```
> R:=x^4-1;
```

$$R := x^4 - 1$$

```
> pt:=I;
```

$$pt := \mathrm{I}$$

```
> degq:=3:
> Q:=add(q[i]*x^i,i=0..degq);
```

$$Q := q_3 x^3 + q_2 x^2 + q_1 x + q_0$$

```
> P:=R-epsilon*Q;
```

$$P := -\left(q_3 x^3 + q_2 x^2 + q_1 x + q_0\right) \epsilon + x^4 - 1$$

```
> eq:=solve(P,epsilon);
```

$$eq := \frac{x^4 - 1}{q_3 x^3 + q_2 x^2 + q_1 x + q_0}$$

```
> Order:=degq+2:
```

Perturbed root at 1

```
> map(normal,series(eq,x=1));
```

$$\frac{4}{q_3 + q_2 + q_1 + q_0} (x-1) + 2 \frac{-3 q_3 - q_2 + q_1 + 3 q_0}{(q_3 + q_2 + q_1 + q_0)^2} (x-1)^2$$
$$+ 2 \frac{2 q_0^2 + q_1 q_0 - 4 q_2 q_0 - 11 q_3 q_0 + q_1^2 + q_2 q_1 - 2 q_3 q_1 + 2 q_2^2 + 5 q_3 q_2 + 5 q_3^2}{(q_3 + q_2 + q_1 + q_0)^3}$$
$$(x-1)^3 + \frac{1}{(q_3 + q_2 + q_1 + q_0)^4} \left(q_0^3 - q_0^2 q_1 - 11 q_0^2 q_2 - 31 q_0^2 q_3 + q_0 q_1^2\right.$$
$$+ 2 q_0 q_1 q_2 - 10 q_0 q_1 q_3 + 15 q_0 q_2^2 + 54 q_0 q_2 q_3 + 61 q_0 q_3^2 - q_1^3 - 5 q_1^2 q_2 - 9 q_1^2 q_3$$
$$\left. - 5 q_1 q_2^2 - 6 q_1 q_2 q_3 + 9 q_1 q_3^2 - 5 q_2^3 - 19 q_2^2 q_3 - 25 q_2 q_3^2 - 15 q_3^3\right) (x-1)^4 +$$
$$\mathrm{O}\left((x-1)^5\right)$$

```
> z1:=series(1+solve(subs(x-1=t,%)=epsilon,t),epsilon);
```

$$z1 := 1 + \left(\frac{q_3}{4} + \frac{q_2}{4} + \frac{q_1}{4} + \frac{q_0}{4}\right) \epsilon - \frac{1}{32} (q_3 + q_2 + q_1 + q_0) \left(-3 q_3 - q_2 + q_1\right.$$
$$\left. + 3 q_0\right) \epsilon^2 + \frac{1}{128} (q_3 + q_2 + q_1 + q_0) \left(7 q_0^2 + 5 q_1 q_0 - 2 q_2 q_0 - 7 q_3 q_0 - 3 q_2 q_1\right.$$
$$\left. - 4 q_3 q_1 - q_2^2 + q_3 q_2 + 4 q_3^2\right) \epsilon^3 - \frac{1}{2048} (q_3 + q_2 + q_1 + q_0) \left(77 q_0^3 + 83 q_0^2 q_1 - 17\right.$$
$$q_0^2 q_2 - 77 q_0^2 q_3 + 7 q_0 q_1^2 - 66 q_0 q_1 q_2 - 90 q_0 q_1 q_3 - 25 q_0 q_2^2 - 2 q_0 q_2 q_3 + 47 q_0 q_3^2$$
$$- 7 q_1^3 - 25 q_1^2 q_2 - 13 q_1^2 q_3 - 5 q_1 q_2^2 + 38 q_1 q_2 q_3 + 43 q_1 q_3^2 + 5 q_2^3 + 27 q_2^2 q_3 + 15 q_2$$
$$\left. q_3^2 - 15 q_3^3\right) \epsilon^4 + \mathrm{O}(\epsilon^5)$$

Square of its modulus:

```
> absz1:=map(normal,series(z1^2,epsilon));
```

$$absz1 := 1 + \left(\frac{q_3}{2} + \frac{q_2}{2} + \frac{q_1}{2} + \frac{q_0}{2}\right)\epsilon + \left(-\frac{1}{8}q_0^2 - \frac{1}{8}q_1 q_0 + \frac{1}{4}q_3 q_1 + \frac{1}{8}q_2^2 + \frac{3}{8}q_3 q_2 + \frac{1}{4}q_3^2 + \frac{1}{8}q_3 q_0 + \frac{1}{8}q_2 q_1\right)\epsilon^2 + \left(-\frac{3}{32}q_0 q_1 q_2 - \frac{3}{32}q_0 q_2 q_3 - \frac{1}{8}q_0 q_1 q_3 + \frac{1}{16}q_0^3 - \frac{1}{64}q_1^3 + \frac{7}{64}q_3^3 + \frac{5}{64}q_0^2 q_1 - \frac{3}{64}q_0^2 q_3 - \frac{1}{16}q_0 q_2^2 - \frac{1}{16}q_1^2 q_2 - \frac{3}{64}q_1^2 q_3 - \frac{3}{64}q_1 q_2^2 + \frac{5}{64}q_1 q_3^2 + \frac{5}{64}q_2^2 q_3 + \frac{3}{16}q_2 q_3^2\right)\epsilon^3 + \left(\frac{5}{128}q_3^4 - \frac{1}{128}q_2^4 - \frac{5}{128}q_0^4 + \frac{1}{128}q_1^4 - \frac{7}{256}q_0 q_3^3 + \frac{15}{256}q_2 q_3^3 - \frac{7}{256}q_2^3 q_3 - \frac{3}{64}q_1^2 q_3^2 + \frac{7}{256}q_0^3 q_3 + \frac{3}{64}q_0^2 q_2^2 - \frac{5}{256}q_1 q_2^3 + \frac{7}{256}q_0 q_1^3 + \frac{5}{256}q_1^3 q_2 - \frac{15}{256}q_0^3 q_1 - \frac{15}{256}q_0 q_2^2 q_3 - \frac{15}{256}q_0 q_1 q_3^2 + \frac{15}{256}q_0 q_1^2 q_3 + \frac{15}{256}q_0 q_1 q_2^2 - \frac{3}{32}q_1 q_2^2 q_3 + \frac{3}{32}q_0^2 q_1 q_3 - \frac{3}{32}q_0 q_2 q_3^2 + \frac{15}{256}q_0^2 q_2 q_3 - \frac{21}{256}q_1 q_2 q_3^2 - \frac{15}{256}q_1^2 q_2 q_3 + \frac{3}{32}q_0 q_1^2 q_2 + \frac{21}{256}q_0^2 q_1 q_2\right)\epsilon^4 + O(\epsilon^5)$$

Perturbation of a nonreal root

```
> map(normal@evalc,series(eq,x=pt));
```

$$-\frac{4\left(I q_0 - I q_2 + q_1 - q_3\right)}{q_0^2 - 2 q_2 q_0 + q_1^2 - 2 q_3 q_1 + q_2^2 + q_3^2}(x - I) + 2\frac{1}{\left(q_0^2 - 2 q_2 q_0 + q_1^2 - 2 q_3 q_1 + q_2^2 + q_3^2\right)^2}\left(I q_1^2 q_3 - 3 I q_1 q_2^2 - I q_2^2 q_3 + 10 I q_0 q_2 q_3 + 5 I q_0^2 q_1 + I q_1^3 - 9 I q_0^2 q_3 - 2 I q_0 q_1 q_2 + 3 I q_3^3 - 5 I q_1 q_3^2 - 3 q_0^3 + 5 q_0^2 q_2 + q_0 q_1^2 - 10 q_0 q_1 q_3 - q_0 q_2^2 + 9 q_0 q_3^2 + 3 q_1^2 q_2 + 2 q_1 q_2 q_3 - q_2^3 - 5 q_2 q_3^2\right)(x - I)^2$$

$$+ 2\left(-24 I q_0^2 q_1 q_2 q_3 - 24 I q_0 q_1 q_2^2 q_3 + \text{[...52 terms...]} + 12 q_0 q_1^2 q_2 q_3 + 36 q_0 q_1 q_2 q_3^2\right) \Big/ \Big(\left(q_0^2 - 2 q_2 q_0 + q_1^2 - 2 q_3 q_1 + q_2^2 + q_3^2\right)\left(q_0^4 - 4 q_0^3 q_2 + 2 q_0^2 q_1^2 - 4 q_0^2 q_1 q_3 + 6 q_0^2 q_2^2 + 2 q_0^2 q_3^2 - 4 q_0 q_1^2 q_2 + 8 q_0 q_1 q_2 q_3 - 4 q_0 q_2^3 - 4 q_0 q_2 q_3^2 + q_1^4 - 4 q_1^3 q_3 + 2 q_1^2 q_2^2 + 6 q_1^2 q_3^2 - 4 q_1 q_2^2 q_3 - 4 q_1 q_3^3 + q_2^4 + 2 q_2^2 q_3^2 + q_3^4\right)\Big)(x - I)^3$$

$$- \left(450 I q_0^2 q_1 q_2^2 q_3^2 - 42 I q_0 q_1^4 q_2 q_3 + \text{[...116 terms...]} - 19 q_2^3 q_3^4 - 35 q_2 q_3^6\right) \Big/ \Big(\Big(q_0^6 - 6 q_2 q_0^5 + \text{[...40 terms...]} + 3 q_2^2 q_3^4 + q_3^6\Big)\left(q_0^2 - 2 q_2 q_0 + q_1^2 - 2 q_3 q_1 + q_2^2 +\right.$$

$q_3^2)\big)\ (x-\mathrm{I})^4 + \mathrm{O}\big((x-\mathrm{I})^5\big)$

```
> S:=subs(x-pt=t,%):
> zz:=series(add(z[i]*t^i,i=1..Order),t):
> iz:=solve(%=epsilon,t):
> iS:=map(normal@evalc,subs([seq(z[j]=coeff(S,t,j),j=1..Order)],
  iz)):
> zj:=series(pt+iS,epsilon);
```

$$zj := \mathrm{I} + \left(-\frac{q_1}{4} + \frac{q_3}{4} + \frac{\mathrm{I}\,q_0}{4} - \frac{\mathrm{I}\,q_2}{4}\right)\epsilon + \left(-\frac{3\,\mathrm{I}\,q_0^2}{32} + \frac{\mathrm{I}\,q_0\,q_2}{16} + \frac{\mathrm{I}\,q_1^2}{32} + \frac{\mathrm{I}\,q_1\,q_3}{16} + \frac{\mathrm{I}\,q_2^2}{32} - \frac{3\,\mathrm{I}\,q_3^2}{32} + \frac{q_1\,q_0}{8} - \frac{q_3\,q_2}{8}\right)\epsilon^2 + \left(\frac{7\,\mathrm{I}\,q_0^3}{128} - \frac{5\,\mathrm{I}\,q_0^2\,q_2}{128} - \frac{5\,\mathrm{I}\,q_0\,q_1^2}{128} - \frac{3\,\mathrm{I}\,q_0\,q_1\,q_3}{64} - \frac{3\,\mathrm{I}\,q_0\,q_2^2}{128} + \frac{3\,\mathrm{I}\,q_0\,q_3^2}{128} - \frac{3\,\mathrm{I}\,q_1^2\,q_2}{128} + \frac{3\,\mathrm{I}\,q_1\,q_2\,q_3}{64} + \frac{\mathrm{I}\,q_2^3}{128} + \frac{5\,\mathrm{I}\,q_2\,q_3^2}{128} - \frac{3\,q_0^2\,q_1}{32} + \frac{q_0\,q_2\,q_3}{16} + \frac{q_1^2\,q_3}{32} + \frac{q_1\,q_2^2}{32} - \frac{q_3^3}{32}\right)\epsilon^3 + \left(-\frac{15\,\mathrm{I}\,q_0\,q_1\,q_2\,q_3}{256} + \frac{21\,\mathrm{I}\,q_0^2\,q_1\,q_3}{512} + \frac{21\,\mathrm{I}\,q_0\,q_1^2\,q_2}{512} - \frac{15\,\mathrm{I}\,q_0\,q_2\,q_3^2}{512} - \frac{15\,\mathrm{I}\,q_1\,q_2^2\,q_3}{512} + \frac{15\,\mathrm{I}\,q_0^3\,q_2}{512} + \frac{45\,\mathrm{I}\,q_0^2\,q_1^2}{1024} + \frac{21\,\mathrm{I}\,q_0^2\,q_2^2}{1024} - \frac{15\,\mathrm{I}\,q_0^2\,q_3^2}{1024} - \frac{5\,\mathrm{I}\,q_0\,q_2^3}{512} - \frac{5\,\mathrm{I}\,q_1^3\,q_3}{512} - \frac{15\,\mathrm{I}\,q_1^2\,q_2^2}{1024} - \frac{15\,\mathrm{I}\,q_1^2\,q_3^2}{1024} + \frac{7\,\mathrm{I}\,q_1\,q_3^3}{512} + \frac{21\,\mathrm{I}\,q_2^2\,q_3^2}{1024} - \frac{77\,\mathrm{I}\,q_0^4}{2048} + \frac{7\,\mathrm{I}\,q_1^4}{2048} - \frac{5\,\mathrm{I}\,q_2^4}{2048} + \frac{15\,\mathrm{I}\,q_3^4}{2048} + \frac{q_0\,q_3^3}{64} + \frac{q_2^3\,q_3}{64} - \frac{q_1^3\,q_2}{64} + \frac{5\,q_0^3\,q_1}{64} - \frac{3\,q_0\,q_1^2\,q_3}{64} - \frac{3\,q_0\,q_1\,q_2^2}{64} - \frac{3\,q_0^2\,q_2\,q_3}{64} + \frac{3\,q_1\,q_2\,q_3^2}{64}\right)\epsilon^4 + \mathrm{O}(\epsilon^5)$$

Square of its modulus:

```
> abszj:=map(normal@evalc,series(zj*subs(I=-I,zj),epsilon));
```

$$abszj := 1 + \left(\frac{q_0}{2} - \frac{q_2}{2}\right)\epsilon + \left(-\frac{q_0^2}{8} + \frac{q_1^2}{8} + \frac{q_2^2}{8} - \frac{q_3^2}{8}\right)\epsilon^2 + \left(\frac{1}{8}\,q_1\,q_2\,q_3 + \frac{1}{16}\,q_0^3 - \frac{1}{8}\,q_0\,q_1^2 - \frac{1}{16}\,q_0\,q_2^2 - \frac{1}{16}\,q_1^2\,q_2 + \frac{1}{16}\,q_2\,q_3^2\right)\epsilon^3 + \left(\frac{1}{128}\,q_3^4 - \frac{1}{128}\,q_2^4 - \frac{5}{128}\,q_0^4 + \frac{1}{128}\,q_1^4 + \frac{1}{32}\,q_1\,q_3^3 + \frac{1}{32}\,q_2^2\,q_3^2 - \frac{1}{64}\,q_1^2\,q_3^2 + \frac{3}{64}\,q_0^2\,q_2^2 - \frac{1}{32}\,q_1^2\,q_2^2 + \frac{1}{8}\,q_0^2\,q_1^2 - \frac{1}{32}\,q_1^3\,q_3 - \frac{1}{16}\,q_1\,q_2^2\,q_3 - \frac{1}{32}\,q_0\,q_2\,q_3^2 + \frac{3}{32}\,q_0\,q_1^2\,q_2 - \frac{1}{8}\,q_0\,q_1\,q_2\,q_3\right)\epsilon^4 + \mathrm{O}(\epsilon^5)$$

System of equations

```
> zero:=map(normal,series(absz1-abszj,epsilon));
```

$$zero := \left(\frac{q_3}{2} + q_2 + \frac{q_1}{2}\right)\epsilon + \left(-\frac{1}{8} q_1 q_0 + \frac{1}{4} q_3 q_1 + \frac{3}{8} q_3 q_2 + \frac{3}{8} q_3^2 + \frac{1}{8} q_3 q_0 + \frac{1}{8} q_2 q_1 - \frac{1}{8} q_1^2\right)\epsilon^2 + \left(-\frac{3}{32} q_0 q_1 q_2 - \frac{3}{32} q_0 q_2 q_3 - \frac{1}{8} q_0 q_1 q_3 - \frac{1}{64} q_1^3 + \frac{7}{64} q_3^3 + \frac{5}{64} q_0^2 q_1 - \frac{3}{64} q_0^2 q_3 - \frac{3}{64} q_1^2 q_3 - \frac{3}{64} q_1 q_2^2 + \frac{5}{64} q_1 q_3^2 + \frac{5}{64} q_2^2 q_3 + \frac{1}{8} q_2 q_3^2 - \frac{1}{8} q_1 q_2 q_3 + \frac{1}{8} q_0 q_1^2\right)\epsilon^3 + \left(\frac{1}{32} q_3^4 - \frac{7}{256} q_0 q_3^3 - \frac{1}{32} q_1 q_3^3 - \frac{1}{32} q_2^2 q_3^2 + \frac{15}{256} q_2 q_3^3 - \frac{7}{256} q_2^3 q_3 - \frac{1}{32} q_1^2 q_3^2 + \frac{7}{256} q_0^3 q_3 - \frac{5}{256} q_1 q_2^3 + \frac{1}{32} q_1^2 q_2^2 - \frac{1}{8} q_0^2 q_1^2 + \frac{7}{256} q_0 q_1^3 + \frac{5}{256} q_1^3 q_2 + \frac{1}{32} q_1^3 q_3 - \frac{15}{256} q_0^3 q_1 - \frac{15}{256} q_0 q_2^2 q_3 - \frac{15}{256} q_0 q_1 q_3^2 + \frac{15}{256} q_0 q_1^2 q_3 + \frac{15}{256} q_0 q_1 q_2^2 - \frac{1}{32} q_1 q_2^2 q_3 + \frac{3}{32} q_0^2 q_1 q_3 - \frac{1}{16} q_0 q_2 q_3^2 + \frac{15}{256} q_0^2 q_2 q_3 - \frac{21}{256} q_1 q_2 q_3^2 - \frac{15}{256} q_1^2 q_2 q_3 + \frac{21}{256} q_0^2 q_1 q_2 + \frac{1}{8} q_0 q_1 q_2 q_3\right)\epsilon^4 + O(\epsilon^5)$$

```
> sys:={seq(coeff(zero,epsilon,i),i=1..Order-1)}:
```

Solution

```
> with(FGb);
Path set to /Users/salvy/lib/maple/FGb/FGblib/../libfgbunid.so
    FGb/Maple interface package Version 1.68
    JC Faugere (Jean-Charles.Faugere@inria.fr)
    Type ?FGb for documentation
```

[*ModuleUnload*, *cpu*, *date*, *fgb*, *fgb_gbasis*, *fgb_gbasis_elim*, *fgb_gbasis_lm*, *fgb_hilbert*, *fgb_interface*, *fgb_matrixn*, *fgb_matrixn_radical*, *fgb_matrixn_radical2*, *fgb_multi*, *fgb_normalForm*, *pseudo_fgb_normalForm*]

Add inequations:

```
> G:=fgb_gbasis_elim([op(sys),1-t*q[degq]*eval(Q,x=1)],0,[t],[seq
  (q[i],i=0..degq)]);
```

$$G := [q_0 + 5 q_3, q_1 - q_3, q_2 + q_3]$$

```
> solve(G,[seq(q[i],i=0..degq)]);
```

$$[[q_0 = -5 q_3, q_1 = q_3, q_2 = -q_3, q_3 = q_3]]$$

```
> subs(%[1],q[degq]=1,Q);
```

$$x^3 - x^2 + x - 5$$