration] \quad (1.1)$$

Check with the web page whether you have the latest version:

```
> NewtonGF:-version();
```

$$2.21 \quad (1.2)$$

Next, we also use one function (equivalent) from Algolib (http://algo.inria.fr/libraries/17.0/algolib.mla) which provides tools for the asymptotic analysis.

```
> libname:="algolib",libname:
```

I. First specifications

Binary trees

In this system, Union stands for disjoint union (+) and Prod for cartesian product (x)

```
> bintrees:={B=Union(Epsilon,Prod(Z,B,B))};
```

$$bintrees := \{B = Union(\mathrm{E}, Prod(Z, B, B))\} \quad (2.1.1)$$

Counting

Using this grammar, one can count

```
> GFSeries(bintrees,unlabelled,z,21);
```

$$[Z = z + O(z^{21}), B = 1 + z + 2 z^2 + 5 z^3 + 14 z^4 + 42 z^5 + 132 z^6 + 429 z^7 + 1430 z^8 + 4862 z^9 + 16796 z^{10} + 58786 z^{11} + 208012 z^{12} + 742900 z^{13} + 2674440 z^{14} + 9694845 z^{15} + 35357670 z^{16} + 129644790 z^{17} + 477638700 z^{18} + 1767263190 z^{19} + 6564120420 z^{20} + O(z^{21})] \quad (2.2.1)$$

For instance, the coefficient of z^20 in this power series is the number of binary trees with 20 internal nodes.

Several variants of binary trees are possible, depending on where the size is. In the example above, the internal nodes have size 1 and the leaves have size 0. The other way round is also possible

```
> GFSeries({B=Union(Z,Prod(B,B))},unlabelled,z,11);
```

$$[Z = z + O(z^{11}), B = z + z^2 + 2 z^3 + 5 z^4 + 14 z^5 + 42 z^6 + 132 z^7 + 429 z^8 \quad (2.2.2)$$

$$+ 1430 z^9 + 4862 z^{10} + O(z^{11})]$$

Since a binary tree with n leaves has n-1 internal nodes, the sequences are just shifted by 1.

One can also put weights at both internal and external nodes:

```
> GFSeries({B=Union(Z,Prod(Z,B,B))},unlabelled,z,11);
```

$$[Z = z + O(z^{11}), B = z + z^3 + 2 z^5 + 5 z^7 + 14 z^9 + O(z^{11})] \tag{2.2.3}$$

The size is necessarily odd and the same sequence (the Catalan numbers) counts these trees.

Labelled vs unlabelled

The unlabelled case have been seen above. In the case of binary trees, labelled enumeration is the same:

```
> GFSeries(bintrees,unlabelled,z,11);
```

$$[Z = z + O(z^{11}), B = 1 + z + 2 z^2 + 5 z^3 + 14 z^4 + 42 z^5 + 132 z^6 + 429 z^7 + 1430 z^8 + 4862 z^9 + 16796 z^{10} + O(z^{11})] \tag{2.3.1}$$

```
> GFSeries(bintrees,labelled,z,11);
```

$$[Z = z + O(z^{11}), B = 1 + z + 2 z^2 + 5 z^3 + 14 z^4 + 42 z^5 + 132 z^6 + 429 z^7 + 1430 z^8 + 4862 z^9 + 16796 z^{10} + O(z^{11})] \tag{2.3.2}$$

Involutions

Grammar:

```
> inv:={Inv=Set(Cycle(Z,card<=2))};
```

$$inv := \{Inv = Set(Cycle(Z, card \leq 2))\} \tag{2.4.1}$$

Enumeration in the labelled and unlabelled universes:

```
> GFSeries(inv,unlabelled,z,11);
```

$$[Z = z + O(z^{11}), Inv = 1 + z + 2 z^2 + 2 z^3 + 3 z^4 + 3 z^5 + 4 z^6 + 4 z^7 + 5 z^8 + 5 z^9 + 6 z^{10} + O(z^{11})] \tag{2.4.2}$$

```
> GFSeries(inv,labelled,z,11);
```

$$\left[Z = z + O(z^{11}), Inv = 1 + z + z^2 + \frac{2}{3} z^3 + \frac{5}{12} z^4 + \frac{13}{60} z^5 + \frac{19}{180} z^6 + \frac{29}{630} z^7 + \frac{191}{10080} z^8 + \frac{131}{18144} z^9 + \frac{1187}{453600} z^{10} + O(z^{11})\right] \tag{2.4.3}$$

Closed-forms for these generating functions:

```
> combstruct[gfeqns](inv,labelled,z);
```

$$\left[Inv(z) = e^{z + \frac{1}{2} z^2}, Z(z) = z\right] \tag{2.4.4}$$

```
> combstruct[gfeqns](inv,unlabelled,z);
```

$$\left[Inv(z) = e^{\sum_{j_1=1}^{\infty} \frac{z^{2 j_1} + z^{j_1}}{j_1}}, Z(z) = z\right] \tag{2.4.5}$$

The last one can be simplified:

```
> simplify(value(%)) assuming z>0,z<1;
```

$$\left[Inv(z) = \frac{1}{(z+1)(-1+z)^2}, Z(z) = z\right] \tag{2.4.6}$$

Other specifications

These will be used in later examples:

Permutations:

```
> perm:={P=Set(Cycle(Z))};
```

$$perm := \{P = Set(Cycle(Z))\} \quad (2.5.1)$$

Cayley trees:

```
> cayley:={T=Prod(Z,Set(T))};
```

$$cayley := \{T = Prod(Z, Set(T))\} \quad (2.5.2)$$

Functional graphs:

```
> func_graphs:={F=Set(Cycle(T)),T=Prod(Z,Set(T))};
```

$$func\_graphs := \{F = Set(Cycle(T)), T = Prod(Z, Set(T))\} \quad (2.5.3)$$

Series-parallel graphs:

```
> spgraphs:={G=Union(Z,S,P),S=Sequence(Union(Z,P),card>1),P=
Set(Union(Z,S),card>1)};
```

$$spgraphs := \{G = Union(Z, S, P), P = Set(Union(Z, S), 1 < card), S = Sequence(Union(Z, P), 1 < card)\} \quad (2.5.4)$$

Parameters

Using named atoms of size 0:

Count permutations by their number of cycles:

```
> perm2:={P=Set(Prod(U,Cycle(Z))),U=Epsilon};
```

$$perm2 := \{P = Set(Prod(U, Cycle(Z))), U = \mathrm{E}\} \quad (2.6.1.1)$$

In this specification, a mark U of size 0 is attached to each cycle in the permutation. This specification can then be translated into bivariate generating functions

```
> combstruct[gfeqns](perm2,labelled,z,[[u,U]]);
```

$$\left[P(z, u) = \mathrm{e}^{u \ln\left(\frac{1}{1-z}\right)}, U(z, u) = u, Z(z, u) = z\right] \quad (2.6.1.2)$$

```
> perm_biv:=subs(%,P(z,u));
```

$$perm\_biv := \mathrm{e}^{u \ln\left(\frac{1}{1-z}\right)} \quad (2.6.1.3)$$

From there, the first values follow

```
> map(series,series(perm_biv,u),z);
```

$$(1) + \left(z + \frac{1}{2} z^2 + \frac{1}{3} z^3 + \frac{1}{4} z^4 + \frac{1}{5} z^5 + \mathrm{O}(z^6)\right) u + \left(\frac{1}{2} z^2 + \frac{1}{2} z^3 + \frac{11}{24} z^4 + \frac{5}{12} z^5 + \mathrm{O}(z^6)\right) u^2 + \left(\frac{1}{6} z^3 + \frac{1}{4} z^4 + \frac{7}{24} z^5 + \mathrm{O}(z^6)\right) u^3 + \left(\frac{1}{24} z^4 + \frac{1}{12} z^5 + \mathrm{O}(z^6)\right) u^4 + \left(\frac{1}{120} z^5 + \mathrm{O}(z^6)\right) u^5 \quad (2.6.1.4)$$

So, for instance, the coefficient 7/24 of z^5u^3 means that there are 35=5! x 7/24 permutations of (1,2,3,4,5) with 3 cycles.

Now we can also compute the average number of cycles in a permutation:

```
> subs(u=1,diff(perm_biv,u));
```

$$\ln\left(\frac{1}{1-z}\right) \mathrm{e}^{\ln\left(\frac{1}{1-z}\right)} \quad (2.6.1.5)$$

```
> simplify(%);
```

$$-\frac{\ln\left(-\frac{1}{-1+z}\right)}{-1+z} \tag{2.6.1.6}$$

```
> series(%,z,15);
```

$$z+\frac{3}{2}z^2+\frac{11}{6}z^3+\frac{25}{12}z^4+\frac{137}{60}z^5+\frac{49}{20}z^6+\frac{363}{140}z^7+\frac{761}{280}z^8+\frac{7129}{2520}z^9+\frac{7381}{2520}z^{10}+\frac{83711}{27720}z^{11}+\frac{86021}{27720}z^{12}+\frac{1145993}{360360}z^{13}+\frac{1171733}{360360}z^{14}+O\left(z^{15}\right) \tag{2.6.1.7}$$

The coefficient of z^n is the nth harmonic number (1+1/2+...+1/n), showing that the average number of cycles in a permutation of (1,...,n) is asymptotically log n.

Using attribute grammars

Recall the grammar for binary trees

```
> bintrees;
```

$$\{B = Union(E, Prod(Z, B, B))\} \tag{2.6.2.1}$$

The attribute computing the internal path length of binary trees (the sum of the distances from each node to the root) is defined recursively on top of this:

```
> att:={pl(B)=Union(0,Prod(0,size(B)+pl(B),size(B)+pl(B)))};
```

$$att := \{pl(B) = Union(0, Prod(0, size(B)+pl(B), size(B)+pl(B)))\} \tag{2.6.2.2}$$

From there, bivariate generating functions can be derived automatically

```
> combstruct[agfseries](bintrees, att, unlabeled, z, [[u,
  pl]]);
```

$$table\left(\left[B(z,u) = 1+z+2uz^2+\left(4u^3+u^2\right)z^3+\left(8u^6+2u^5+4u^4\right)z^4+\left(16u^{10}+4u^9+8u^8+8u^7+6u^6\right)z^5+O\left(z^6\right), Z(z,u) = uz\right]\right) \tag{2.6.2.3}$$

The coefficient of u^kz^n in there is the number of binary trees with n internal nodes and path length k.

Equations can be derived automatically as well

```
> combstruct[agfeqns](bintrees, att, unlabeled, z, [[u, pl]]
  );
```

$$\left[B(z,u) = 1+zB(zu,u)^2, Z(z,u) = zu\right] \tag{2.6.2.4}$$

```
> eqB:=op(1,%);
```

$$eqB := B(z,u) = 1+zB(zu,u)^2 \tag{2.6.2.5}$$

At u=1 this is the same as before: the generating function of the Catalan numbers.

This equation can be differentiated and evaluated at u=1:

```
> map(diff,eqB,u);
```

$$\frac{\partial}{\partial u}B(z,u) = 2zB(zu,u)\left(D_1(B)(zu,u)z+D_2(B)(zu,u)\right) \tag{2.6.2.6}$$

```
> convert(%,D);
```

$$D_2(B)(z,u) = 2zB(zu,u)\left(D_1(B)(zu,u)z+D_2(B)(zu,u)\right) \tag{2.6.2.7}$$

```
> eval(%,u=1);
```

$$D_2(B)(z,1) = 2zB(z,1)\left(D_1(B)(z,1)z+D_2(B)(z,1)\right) \tag{2.6.2.8}$$

```
> gfpl:=solve(%,op(1,%));
```

$$gfpl := -\frac{2B(z,1)D_1(B)(z,1)z^2}{2zB(z,1)-1} \tag{2.6.2.9}$$

In this expression of the generating function, B(z,1) is the generating function of the number of binary trees and D_1(B)(z,1) is its derivative.

II. Enumeration

Series-parallel graphs

Recall their grammar

```
> spgraphs;
```

$$\{G = Union(Z, S, P), P = Set(Union(Z, S), 1 < card), S = Sequence(Union(Z, P), 1 < card)\} \quad \textbf{(3.1.1)}$$

Count up to size 20

```
> GFSeries(spgraphs,labelled,z,21);
```

$$\Big[Z = z + O(z^{21}), P = \frac{1}{2} z^2 + \frac{7}{6} z^3 + \frac{73}{24} z^4 + \frac{1051}{120} z^5 + \frac{19381}{720} z^6 + \frac{436087}{5040} z^7 + \frac{11585953}{40320} z^8 + \frac{50711653}{51840} z^9 + \frac{12322179901}{3628800} z^{10} + \frac{477938035807}{39916800} z^{11} + \frac{20485584143113}{479001600} z^{12} + \frac{961567521142411}{6227020800} z^{13} + \frac{4905491228765946 1}{87178291200} z^{14} + \frac{386081655546862081}{186810624000} z^{15} + \frac{15991196823309586 7953}{20922789888000} z^{16} + \frac{1011412085415424373 8771}{355687428096000} z^{17} + \frac{6809433238458078481 42861}{6402373705728000} z^{18} + \frac{4862215027002682021 6099567}{121645100408832000} z^{19} + \frac{3670113810844512283 440027673}{2432902008176640000} z^{20} + O(z^{21}), S = z^2 + 2 z^3 + \frac{61}{12} z^4 + \frac{29}{2} z^5 + \frac{15961}{360} z^6 + \frac{2841}{20} z^7 + \frac{9489061}{20160} z^8 + \frac{9675221}{6048} z^9 + \frac{10062777121}{1814400} z^{10} + \frac{5907821951}{302400} z^{11} + \frac{16699361378701}{239500800} z^{12} + \frac{31881496079}{126720} z^{13} + \frac{3993974290192068 1}{43589145600} z^{14} + \frac{366558482492939101}{108972864000} z^{15} + \frac{1182574676807006227 1}{951035904000} z^{16} + \frac{1612664890101185883 7}{348713164800} z^{17} + \frac{5535562257490782827 90641}{3201186852864000} z^{18} + \frac{5501988125434716665 273}{8468748288000} z^{19} + \frac{2981971748932306048 579052701}{1216451004088320000} z^{20} + O(z^{21}), G = z + \frac{3}{2} z^2 + \frac{19}{6} z^3 + \frac{65}{8} z^4 + \frac{2791}{120} z^5 + \frac{17101}{240} z^6 + \frac{1152019}{5040} z^7 + \frac{2037605}{2688} z^8 + \frac{935494831}{362880} z^9 + \frac{10815911381}{1209600} z^{10} + \frac{1257770533339}{39916800} z^{11} + \frac{513183875243}{4561920} z^{12} + \frac{252822423846447 1}{6227020800} z^{13} \quad \textbf{(3.1.2)}$$

$$+ \frac{42978132697166941}{29059430400} z^{14} + \frac{7101273378743303779}{1307674368000} z^{15}$$
$$+ \frac{2154248190413524297}{107296358400} z^{16} + \frac{2414845703016939977501}{32335220736000} z^{17}$$
$$+ \frac{85145513111617353034483}{304874938368000} z^{18} + \frac{127652707703771090396080939}{121645100408832000} z^{19}$$
$$+ \frac{11677645222677726521937131}{2948972131123200} z^{20} + \mathrm{O}\left(z^{21}\right) \Big]$$

Complexity wrt size is good:

```
> GFSeries(spgraphs,labelled,z,100);
```

$$\Big[Z = z + \mathrm{O}\left(z^{100}\right), P = \frac{1}{2} z^2 + \frac{7}{6} z^3 + \frac{73}{24} z^4 + \frac{1051}{120} z^5 + \frac{19381}{720} z^6 \tag{3.1.3}$$
$$+ \frac{436087}{5040} z^7 + \frac{11585953}{40320} z^8 + \frac{50711653}{51840} z^9 + \frac{12322179901}{3628800} z^{10}$$
$$+ \frac{477938035807}{39916800} z^{11} + \frac{20485584143113}{479001600} z^{12} + \frac{961567521142411}{6227020800} z^{13}$$
$$+ \frac{49054912287659461}{87178291200} z^{14} + \frac{386081655546862081}{186810624000} z^{15}$$
$$+ \frac{1599119682330958 67953}{20922789888000} z^{16} + \frac{10114120854154243738771}{355687428096000} z^{17}$$
$$+ \frac{680943323845807848142861}{6402373705728000} z^{18} + \frac{48622150270026820216099567}{121645100408832000} z^{19}$$
$$+ \frac{3670113810844512283440027673}{243290200817664000} z^{20}$$
$$+ \frac{41713825110768176151124164493}{7298706024529920000} z^{21}$$
$$+ \frac{2442235428251304102553443324 9301}{1124000727777607680000} z^{22}$$
$$+ \frac{2142271229523782591975604485014807}{25852016738884976640000} z^{23}$$
$$+ \frac{196652401492001818227840067840750273}{620448401733239439360000} z^{24}$$
$$+ \frac{18854021652578705866643315410059626851}{15511210043330985984000000} z^{25}$$
$$+ \frac{991856025081685816655836986453218 99599}{21225866375084507136000000} z^{26}$$
$$+ \frac{28007371496957409845800458276276058474441}{1555552778631193165824000000} z^{27}$$
$$+ \frac{2119534735730151032006671904109496114 5024553}{304888344611713860501504000000} z^{28}$$
$$+ \frac{2377903393436843018289227615684919343188476971}{8841761993739701954543616000000} z^{29}$$

$$+ \frac{2764757793545808787814415090463307056264385601 01}{265252859812191058636308480000000} z^{30}$$

$$+ \frac{332731931853809206447111218860505756968710548360 07}{822283865417792281772556288000000 0} z^{31}$$

$$+ \frac{41400664025740769967908058425333144277562790348601 13}{263130836933693530167218012160000000} z^{32}$$

$$+ \frac{76002984345447747263532793145504776997345590715462773}{1240473945544555213645456343040000000} z^{33}$$

$$+ \frac{705376376321529395421129893495422072335468365726513148 61}{29523279903960414084761860964352000000 00} z^{34}$$

$$+ \frac{963990594199339983311454979447292072789046939044084745396 7}{103331479663861449296666513375232000000 00} z^{35}$$

$$+ \frac{13567422218498669104427709482336851077272549268828795172073 53}{37199332678990121746799944815083520000000 0}$$

$$z^{36}$$

$$+ 19648511282562982842919346263956989108709380764708810127659 7\backslash$$

$$371 \,/\, 1376375309122634504631597958158090240000000 0\, z^{37}$$

$$+ 29256684137149295671344715092998878162359931761209751407389 3\backslash$$

$$13461 \,/\, 523022617466601111760007224100074291200000000\, z^{38}$$

$$+ 63938090084432844226366828355208454204042062780276241324441 6\backslash$$

$$460881 \,/\, 29139831544567776226628973914146996224000000000\, z^{39}$$

$$+ 70294054860401877755511552947450057462829902529390639427252427 5\backslash$$

$$637473 \,/\, 815915283247897734345611269596115894272000000000\, z^{40}$$

$$+ 11326997013759058749181903634118671601122116705743795339206185 3\backslash$$

$$325659651 \,/$$

$$33452526613163807108170062053440751665152000000000\, z^{41}$$

$$+ 18714055377628403437272091524306584119946728622887655983693233 4\backslash$$

$$14152447101 \,/$$

$1405006117752879898543142606244511569936384000000000\, z^{42}$

$+ 316820435490927777382129055992915809544798215271331751715704144\backslash$

$6928345320607 /$

$6041526306337383563735513206851399750726451200000000\, z^{43}$

$+ 289097834755580545046918517988431913323155351884072778589626047\backslash$

$51640848114867 /$

$1399090302520236193707171479481376784378757120000000000\, z^{44}$

$+ 139246786973858916524299456070737858806116689337797562667259297\backslash$

$87627663603000093 /$

$17088888695068599223137594499379673580626247680000000000\, z^{45}$

$+ 176944886492693306984088501943791581551916375621675042950369209\backslash$

$2221268521668570138 1 /$

$55026221598120889498503054288002548929616517529600000000000\, z^{46}$

$+ 887650040870300881645688864599924662387774701574098127191483734\backslash$

$650987740933759746 51 /$

$698981733813968055791795554469221567484317925376000000000000\, z^{47}$

$+ 623002648789967845429646905016029482905628278457468490744048069\backslash$

$003333819622534266135153 /$

$1241391559253607267086228904737337503852148635467776000000000000\, z^{48}$

$+ 120719195648858220709336935811180310739645322162462829137520\backslash$

$884450652382970987335041660371 /$

$$608281864034267560872252163321295376887552831379210240000000000\, z^{49}$$

$$+\ 238841797631809198000303681588829854290851147302543296141880\backslash$$

$$33176213987330956618025822036301 /$$

$$3041409320171337804361260816606476884437764156896051200000000000000$$

$$z^{50}$$

$$+\ 688984122074322568383226988386675823292157185776172975830347410\backslash$$

$$0925272580015639032477602646 01 /$$

$$2215883933267688971748918594956147444376085314309980160000000000000$$

$$z^{51}$$

$$+\ 993550281579880441950860264114066252672951998086288159955132963\backslash$$

$$6804210043091203152962801950091 13 /$$

$$8065817517094387857166063685640376697528950544088327782400000000000\backslash$$

$$00\, z^{52}$$

$$+\ 208731418282399884891568714119080902541226890199448445461510646\backslash$$

$$682039050020235812829500523225104411 /$$

$$4274883284060025564298013753389399649690343788366813724672000000000\backslash$$

$$0000\, z^{53}$$

$$+\ 447030776676267349359277825172358921163541767242067724460373672\backslash$$

$$5416696064641992492298202086869198626 1 /$$

$$2308436973392413804720927426830275810832785645718079411322880000000\backslash$$

$$000000\, z^{54}$$

$+ 9756208654805312596411244209040639793452788360917160712804782577355532526499987157289472958166898369367 / 12696403353658275925965100847566516959580321051449436762275840000000000000 z^{55}$

$+ 2169037481781621884630168284557788416619355591883822751561547384515528142875397948227089054881107257258753 / 710998587804863451854045647463724949736497978881168458687447040000000000000 z^{56}$

$+ 7015378329482373273581007156092421999536691840522768607504881196996158347459504494348041865059171443513 4853 / 57895599292681738222400859864903317335686263994609431635977830400000000000000 z^{57}$

$+ 11318433512094903910862991248960416834679553232651700642318114551383305265362051929085232931418049309266 4780061 / 2350561331282878571829474910515074683828862318181142924420699914240000000000000 z^{58}$

$+ 26548657760674235648288879979950976250936248017249897957575062618753448628459851948534397945704069381024 025048767 / 1386831185456898357379390197203894063459028767726874325408212949401600000000000000 z^{59}$

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$$59232424180914028 1 /$$

$$13332316491992021811671319836609528641530852609197374495513280556 4\backslash$$

$$5965714188998794413449485373664546946481398868960339403588360155 95\backslash$$

$$5200000000000000000000\, z^{99} + \mathrm{O}\left(z^{100}\right), S = z^2 + 2\, z^3 + \frac{61}{12}\, z^4 + \frac{29}{2}\, z^5$$

$$+ \frac{15961}{360}\, z^6 + \frac{2841}{20}\, z^7 + \frac{9489061}{20160}\, z^8 + \frac{9675221}{6048}\, z^9 + \frac{10062777121}{1814400}\, z^{10}$$

$$+ \frac{5907821951}{302400}\, z^{11} + \frac{16699361378701}{239500800}\, z^{12} + \frac{31881496079}{126720}\, z^{13}$$

$$+ \frac{3993974290192068 1}{43589145600}\, z^{14} + \frac{366558482492939101}{108972864000}\, z^{15}$$

$$+ \frac{1182574676807006227 1}{951035904000}\, z^{16} + \frac{1612664890101185883 7}{348713164800}\, z^{17}$$

$$+ \frac{553556225749078282790641}{3201186852864000}\, z^{18} + \frac{55019881254347166652 73}{8468748288000}\, z^{19}$$

$$+ \frac{29819717489323060485790527 01}{1216451004088320000}\, z^{20}$$

$$+ \frac{2875094944997145796073494 3}{3096420737679360}\, z^{21}$$

$$+ \frac{1983485716071573353386070664780 1}{562000363888803840000}\, z^{22}$$

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$$+ \frac{15965806269275698352930918019879546 1}{3102242008666197196800000}\, z^{24}$$

$$+ \frac{8898709287665305830510340094779 3}{4509334857646080000 0}\, z^{25}$$

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$$+ \frac{2651734210724574173425253551898653731906 1}{907405787534862680064000000}\, z^{27}$$

$$+ \frac{171988831665616115718605155434838732111356 61}{1524441723058569302507520000000}\, z^{28}$$

$$+ \frac{2217620588256456939795527007870759880147969}{50814724101952310083584000000}\, z^{29}$$

$$+ \frac{224297946226358903448558883274803468423176430921}{13262642990609552931815424000000 0}\, z^{30}$$

$$+\frac{2093993922290982978624596000416094537804869 67}{3189668828910426390528000000}z^{31}$$

$$+\frac{335811857610939536072898139578716041392540188844 4501}{1315654184668467650836090060800000000}z^{32}$$

$$+\frac{29175162189420861883810932963519856789521463672697}{29355367203556073345227161600000 0}z^{33}$$

$$+\frac{5720578377839705325173726154049703885106208139330 4942481}{147616399519802070423809304821760000000}z^{34}$$

$$+\frac{130289331132262226419126613775272824027639856772003497969 01}{8610956638655120774722209447936000000000}z^{35}$$

$$+\frac{344876883895863713883240106848908035705608680285680566169 9}{5830616407365222844326010159104000000000}z^{36}$$

$$+\frac{92235562942589031697786769001341407440694201221662333 77}{398424830012961053775463710720000000}z^{37}$$

$+ 2372071024417969027226854574230294512741879742483481771692 38\backslash$

$52441 / 261511308733300555880003612050037145600000000\, z^{38}$

$+ 262939584882204000067346037067684636350620807200578256162684\backslash$

$77947 / 739053698594110266617401512315322368000000000\, z^{39}$

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$08918805601 /$

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z^{50}

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\end{aligned}
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$$00000000000000000000\, z^{99} + \mathrm{O}\left(z^{100}\right),\; G = z + \frac{3}{2} z^{2} + \frac{19}{6} z^{3} + \frac{65}{8} z^{4}$$

$$+ \frac{2791}{120} z^{5} + \frac{17101}{240} z^{6} + \frac{1152019}{5040} z^{7} + \frac{2037605}{2688} z^{8} + \frac{935494831}{362880} z^{9}$$

$$+ \frac{10815911381}{1209600} z^{10} + \frac{1257770533339}{39916800} z^{11} + \frac{513183875243}{4561920} z^{12}$$

$$+ \frac{2528224238464471}{6227020800} z^{13} + \frac{42978132697166941}{29059430400} z^{14}$$

$$+ \frac{7101273378743303779}{1307674368000} z^{15} + \frac{2154248190413524297}{107296358400} z^{16}$$

$$+ \frac{2414845703016939977501}{32335220736000} z^{17} + \frac{8514551311161735303448 3}{304874938368000} z^{18}$$

$$+ \frac{127652707703771090396080939}{121645100408832000} z^{19}$$

$$+ \frac{11677645222677726521937131}{2948972131123200} z^{20}$$

$$+ \frac{40336181142100331021578721629}{2688996956405760000} z^{21}$$

$$+ \frac{2136402286798150269775194884830 1}{3746669092592025600 00} z^{22}$$

$$+ \frac{562137620301761921158196997677521 9}{2585201673888497664 0000} z^{23}$$

$$+ \frac{377998920789388853689713134240543}{45454095365072486 4000} z^{24}$$

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$$+ \frac{2527462567786487153606648632480030956105829751684504501677 9}{103331479663861449296666513375232 00000000} z^{35}$$

$$+ \frac{137152756549276167534908148445333270697090353079063872349}{1434329388046659793591669358592 0000000} z^{36}$$

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z^{37}

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$72781 / 1743408724888670372533357413666914304000000000\, z^{38}$

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z^{52}

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427488328406002556429801375338939964969034378836681372467200000000\

0000 z^{53}

$+$ 293593078791970158264977111860001127185584903854996500277180789\

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000 z^{54}

$+$ 255655280965724751310982941177843358779558568124037014035923846\

68609441836308749115891783559738435537779 /

126964033536582759259651008475665169595803210514494367622758400000\

00000000 z^{55}

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67141893273716465736177337056428225078089 /

163447951219508839506677160336488494192298385949693898548838400000\

0000000 z^{56}

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4850931144065486913633361878401716235443679 /

193908705764962759596557903853743168109953994240318670551121920000\

$000000000\, z^{57}$

$+\ 98859311727647577488227076246314233783409634947035734626796323\backslash$

$8602275626987867266601183433663201798261332161 81/$

$78352044376095952394315830350502489460962077272704764147356663808\backslash$

$0000000000000\, z^{58}$

$+\ 69564635743080565793785840025728091837403306588075523714816178\backslash$

$8085194797914132897457749425723663752655877703661 39/$

$13868311854568983573793901972038940634590287677268743254082129494\backslash$

$0160000000000000\, z^{59}$

$+\ 41958520104682743013488598186136744407623448055107927767908345\backslash$

$3492257367687546517530933826971646723697806319/$

$21031434379008442902515172830264901147566342844660073519015526400\backslash$

$0000000000\, z^{60}$

$+\ 40292547681694826637274179720911706921230328479073892210417897\backslash$

$67637667147032719395088614526891561121550655934537175351/$

$50758021387722479880085681217662522722600452898803600309940593948\backslash$

$0985600000000000000\, z^{61}$

$+\ 33146708249026423477096756506878449890233879319194459322089972\backslash$

$5295992575932761181032769863121446783046918358664591875901/$

$10489991086795979175217707451650254696004093599086077397387722749\backslash$

$2737024000000000000000\, z^{62}$

$+ 2494699559605362053651179866391607439314584784438283913277617122523966723586174260850515530743107535952701977455064937196 19 /$

$$
\begin{aligned}
&+ 24946995596053620536511798663916074393145847844382839132776171 2\backslash \\
&2523966723586174260850515530743107535952701977455064937196 19 / \\
&19826083154044400641161467083618981375447736902272686281062795996 1\backslash \\
&27297536000000000000000\, z^{63} \\
&+ 11286186137714443431421317414376907507582277564957526087838111 7\backslash \\
&030216817852302827548850513421488946538283882064294420160 9 / \\
&22515647624147664644385305533699135977795318281349515073871332512 6\backslash \\
&8116480000000000000000\, z^{64} \\
&+ 16475352206651928360151941420053303836231259384720924584526651 9\backslash \\
&1478069633816816027341204850744672679854740493442855198979606339 1 \\
&/ \\
&8247650592082470666723170306785496252186258551345437492922123143 8\backslash \\
&8955774976000000000000000000\, z^{65} \\
&+ 8974877700575003007786679577374508574125308042057237120037307 38\backslash \\
&731719313284681673799793939498659695343053601198970915103423114 1 / \\
&1127008155439840712223041905275036754957128497699376551827867757 49\backslash \\
&4143025152000000000000000000\, z^{66} \\
&+ 1052944460001893179730132927027849992154281277113841451660661 04\backslash \\
&5778920581921655450036527767402402961463798166830689446990821050 08\backslash \\
&129 / \\
&3315555538017153208022714463327769493378875937640865872154693500 02\backslash
\end{aligned}
$$

$4360221540352000000000000000\, z^{67}$

$+ 418928247151681263539225969211163205741485236520950393582295305\backslash$

$488163120431664854712791787230180848307432598818667496605424755146\backslash$

$8153 \Big/$

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$+ 865138684512870196506811021274424748164223496630917660213343151\backslash$

$070606638481817382696069437271182553059869291111600897320740428291\backslash$

$29839431 \Big/$

$171122452428141311372468338881272839092270544893520369393648040923\backslash$

$25727975414064742400000000000000\, z^{69}$

$+ 385558030973307481478780463404809588114334891669781432192495046\backslash$

$624584155279879469439589990681504687667094026792200992608905933057\backslash$

$99283709 \Big/$

$191045800159009438533856199707960107439536493501537892464997812832\backslash$

$98260897591459840000000000000000\, z^{70}$

$+ 685366752914057775944819811768721482744245348944128608427771279\backslash$

$108453216015017468159220836976855134639561677557647314683439640362\backslash$

$6958033579539 \Big/$

$850478588567862317521167644239926010288584608120796235886430763388\backslash$

$5886803780790176972800000000000000000\, z^{71}$

$+\ 2054021819387341965615302470782172174533634914224869894925298 54\backslash$

$549502067372975582516846251483903340807486956630313964009950736575\backslash$

$4040274177\,/$

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$+\ 56673159564443412405431403267674234858848496084600523231613914 5\backslash$

$499991895731704888869529713307134959721137051627470187299795260340\backslash$

$370276610601818819 41\,/$

$110262848050646213741984342740417927381894417273644990390203975611\backslash$

$8037452336571884864169574400000000000000000\,z^{74}$

$+\ 10360093846910455215051810650483243570084878154868913923044193 0\backslash$

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$693098012902669553\,/$

$504206762648699363509824895722407059157964728419469900756557750291\backslash$

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$+\ 88676706248167070016272705617298543085941181117407492168950622 03\backslash$

$$
\begin{aligned}
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&10792726418447859364958499036128382908064914155250439098185110992595007132180720279093248000000000000000000\, z^{76} \\
&+ 4337491183678813300161611830563492103635446229812094695516206282450378574686568066198852617498488755072613922558839441233142044940921772531258946602335361 01 \big/ \\
&13198462911662351784915525826028025907612761747655305349707415880732908304468765461824109805568000000000000000000\, z^{77} \\
&+ 7090127171306366787401121632993534186151644900560223747329163144647070619910617450744216502441910995398713827170989329526329080930487786838871883645324415 3083 \big/ \\
&53925148467649037292654862660628791565389283711848819000233156312708739643972384601167077205606400000000000000000000\, z^{78} \\
&+ 4707123309123088057331104625439210539836308016542787589511731115124784624118477628765862515394063402097129947618015843845819019866896076688901523464935735 43360739 \big/ \\
&89461821307829752868514417153983165206980821677957190721386806322783799069350186053336181084101017600000000000000000000\, z^{79} \\
&+ 26213138315972426527916051544024279707280392625565420886494086763740366083121022396411470423670494366397722700591705274281740050 4\backslash
\end{aligned}
$$

3873972817426111588372472309877 /

124463209506132433015628074819680069850153745215191952657900865281\

034805887535583396667875078963200000000000000000 z^{80}

+ 488824933673178676115631532144417289093989215111319382893732218\

767248214138721634949375285084499616250795076249506672897634011861\

685019196875514146081547010485122075551 /

579712602074736798587973423157810910541235724473162595874586504971\

63901796938920562561845342497459404800000000000000000000 z^{81}

+ 535067305242163898317783468872369892879547332502173188159532489\

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55480014548766209617826939706198416177701 /

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z^{86}

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z^{87}

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$+ 92626838089777420236868638806721632103143656634884317614029 2\backslash$

$126599585006531837199164054897688395076487061072950581508157097302\backslash$

$43361392170533137678509283843097325838347719562251300200876 31/$

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$00000\, z^{89}$

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z^{90}

$+ 122085334568526152566608669551717734913220377611627329593788047\backslash$

$398802536430705207565870195111878109663373507902893463805485830792\backslash$

$50070903179281382805214591286726039421377371102149692916565309 39/$

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$000000000\, z^{91}$

+ 376057431821518350937328475237582856807305544413618261717931285\
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92773321746408749932982223599665996404860801517713122523713939/
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903109812127822431731715676571058384256215994849034240000000000000\
0000000 z^{92}

+ 885400615969422914464519364470379573705184435768127936378755377\
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853328342661104909195373327139986401914790429451521771354947110490\
9/
608827635306127144610107980161179176955133278724428176413759095375\
393056379719639586949099745707813317809998047045830574080000000000\
00000000000 z^{93}

+ 358626167885720058369197265646060067443018639560732111001316859\
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896097943874293323892861824888321393287666899480458254555216214008\
999/
6143311619529621922450016111620263282509529963729846736200550138045\
453106606880947063441294997646415020117885035041725087744000000000\
000000000000 z^{94}

+ 242106761228598489826648746723284848306689516234759021899511476\

$$
9484889789473776436362081662444846317468841175752207960816226499066\
765048913931274618937102352005704357759452484097376572979757254211\
71003779 /
103299784882390592625997020993947270953977463401173728692122505712\
342939875947031248717653753854244685632822368642266073504153600000\
00000000000000000 z^{95}
+ 402195490090108498760595935668852904463802390393679542421308227\
381737698187463697208631637907812634776201648431397761498346380062\
697270913001633770398604888591615565964106509419544058243594080947\
789983 /
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036726054338073685709750500754470580510706631745638571704320000000\
00000000000000 z^{96}
+ 330528929093226329540189692245433870559902533764016118311442556\
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730764068579455832543092563725131627774219700631907331780210859204\
631812162301 /
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852232840744322716417083414446115011466219906178892433155162112000\
0000000000000000000 z^{97}
+ 477076714878386960239254307357980761652801757221625848900119760\
$$

$9232256313395554511169495965489635401843050423229185942129921389 95\backslash$

$3956740714617721109366288436036668588411443708570166830109487333 87\backslash$

$777425308438981\,/$

$3142296816294415915208728581019080824603231254692983887831412925 76\backslash$

$4902356674407932960992053069243039941201950196202820143137549189 12\backslash$

$0000000000000000000000\ z^{98}$

$+\ 569238243367782252127016194928736099185611051549411947890864893\backslash$

$2331267657438973367844964536796168401191110774083817953880153244 51\backslash$

$6841366217986010207831985482545160347658275752871706387883058124 19\backslash$

$2519631242237233 39\,/$

$9332621544394415268169923885626670049071596826438162146859296389 52\backslash$

$1759999322991560894146397615651828625369792082722375825118521091 68\backslash$

$640000000000000000000000\ z^{99} + \mathrm{O}\left(z^{100}\right)\Big]$

Same experiments in the unlabelled universe

```
> GFSeries(spgraphs,unlabelled,z,21);
```

$$\Big[Z = z + \mathrm{O}\left(z^{21}\right), P = z^{2} + 2 z^{3} + 6 z^{4} + 18 z^{5} + 64 z^{6} + 227 z^{7} + 856 z^{8} + 3280 z^{9} + 12885 z^{10} + 51342 z^{11} + 207544 z^{12} + 847886 z^{13} + 3497384 z^{14} + 14541132 z^{15} + 60884173 z^{16} + 256480895 z^{17} + 1086310549 z^{18} + 4623128656 z^{19} + 19759964149 z^{20} + \mathrm{O}\left(z^{21}\right), S = z^{2} + 3 z^{3} + 9 z^{4} + 30 z^{5} + 103 z^{6} + 375 z^{7} + 1400 z^{8} + 5380 z^{9} + 21073 z^{10} + 83950 z^{11} + 338878 z^{12} + 1383576 z^{13} + 5702485 z^{14} + 23696081 z^{15} + 99163323 z^{16} + 417553252 z^{17} + 1767827220 z^{18} + 7520966100 z^{19} + 32135955585 z^{20} + \mathrm{O}\left(z^{21}\right), G = z + 2 z^{2} + 5 z^{3} + 15 z^{4} + 48 z^{5} + 167 z^{6} + 602 z^{7} + 2256 z^{8} + 8660 z^{9} + 33958 z^{10} + 135292 z^{11} + 546422 z^{12} + 2231462 z^{13} + 9199869 z^{14} + 38237213 z^{15} + 160047496 z^{16} + 674034147 z^{17} + 2854137769 z^{18} + 12144094756 z^{19} + 51895919734 z^{20} + \mathrm{O}\left(z^{21}\right)\Big] \qquad \textbf{(3.1.4)}$$

Complexity wrt size is good:

```
> GFSeries(spgraphs,unlabelled,z,100);
```

$$\Big[Z = z + \mathrm{O}\left(z^{100}\right), P = z^{2} + 2 z^{3} + 6 z^{4} + 18 z^{5} + 64 z^{6} + 227 z^{7} + 856 z^{8} + 3280 z^{9} + 12885 z^{10} + 51342 z^{11} + 207544 z^{12} + 847886 z^{13} + 3497384 z^{14} + 14541132 z^{15} + 60884173 z^{16} + 256480895 z^{17} \qquad \textbf{(3.1.5)}$$

$+ 1086310549 z^{18} + 4623128656 z^{19} + 19759964149 z^{20} + 84784735379 z^{21}$
$+ 365066645854 z^{22} + 1576927900803 z^{23} + 6831518134251 z^{24}$
$+ 29674505668536 z^{25} + 129216630647787 z^{26} + 563949605921815 z^{27}$
$+ 2466473805675492 z^{28} + 10808418041305028 z^{29}$
$+ 47450298633932617 z^{30} + 208667359499177355 z^{31}$
$+ 919097231317536374 z^{32} + 4054308523612570117 z^{33}$
$+ 17909417025104806104 z^{34} + 79217251907238038766 z^{35}$
$+ 350831964278618118793 z^{36} + 1555570478952302523435 z^{37}$
$+ 6905003358779377150283 z^{38} + 30682901704698511000040 z^{39}$
$+ 136478394667053366526069 z^{40} + 607637655947410656650762 z^{41}$
$+ 2707813093560057509717917 z^{42} + 12077228348020819675319519 z^{43}$
$+ 53910469708722595692882989 z^{44} + 240835237458096124437607418 z^{45}$
$+ 1076695573403722102397908626 z^{46}$
$+ 4817011615224396450951597848 z^{47}$
$+ 21565578467318715329650136511 z^{48}$
$+ 96611936993412540588598040408 z^{49}$
$+ 433086940800755356965500371812 z^{50}$
$+ 1942598578539538922219610962604 z^{51}$
$+ 8718553235026366943178551313372 z^{52}$
$+ 39151594227310908742094794392764 z^{53}$
$+ 175909375754333658552024999315348 z^{54}$
$+ 790777637498303647409476517126864 z^{55}$
$+ 3556619810880625116055583952893182 z^{56}$
$+ 16004069643783836219550291150548078 z^{57}$
$+ 72048676468471050226624333435995638 z^{58}$
$+ 324501871179990298218247630032201086 z^{59}$
$+ 1462168482061614209988602239443014373 z^{60}$
$+ 6591137413326026111323785428385412932 z^{61}$
$+ 29723508898940372305636310412565355015 z^{62}$
$+ 134094469969726159687180508232216335184 z^{63}$
$+ 605183842824469995712505385480157591751 z^{64}$
$+ 2732274233133304133201945518052056164901 z^{65}$
$+ 12340047459169172289223442249381301507991 z^{66}$
$+ 55751974762654642190830700501961630705116 z^{67}$
$+ 251970716513559414300972052318592561027140 z^{68}$

$+ 113915280857045860400069380838260814213593 2 z^{69}$

$+ 515171573312181281483447489842602674438021 8 z^{70}$

$+ 2330536221287029330652156749509874078268086 2 z^{71}$

$+ 1054605708864815212083378125456612853625695 31 z^{72}$

$+ 4773655393017680493080157268177466544498575 59 z^{73}$

$+ 2161400438365001438344843051541749391720127 047 z^{74}$

$+ 9789022105375380573446876217038329484570391 735 z^{75}$

$+ 4434657641674989589381490322535331593704919 9857 z^{76}$

$+ 2009530199942698790757765895754456005612383 73797 z^{77}$

$+ 9108349442743289957820825900692353291445136 42864 z^{78}$

$+ 4129454702583341015256925720516448522211136 389158 z^{79}$

$+ 1872625275586402504765819454089609309161056 0306515 z^{80}$

$+ 8493986865642636303906244550896901243729140 5113441 z^{81}$

$+ 3853650227501337909094120000873646631902927 25750276 z^{82}$

$+ 1748761462178382427401130823015191026861668 245364808 z^{83}$

$+ 7937506254641072637347369219183118564187658 559244881 z^{84}$

$+ 3603549992062411370471258929803206389808605 2343695028 z^{85}$

$+ 1636318364844920671662818434376243840003694 83513702290 z^{86}$

$+ 7431795213702573136164314793558383232689019 81325477175 z^{87}$

$+ 3376030066729889978483097020460394811091500 275656850534 z^{88}$

$+ 1533922986025538999193756478422943340230161 8664779086271 z^{89}$

$+ 6970815441248245740616340087937484580604283 3554083485895 z^{90}$

$+ 3168433217328619989693361439186959525828981 36752417849642 z^{91}$

$+ 1440405264519961532607041318250611565784785 600625493661172 z^{92}$

$+ 6549411743598347713171757459601288133828141 081912017032214 z^{93}$

$+ 2978486790495961358753938360899020988580870 5389788529090706 z^{94}$

$+ 1354762739152137643815878893470678822541320 06377251918402085 z^{95}$

$+ 6163159840755928291829022103298044609707481 1033245107207857 4 z^{96}$

$+ 2804237108000119406185286197202885041061777 66922216758035787 2$
z^{97}

$+ 1276132245073842217093402190105303588328497 042873205567706891 9$
z^{98}

$+ 5808244159635144604852823147126511549105125 885870918152023358 4$
$z^{99} + \mathrm{O}\left(z^{100}\right), S = z^2 + 3 z^3 + 9 z^4 + 30 z^5 + 103 z^6 + 375 z^7 + 1400 z^8$

$+ 5380 z^9 + 21073 z^{10} + 83950 z^{11} + 338878 z^{12} + 1383576 z^{13}$

$$+ 5702485\, z^{14} + 23696081\, z^{15} + 99163323\, z^{16} + 417553252\, z^{17}$$
$$+ 1767827220\, z^{18} + 7520966100\, z^{19} + 32135955585\, z^{20} + 137849390424\, z^{21}$$
$$+ 593407692685\, z^{22} + 2562695780058\, z^{23} + 11099806544050\, z^{24}$$
$$+ 48206136562750\, z^{25} + 209876865026303\, z^{26} + 915840095739301\, z^{27}$$
$$+ 4004923697094450\, z^{28} + 17547807425910789\, z^{29}$$
$$+ 77027671121229420\, z^{30} + 338698369075550442\, z^{31}$$
$$+ 1491674669942837919\, z^{32} + 6579403269510266993\, z^{33}$$
$$+ 29061024206625258634\, z^{34} + 128532053947146677203\, z^{35}$$
$$+ 569187394850639272008\, z^{36} + 2523552225750687574021\, z^{37}$$
$$+ 11200949914418953247199\, z^{38} + 49768833881033655476666\, z^{39}$$
$$+ 221358701663889116137490\, z^{40} + 985486070235332704928186\, z^{41}$$
$$+ 4391359180109334073113040\, z^{42} + 19584986258357898621582742\, z^{43}$$
$$+ 87419030679735859329888021\, z^{44} + 390508829695036653714110749\, z^{45}$$
$$+ 1745752679750356745543747250\, z^{46}$$
$$+ 7809934356810025475378587350\, z^{47}$$
$$+ 34963232567662210382916046591\, z^{48}$$
$$+ 156625638327078888491533468471\, z^{49}$$
$$+ 702084716905161319561621324283\, z^{50}$$
$$+ 3149057465736713397211987861575\, z^{51}$$
$$+ 14132718533891407074808911741342\, z^{52}$$
$$+ 63462202701727190103524926348643\, z^{53}$$
$$+ 285127879821108040685756873728863\, z^{54}$$
$$+ 1281712589598882681924816470735910\, z^{55}$$
$$+ 5764475839159343147545846518732213\, z^{56}$$
$$+ 25938175720828538634824466746266299\, z^{57}$$
$$+ 116767531006815301302183719595991655\, z^{58}$$
$$+ 525897182367663152180504846644676004\, z^{59}$$
$$+ 2369567231893298635569125356330229444\, z^{60}$$
$$+ 10681207951466154055431745599080444099\, z^{61}$$
$$+ 48166917781432669922563756209229042847\, z^{62}$$
$$+ 217294528647512546180684688898164919125\, z^{63}$$
$$+ 980651602953915019651563811230560441110\, z^{64}$$
$$+ 4427326319159664665756713444661137196201\, z^{65}$$
$$+ 19995129621058570665916094499612772912981\, z^{66}$$
$$+ 90335426587757689763099394841120580323647\, z^{67}$$

$+ 408261715972987352616665624690566012269577\, z^{68}$

$+ 1845702024670773129617685982003117610053737\, z^{69}$

$+ 8346853225831594512729800483796769621612861\, z^{70}$

$+ 37758808228532369076910136712166705852706975\, z^{71}$

$+ 170861540162522873937299969136044817998334441\, z^{72}$

$+ 773387721651963815288956594660572255213218056\, z^{73}$

$+ 3501657585398428239872261145753562309006886839\, z^{74}$

$+ 15858795351951138351691511268729129482085808589\, z^{75}$

$+ 71842863309805800491029087228029597006420103768\, z^{76}$

$+ 325544916779290540347827499425366360882470009651\, z^{77}$

$+ 1475533586881973835551145749657953530760728487457\, z^{78}$

$+ 6689525872293286072367037544308386152101695633399\, z^{79}$

$+ 30335202749415723436354484401402100799230540903610\, z^{80}$

$+ 137594528232927948510688439426951923628706493165060\, z^{81}$

$+ 624245745182119760012179757555922593488849396987332\, z^{82}$

$+ 2832746718504023125446381441363246758490345127289020\, z^{83}$

$+ 12857462760960891623177089991982641863273570972892705\, z^{84}$

$+ 58370836052453520345201369865260093547175671577112158\, z^{85}$

$+ 265049795708745236390496093152874910223085178347917436\, z^{86}$

$+ 1203781997291115226934768952542768944480000717779683156\, z^{87}$

$+ 5468333412158948616685815657790072095685378127909622000\, z^{88}$

$+ 24845453672126760098678986312675550548870958338636151799\, z^{89}$

$+ 112907236130541012438791603473989080889986283287928115554\, z^{90}$

$+ 513189405708801911720198722221041910758745848576446671284\, z^{91}$

$+ 2332989960387428294188669071660017044125201017176306337426\, z^{92}$

$+ 10607807219029831940603755807268706460134307825024052043380\, z^{93}$

$+ 48240773474137826179149084750182765888262764107566717525710\, z^{94}$

$+ 219420494523887829659207143291986944240113432458414978009005\, z^{95}$

$+ 998189134078763949005153472797426448869712564891474030193118\, z^{96}$

$+ 4541713086168578938593828201507688104959237222971593038705643$
z^{97}

$+ 20667897288184371858965417542167671460581865561995160181622373$
z^{98}

$+ 94067849429916606801713431849035945464708064179305500024722049$
$z^{99} + \mathrm{O}\left(z^{100}\right), G = z + 2\, z^{2} + 5\, z^{3} + 15\, z^{4} + 48\, z^{5} + 167\, z^{6} + 602\, z^{7}$

$$+ 2256 z^{8} + 8660 z^{9} + 33958 z^{10} + 135292 z^{11} + 546422 z^{12} + 2231462 z^{13}$$
$$+ 9199869 z^{14} + 38237213 z^{15} + 160047496 z^{16} + 674034147 z^{17}$$
$$+ 2854137769 z^{18} + 12144094756 z^{19} + 51895919734 z^{20}$$
$$+ 222634125803 z^{21} + 958474338539 z^{22} + 4139623680861 z^{23}$$
$$+ 17931324678301 z^{24} + 77880642231286 z^{25} + 339093495674090 z^{26}$$
$$+ 1479789701661116 z^{27} + 6471397502769942 z^{28} + 28356225467215817 z^{29}$$
$$+ 124477969755162037 z^{30} + 547365728574727797 z^{31}$$
$$+ 2410771901260374293 z^{32} + 10633711793122837110 z^{33}$$
$$+ 46970441231730064738 z^{34} + 207749305854384715969 z^{35}$$
$$+ 920019359129257390801 z^{36} + 4079122704702990097456 z^{37}$$
$$+ 18105953273198330397482 z^{38} + 80451735585732166476706 z^{39}$$
$$+ 357837096330942482663559 z^{40} + 1593123726182743361578948 z^{41}$$
$$+ 7099172273669391582830957 z^{42} + 31662214606378718296902261 z^{43}$$
$$+ 141329500388458455022771010 z^{44} + 631344067153132778151718167 z^{45}$$
$$+ 2822448253154078847941655876 z^{46}$$
$$+ 12626945972034421926330185198 z^{47}$$
$$+ 56528811034980925712566183102 z^{48}$$
$$+ 253237575320491429080131508879 z^{49}$$
$$+ 1135171657705916676527121696095 z^{50}$$
$$+ 5091656044276252319431598824179 z^{51}$$
$$+ 22851271768917774017987463054714 z^{52}$$
$$+ 102613796929038098845619720741407 z^{53}$$
$$+ 461037255575441699237781873044211 z^{54}$$
$$+ 2072490227097186329334292987862774 z^{55}$$
$$+ 9321095650039968263601430471625395 z^{56}$$
$$+ 41942245364612374854374757896814377 z^{57}$$
$$+ 188816207475286351528808053031987293 z^{58}$$
$$+ 850399053547653450398752476676877090 z^{59}$$
$$+ 3831735713954912845557727595773243817 z^{60}$$
$$+ 17272345364792180166755531027465857031 z^{61}$$
$$+ 77890426680373042228200066621794397862 z^{62}$$
$$+ 351388998617238705867865197130381254309 z^{63}$$
$$+ 1585835445778385015364069196710718032861 z^{64}$$
$$+ 7159600552292968798958658962713193361102 z^{65}$$
$$+ 32335177080227742955139536748994074420972 z^{66}$$

$+ 1460874013504123319539300953430822110287 63 z^{67}$

$+ 6602324324865467669176376770091585732967 17 z^{68}$

$+ 29848548332412317336183797903857257521896 69 z^{69}$

$+ 134985689589534073275642753822227963659930 79 z^{70}$

$+ 610641704414026623834317042072654466353878 37 z^{71}$

$+ 2763221110490043951456377816817061033609039 72 z^{72}$

$+ 12507532609537318645969723214783189096630756 15 z^{73}$

$+ 56630580237634296782171041972953117007270138 86 z^{74}$

$+ 256478174573265189251383874857674589666562003 24 z^{75}$

$+ 1161894397265556963848439904533829129434693036 25 z^{76}$

$+ 5264979367735604194236040890008119614437083834 48 z^{77}$

$+ 23863685311563028313332283397271888599052421303 21 z^{78}$

$+ 108189805748766270876239632648248346743128320225 57 z^{79}$

$+ 490614555052797484840126789422981938908411012101 25 z^{80}$

$+ 2225343968893543115497508849359209360659978982785 01 z^{81}$

$+ 10096107679322535509215917576432872566791421227376 08 z^{82}$

$+ 45815081806824055528475122643784377853520133726538 28 z^{83}$

$+ 207949690156019642605244592111657604274612295321375 86 z^{84}$

$+ 944063359730776340499139591632921574452617239208071 86 z^{85}$

$+ 4286816321932373035567779365904992942234546618616197 26 z^{86}$

$+ 19469615186613725405512004318986072677489026991051603 31 z^{87}$

$+ 88443634788888385951689126782504669067768784035664725 34 z^{88}$

$+ 401846835323821500906165510969049839511725770034152380 70 z^{89}$

$+ 1826153905430234698449550043533639266960291168420116014 49 z^{90}$

$+ 8300327274416639106895348661397378633416439853288645209 26 z^{91}$

$+ 37733952249073898267957103899106286099099866178017999985 98 z^{92}$

$+ 171572189626281796537755132668699945939624489069360690755 94 z^{93}$

$+ 780256413790974397666884683591729757740714694973552466164 16 z^{94}$

$+ 3548967684391015940407950326390548264942454388356668964110 90 z^{95}$

$+ 16145051181543567781880556831272309098404606752239251022716 92 z^{96}$

$+ 73459501941686983447791143987105731460210148921937606190635 15 z^{97}$

$+ 334292197389227940298994394432207073438668359907272158586912 92 z^{98}$

$$+ 152150291026268052850241663320301060955759323038014681544955633\, z^{99} + O\left(z^{100}\right)]$$

Labelled and unlabelled rooted trees

Recall their specification:

```
> cayley;
```

$$\{T = Prod(Z, Set(T))\} \tag{3.2.1}$$

Enumerate in the labelled universe

```
> GFSeries(cayley,labelled,z,41);
```

$$\left[Z = z + O\left(z^{41}\right), T = z + z^2 + \frac{3}{2} z^3 + \frac{8}{3} z^4 + \frac{125}{24} z^5 + \frac{54}{5} z^6 + \frac{16807}{720} z^7 \right. \tag{3.2.2}$$

$$+ \frac{16384}{315} z^8 + \frac{531441}{4480} z^9 + \frac{156250}{567} z^{10} + \frac{2357947691}{3628800} z^{11}$$

$$+ \frac{2985984}{1925} z^{12} + \frac{1792160394037}{479001600} z^{13} + \frac{7909306972}{868725} z^{14}$$

$$+ \frac{320361328125}{14350336} z^{15} + \frac{35184372088832}{638512875} z^{16} + \frac{2862423051509815793}{20922789888000} z^{17}$$

$$+ \frac{5083731656658}{14889875} z^{18} + \frac{5480386857784802185939}{6402373705728000} z^{19}$$

$$+ \frac{3200000000000000}{14849255421} z^{20} + \frac{41209797661291758429}{7567605760000} z^{21}$$

$$+ \frac{2446363617936581851 64}{17717861581875} z^{22} + \frac{39471584120695485887249589623}{1124000727777607680000} z^{23}$$

$$+ \frac{2243810146471316029 44}{2505147019375} z^{24} + \frac{2273736754432320594787597 65625}{9927174427731831 02976} z^{25}$$

$$+ \frac{1670156216536524654681831 88}{284473896821296875} z^{26}$$

$$+ \frac{1030105146087753745397354726 7843}{682977630656921600 0000} z^{27}$$

$$+ \frac{146934843447668157108621148 16}{3784415134680984375} z^{28}$$

$$+ \frac{30531345459705245357453367594899121599 09}{304888344611713860501504000000} z^{29}$$

$$+ \frac{27368368148803710937 5000}{10577732774609} z^{30}$$

$$+ \frac{177618877530938979798237700614561027638342 71}{265252859812191058636308480000000} z^{31}$$

$$+ \frac{21267647932558653966460912964485513216}{1225298442569065513867 96875} z^{32}$$

$$+ \frac{204856241205141860873006985622806637233}{45466215983857270784000000 0} z^{33}$$

$$+ \frac{278577834677702028922836034978935440866}{2378520506163480115155 46875} z^{34}$$

$$+ \frac{4798033469637006114181290566921234130859375}{157391912857431611947085306265 6} z^{35}$$

$$+ \frac{1332006499728309914389939527119536128}{16766946025814789492187 5} z^{36}$$

$$+ \frac{77101058844249696231397590109538589818315530192623808 93}{371993326789901217467999448150835200000000} z^{37}$$

$$+ \frac{22826326484345086991158843019660480874276907 96}{42166098570041680286724155859375} z^{38}$$

$$+ \frac{33918951985613054247250097775181717149895760931 57}{23964724184035124219320729600000000} z^{39}$$

$$+ \frac{5629499534213120000000000000000000000000000}{1519760644525099050897380839281} z^{40} + O(z^{41})]$$

the number is obtained by multiplying back by a factorial:

```
> coeff(subs(%,T),z,40)*40!;
```

302231454903657293676544000000000000000000000000000000000000000 **(3.2.3)**

Those numbers are known:

```
> 40^39;
```

302231454903657293676544000000000000000000000000000000000000000 **(3.2.4)**

In the unlabelled case, such a nice formula is not known, but it is equally easy to compute values for high indices

```
> GFSeries(cayley,unlabelled,z,41);
```

$$[Z = z + O(z^{41}), T = z + z^2 + 2 z^3 + 4 z^4 + 9 z^5 + 20 z^6 + 48 z^7 + 115 z^8 + 286 z^9 + 719 z^{10} + 1842 z^{11} + 4766 z^{12} + 12486 z^{13} + 32973 z^{14} + 87811 z^{15} + 235381 z^{16} + 634847 z^{17} + 1721159 z^{18} + 4688676 z^{19} + 12826228 z^{20} + 35221832 z^{21} + 97055181 z^{22} + 268282855 z^{23} + 743724984 z^{24} + 2067174645 z^{25} + 5759636510 z^{26} + 16083734329 z^{27} + 45007066269 z^{28} + 126186554308 z^{29} + 354426847597 z^{30} + 997171512998 z^{31} + 2809934352700 z^{32} + 7929819784355 z^{33} + 22409533673568 z^{34} + 63411730258053 z^{35} + 179655930440464 z^{36} + 509588049810620 z^{37} + 1447023384581029 z^{38} + 4113254119923150 z^{39} + 11703780079612453 z^{40} + O(z^{41})]$$ **(3.2.5)**

This one is also fast:

```
> GFSeries(cayley,unlabelled,z,101);
```

$$[Z = z + O(z^{101}), T = z + z^2 + 2 z^3 + 4 z^4 + 9 z^5 + 20 z^6 + 48 z^7 + 115 z^8 + 286 z^9 + 719 z^{10} + 1842 z^{11} + 4766 z^{12} + 12486 z^{13} + 32973 z^{14} + 87811 z^{15} + 235381 z^{16} + 634847 z^{17} + 1721159 z^{18} + 4688676 z^{19} + 12826228 z^{20} + 35221832 z^{21} + 97055181 z^{22} + 268282855 z^{23} + 743724984 z^{24} + 2067174645 z^{25} + 5759636510 z^{26} + 16083734329 z^{27} + 45007066269 z^{28} + 126186554308 z^{29} + 354426847597 z^{30} + 997171512998 z^{31} + 2809934352700 z^{32} + 7929819784355 z^{33}$$ **(3.2.6)**

$$+ 22409533673568 z^{34} + 63411730258053 z^{35} + 179655930440464 z^{36}$$
$$+ 509588049810620 z^{37} + 1447023384581029 z^{38} + 4113254119923150 z^{39}$$
$$+ 11703780079612453 z^{40} + 33333125878283632 z^{41}$$
$$+ 95020085893954917 z^{42} + 271097737169671824 z^{43}$$
$$+ 774088023431472074 z^{44} + 2212039245722726118 z^{45}$$
$$+ 6325843306177425928 z^{46} + 18103111141539779470 z^{47}$$
$$+ 51842285219378800562 z^{48} + 148558992149369434381 z^{49}$$
$$+ 425976989835141038353 z^{50} + 1222179262369751914558 z^{51}$$
$$+ 3508609802706585591648 z^{52} + 10078062032127180323468 z^{53}$$
$$+ 28963544938490115587690 z^{54} + 83281891024323882188934 z^{55}$$
$$+ 239588251950971630070883 z^{56} + 689586695750027771528858 z^{57}$$
$$+ 1985698827814122851389544 z^{58} + 5720475695410470698034352 z^{59}$$
$$+ 16486885726043465205200778 z^{60} + 47536435298225838513777689 z^{61}$$
$$+ 137116646299836640013582158 z^{62} + 395661426200172120893172166 z^{63}$$
$$+ 1142146565612503377367247619 z^{64}$$
$$+ 3298218689025396468807928287 z^{65}$$
$$+ 9527778769277367435762139714 z^{66}$$
$$+ 27533018688066675122704256503 z^{67}$$
$$+ 79590547737981375530085744985 z^{68}$$
$$+ 230149693903609741713900061706 z^{69}$$
$$+ 665727478405857651614359412994 z^{70}$$
$$+ 1926264145029683789201902595481 z^{71}$$
$$+ 5575255070241458769777337977216 z^{72}$$
$$+ 16141340753693289446826761342965 z^{73}$$
$$+ 46745197819192522341803684458539 z^{74}$$
$$+ 135410903503191503503384705970501 z^{75}$$
$$+ 392361462360303212219218458046293 z^{76}$$
$$+ 1137187531620128717310046733793377 z^{77}$$
$$+ 3296764685809404767411359346950251 z^{78}$$
$$+ 9559850833672724865816863589301148 z^{79}$$
$$+ 27728021718947447790064072086211720 z^{80}$$
$$+ 80443073708065346515907947187548704 z^{81}$$
$$+ 233430673259657258534788317265918255 z^{82}$$
$$+ 677523352768124889000758550090536896 z^{83}$$
$$+ 1966913982276181876912481643993713829 z^{84}$$

$+ 5711352604881678383437989565512293584\, z^{85}$
$+ 16587575765193708421040000378631758598\, z^{86}$
$+ 48185360814676586544379947795827489489\, z^{87}$
$+ 140001774817536903942967946008051857304\, z^{88}$
$+ 406851804555194134855599602163662891898\, z^{89}$
$+ 1182554992433068847214862126968409557746\, z^{90}$
$+ 3437850734230350035768412388914235959988\, z^{91}$
$+ 9996120786153335036227014784756370016600\, z^{92}$
$+ 29070538491469576103323086044968128987841\, z^{93}$
$+ 84557105785656201807648937916789795631644\, z^{94}$
$+ 245992006962808524149265893171336435342958\, z^{95}$
$+ 715754664594286649942853831332902539636094\, z^{96}$
$+ 2082946825910124637547671590789256847274970\, z^{97}$
$+ 6062636513439329551550570219718718950653609\, z^{98}$
$+ 17648704032743316047867540848170516567878799\, z^{99}$
$+ 51384328351659326880337136395054298255277970\, z^{100} + \mathrm{O}(z^{101})\,]$

III. Random Generation

Recursive method

Draw a random binary tree of size 30:

```
> combstruct[draw]([B,bintrees,unlabelled],size=30);
```

$Prod(Z, E, Prod(Z, E, Prod(Z, Prod(Z, E, Prod(Z, Prod(Z, Prod(Z, Prod(Z, Prod(Z, Prod(Z, Prod(Z, Prod(Z, Prod(Z, Prod(Z, E, Prod(Z, E, E)), E), E), E), Prod(Z, E, E)), Prod(Z, E, Prod(Z, E, E))), E), Prod(Z, Prod(Z, E, Prod(Z, Prod(Z, Prod(Z, E, E), Prod(Z, E, E)), Prod(Z, Prod(Z, Prod(Z, E, Prod(Z, E, Prod(Z, E, E))), E), E))), E)), E), Prod(Z, E, E))), E)))$ **(4.1.1)**

and similarly a random labelled series-parallel graph:

```
> combstruct[draw]([G,spgraphs,labelled],size=30);
```

$Set(Sequence(Z_{16}, Set(Sequence(Set(Sequence(Z_6, Z_7, Set(Sequence(Z_{19}, Set(Sequence(Z_{12}, Set(Sequence(Z_{26}, Z_{10}, Set(Z_{14}, Z_{13})), Z_9)), Z_{15}, Z_{27}), Z_{24}, Z_{21}, Z_{29}, Z_5), Z_4)), Sequence(Z_{23}, Z_{22}), Sequence(Set(Z_{25}, Sequence(Z_3, Z_8)), Z_{18})), Z_2, Z_{30}, Z_{20}, Z_1), Sequence(Z_{28}, Z_{17}))), Z_{11})$ **(4.1.2)**

Time for size 500

```
> time(combstruct[draw]([G,spgraphs,labelled],size=500));
```

2.688 **(4.1.3)**

The next one is much faster, since the precomputation of the enumeration sequences has been done

```
> time(combstruct[draw]([G,spgraphs,labelled],size=500));
```
$$0.057 \tag{4.1.4}$$

But increasing size becomes expensive:

```
> time(combstruct[draw]([G,spgraphs,labelled],size=1000));
```
$$14.406 \tag{4.1.5}$$

Boltzmann Sampling

Find the value of the parameter x leading to an expected size of 10000 for labelled series-parallel graphs

```
> val:=BoltzmannParameter(spgraphs,labelled,G,10000);
```
$$val := 0.2451438466 \tag{4.2.1}$$

Compute the values of the generating functions at that point

```
> pp:=NumericalNewtonIteration(spgraphs,labelled);
```
$$pp := \mathbf{proc}(x, \{ending\_block::integer := 2\}) \ \ldots \ \mathbf{end\ proc} \tag{4.2.2}$$

What has been computed is a procedure that takes as input a real number (smaller than the radius of convergence of the generating functions) and returns the values of the generating functions at that point:

```
> pp(%%);
```
$$[G = 0.6179865811, P = 0.1368040560, S = 0.2360386785] \tag{4.2.3}$$

We can check the expected size for that approximation

```
> BoltzmannExpectedSize(spgraphs,labelled)(val);
```
$$[E(G) = 9790.181289, E(P) = 16891.76075, E(S) = 15841.02300, E(Z) = 1.000000000] \tag{4.2.4}$$

This is slightly smaller than 10000, due to the precision used to compute the value val above. If a more precise expectation is needed, one can compute this value with increased precision:

```
> Digits:=20:
> val:=BoltzmannParameter(spgraphs,labelled,G,10000);
```
$$val := 0.24514384663868229347 \tag{4.2.5}$$

then obtaining the values is still fast

```
> pp(val);
```
$$[G = 0.61798754624448557104, P = 0.13680442464368809001, S = 0.23603927496211518756] \tag{4.2.6}$$

and the expectations behave better

```
> BoltzmannExpectedSize(spgraphs,labelled)(val);
```
$$[E(G) = 9999.9999999474851527, E(P) = 17253.774910836998107, E(S) = 16180.517287091324387, E(Z) = 1.0000000000000000000] \tag{4.2.7}$$

Note that the first value of val was perfect for its precision, so there was no roundoff error, but a high dependency on the precision so close to the singularity.

IV. Asymptotics

Fibonacci numbers

Their generating series is

```
> f:=1/(1-z-z^2);
```
$$f := \frac{1}{-z^2 - z + 1} \tag{5.1.1}$$

```
> series(f,z,11);
```

$$1+z+2z^2+3z^3+5z^4+8z^5+13z^6+21z^7+34z^8+55z^9+89z^{10}+\mathrm{O}(z^{11}) \quad \textbf{(5.1.2)}$$

leading to an asymptotic behaviour

```
> equivalent(f,z,n);
```

$$\frac{(e^{-n})^{-\ln(2)+\ln(\sqrt{5}-1)}}{2\left(\frac{\sqrt{5}}{2}-\frac{1}{2}\right)^2-\frac{1}{2}+\frac{\sqrt{5}}{2}}+\mathrm{O}\left(\frac{(e^{-n})^{-\ln(2)+\ln(\sqrt{5}-1)}}{n}\right) \quad \textbf{(5.1.3)}$$

Conway's sequence

Here is the generating function:

```
> GFconway:=-(-1-4*z^69+18*z^74+31*z^71-4*z^68-z^18+3*z^19-
  z-36*z^24+58*z^27+13*z^22+8*z^12-4*z^17-23*z^31+15*z^70-6*
  z^23-20*z^25+8*z^21-z^13-z^16+6*z^20-6*z^9-18*z^77-5*z^14
  -18*z^75-22*z^72+12*z^78+18*z^76-20*z^73+z^2+z^3+6*z^11-20*
  z^30-30*z^29+z^4-19*z^67-50*z^65+34*z^28+z^5-4*z^8+35*
  z^32+7*z^38+12*z^36-79*z^39+107*z^43+8*z^35-13*z^40-26*z^34
  -9*z^33+42*z^37-39*z^47-32*z^46-66*z^51-33*z^45+14*z^42-65*
  z^44+38*z^49+16*z^41-z^26+z^7-64*z^52-15*z^56+89*z^53-25*
  z^50-8*z^54+126*z^48-4*z^15+45*z^55-11*z^63+41*z^62+54*z^61
  -56*z^60+15*z^58-44*z^59-27*z^57+62*z^66-21*z^64)/
  ((z-1)*(-1+6*z^69+6*z^71-12*z^68+z^10+3*z^18+2*z^19+3*
  z^24+8*z^27-z^22+z^12+10*z^17+5*z^31-3*z^70-9*z^23+7*z^25-6*
  z^21-2*z^13+2*z^16-6*z^20+z^9-5*z^14+z^2+2*z^3+z^11-8*z^30
  -6*z^29+z^4+4*z^67+7*z^65-10*z^28-2*z^5+z^8+12*z^32-10*z^38-
  z^36-z^39+3*z^43-7*z^35+6*z^40+7*z^34-7*z^33+3*z^37-14*
  z^47+3*z^46-9*z^51-9*z^45+10*z^42-2*z^44+2*z^41+8*z^26-z^7
  -2*z^6-3*z^52-12*z^56+4*z^53+7*z^50+10*z^54+8*z^48-3*z^15+7*
  z^55-5*z^62+2*z^61+4*z^60-2*z^58+12*z^59-7*z^57-7*z^66-z^64)
  ):
```

Check the first few coefficients:

```
> series(GFconway,z,21);
```

$$1+2z+2z^2+4z^3+6z^4+6z^5+8z^6+10z^7+14z^8+20z^9+26z^{10}+34z^{11}+46z^{12}+62z^{13}+78z^{14}+102z^{15}+134z^{16}+176z^{17}+226z^{18}+302z^{19}+408z^{20}+\mathrm{O}(z^{21}) \quad \textbf{(5.2.1)}$$

Compute the asymptotic behaviour:

```
> equivalent(GFconway,z,n):
```

evaluate numerically

```
> evalf(%);
```

$$2.0421600768578803651\ 1.3035772690342963913^{-1.\ln(e^{-1.n})}+\mathrm{O}\left(\frac{1.3035772690342963913^n}{n}\right) \quad \textbf{(5.2.2)}$$

Catalan numbers

Recall the specificiation for binary trees

```
> bintrees;
```

$$\{B=Union(\mathrm{E},Prod(Z,B,B))\} \quad \textbf{(5.3.1)}$$

Deduce a closed-form for their generating function

```
> combstruct[gfsolve](%,unlabelled,z);
```

$$\left\{B(z) = -\frac{-1+\sqrt{1-4z}}{2z}, Z(z) = z\right\} \quad (5.3.2)$$

```
> solB:=subs(%,B(z));
```

$$solB := -\frac{-1+\sqrt{1-4z}}{2z} \quad (5.3.3)$$

Apply singularity analysis to obtain a few terms of the asymptotic behaviour (which could also be computed by Stirling's formula)

```
> as_B:=equivalent(solB,z,n,3);
```

$$as\_B := \frac{\left(\frac{1}{n}\right)^{3/2}\left(e^{-n}\right)^{-2\ln(2)}}{\sqrt{\pi}} - \frac{9\left(\frac{1}{n}\right)^{5/2}\left(e^{-n}\right)^{-2\ln(2)}}{8\sqrt{\pi}} + \frac{145\left(\frac{1}{n}\right)^{7/2}\left(e^{-n}\right)^{-2\ln(2)}}{128\sqrt{\pi}} + O\left(\left(\frac{1}{n}\right)^{9/2}\left(e^{-n}\right)^{-2\ln(2)}\right) \quad (5.3.4)$$

Root with an only child

```
> B1:=2*z*solB;
```

$$B1 := 1-\sqrt{1-4z} \quad (5.4.1)$$

Asymptotic behaviour:

```
> equivalent(B1,z,n,3);
```

$$\frac{\left(\frac{1}{n}\right)^{3/2}\left(e^{-n}\right)^{-2\ln(2)}}{2\sqrt{\pi}} + \frac{3\left(\frac{1}{n}\right)^{5/2}\left(e^{-n}\right)^{-2\ln(2)}}{16\sqrt{\pi}} + \frac{25\left(\frac{1}{n}\right)^{7/2}\left(e^{-n}\right)^{-2\ln(2)}}{256\sqrt{\pi}} + O\left(\left(\frac{1}{n}\right)^{9/2}\left(e^{-n}\right)^{-2\ln(2)}\right) \quad (5.4.2)$$

Ratio (asymptotic probability that the root has a leaf for a child):

```
> asympt(%/as_B,n,3);
```

$$\frac{1}{2} + \frac{3}{4n} + O\left(\frac{1}{n^2}\right) \quad (5.4.3)$$

Path length

Recall that the generating function for expected path length was

```
> gfpl;
```

$$-\frac{2B(z,1)\,D_1(B)(z,1)\,z^2}{2zB(z,1)-1} \quad (5.5.1)$$

in which B(z,1) denotes the generating function of binary trees and D[1](B)(z,1) its derivative. I.e., we have

```
> gfpl:=subs(B(z,1)=solB,D[1](B)(z,1)=diff(solB,z),gfpl);
```

(5.5.2)

$$gfpl := -\frac{\left(-1+\sqrt{1-4z}\right)z\left(\frac{1}{\sqrt{1-4z}\,z}+\frac{-1+\sqrt{1-4z}}{2z^2}\right)}{\sqrt{1-4z}} \quad \textbf{(5.5.2)}$$

From there, the asymptotic behaviour follows

```
> equivalent(%,z,n,3);
```

$$\left(e^{-n}\right)^{-2\ln(2)}-\frac{3\sqrt{\frac{1}{n}}\left(e^{-n}\right)^{-2\ln(2)}}{\sqrt{\pi}}+\frac{19\left(\frac{1}{n}\right)^{3/2}\left(e^{-n}\right)^{-2\ln(2)}}{8\sqrt{\pi}}+O\left(\left(\frac{1}{n}\right)^{5/2}\left(e^{-n}\right)^{-2\ln(2)}\right) \quad \textbf{(5.5.3)}$$

and the ratio divided by n gives the average distance from a node to its root:

```
> asympt(%/as_B/n,n,3);
```

$$\frac{\sqrt{\pi}}{\sqrt{\frac{1}{n}}}-3+\frac{9\sqrt{\pi}\sqrt{\frac{1}{n}}}{8}-\frac{1}{n}+O\left(\left(\frac{1}{n}\right)^{3/2}\right) \quad \textbf{(5.5.4)}$$

Cayley trees

Recall the specification

```
> cayley;
```

$$\{T=Prod(Z,Set(T))\} \quad \textbf{(5.6.1)}$$

Closed-form for the generating function

```
> combstruct[gfsolve](cayley,labelled,z)
```

$$\{T(z)=-\mathrm{LambertW}(-z),Z(z)=z\} \quad \textbf{(5.6.2)}$$

```
> subs(%,T(z));
```

$$-\mathrm{LambertW}(-z) \quad \textbf{(5.6.3)}$$

Asymptotic behaviour

```
> equivalent(%,z,n);
```

$$\frac{\sqrt{2}\sqrt{e}\sqrt{e^{-1}}\left(\frac{1}{n}\right)^{3/2}e^{n}}{2\sqrt{\pi}}+O\left(\frac{e^{n}}{n^2}\right) \quad \textbf{(5.6.4)}$$

```
> map(simplify,%) assuming n::posint;
```

$$\frac{\sqrt{2}\,e^{n}}{2\sqrt{\pi}\,n^{3/2}}+O\left(\frac{e^{n}}{n^2}\right) \quad \textbf{(5.6.5)}$$

Hayman admissibility

```
> equivalent(exp(z),z,n);
```

$$\frac{\sqrt{2}\sqrt{\frac{1}{n}}\,e^{n}\,n^{-n}}{2\sqrt{\pi}}+O\left(\left(\frac{1}{n}\right)^{3/2}e^{n}\,n^{-n}\right) \quad \textbf{(5.7.1)}$$

```
> equivalent(exp(z+z^2/2),z,n) assuming n::posint;
```

$$\frac{e^{-\frac{1}{4}}\sqrt{\frac{1}{n}}\,e^{\frac{1}{\sqrt{\frac{1}{n}}}}\sqrt{n^{-n}}}{2\sqrt{\pi}\sqrt{e^{-n}}} + O\left(\frac{e^{\frac{\cos\left(\left(\frac{1}{4}-\frac{1}{4\,\mathrm{signum}(n)}\right)\pi\right)}{\sqrt{\frac{1}{|n|}}}}\sqrt{n^{-n}}}{n\sqrt{e^{-n}}}\right) \tag{5.7.2}$$

```
> map(simplify,%) assuming n::posint;
```

$$\frac{e^{-\frac{1}{4}+\sqrt{n}+\frac{n}{2}}\,n^{-\frac{1}{2}-\frac{n}{2}}}{2\sqrt{\pi}} + O\left(e^{\sqrt{n}+\frac{n}{2}}\,n^{-1-\frac{n}{2}}\right) \tag{5.7.3}$$

```
> equivalent(exp(exp(z)-1),z,n);
```

The saddle point is , LambertW$(n+1)$

Saddle point's expansion:

$$\ln(n) - \ln(\ln(n)) + \frac{\ln(\ln(n))}{\ln(n)} + O\left(\frac{\ln(\ln(n))^2}{\ln(n)^2}\right)$$

$$\frac{\sqrt{2}\,e^{-1}\sqrt{e^{-\_saddlepoint}}\,e^{e^{\_saddlepoint}}}{2\,\_saddlepoint^{n}\sqrt{\pi}\,\_saddlepoint} + O\left(\frac{\sqrt{e^{-\_saddlepoint}}\,e^{e^{\_saddlepoint}}}{\_saddlepoint^{2}\,\_saddlepoint^{n}}\right) \tag{5.7.4}$$