Automatic Proofs of Identities: Beyond A=B

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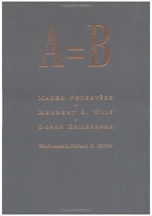

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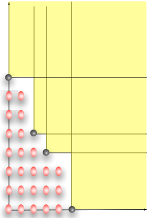
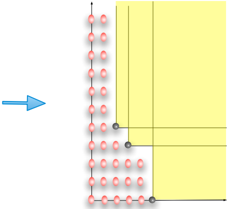




FPSAC, Linz, July 20, 2009
Joint work with F. Chyzak and M. Kauers

I Introduction

Examples of Identities: Definite Sums, q-Sums, Integrals

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\frac{u^n}{n!} = \frac{\exp\left(\frac{4u(xy-u(x^2+y^2))}{1-4u^2}\right)}{\sqrt{1-4u^2}}$$ [Mehler1866]

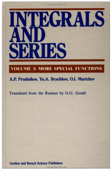
$$\sum_{k=0}^{n} \frac{q^{k^2}}{(q;q)_k(q;q)_{n-k}} = \sum_{k=-n}^{n} \frac{(-1)^k q^{(5k^2-k)/2}}{(q;q)_{n-k}(q;q)_{n+k}}$$ [Andrews1974]

$$\int_0^{+\infty} xJ_1(ax)I_1(ax)Y_0(x)K_0(x)\,dx = -\frac{\ln(1-a^4)}{2\pi a^2}$$ [GlasserMontaldi1994]

$$\frac{1}{2\pi i}\oint \frac{(1+2xy+4y^2)\exp\left(\frac{4x^2y^2}{1+4y^2}\right)}{y^{n+1}(1+4y^2)^{\frac{3}{2}}}\,dy = \frac{H_n(x)}{\lfloor n/2\rfloor!}$$ [Doetsch1930]

+ multiple sums/integrals

& many, many more in , e.g.,



Examples of Non-"Holonomic" Identities

$$\sum_{k=0}^{n} \binom{n}{k} i(k+i)^{k-1}(n-k+j)^{n-k} = (n+i+j)^n \quad \text{[Abel1826]}$$

$$\sum_{k=0}^{n} (-1)^{m-k} k! \binom{n-k}{m-k} \left\{ {n+1 \atop k+1} \right\} = \left\langle {n \atop m} \right\rangle \quad \text{[Frobenius1910]}$$

$$\sum_{k=0}^{m} \binom{m}{k} B_{n+k} = (-1)^{m+n} \sum_{k=0}^{n} \binom{n}{k} B_{m+k} \quad \text{[Gessel2003]}$$

$$\int_0^\infty x^{k-1} \zeta(n, \alpha + \beta x)\, dx = \beta^{-k} B(k, n-k) \zeta(n-k, \alpha)$$

$$\int_0^\infty x^{\alpha-1} \operatorname{Li}_n(-xy)\, dx = \frac{\pi(-\alpha)^n y^{-\alpha}}{\sin(\alpha\pi)}$$

$$\int_0^\infty x^{s-1} \exp(xy) \Gamma(a, xy)\, dx = \frac{\pi y^{-s}}{\sin((a+s)\pi)} \frac{\Gamma(s)}{\Gamma(1-a)}$$

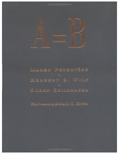
Computer Algebra Algorithms

Aim

- Prove these identities automatically (fast?);
- Compute the rhs given the lhs;
- Explain why these identities exist.

Examples:

- 1st slide: Zeilberger's algorithm and variants (many people)

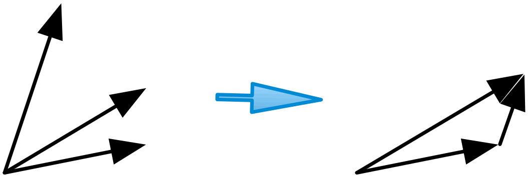


- 2nd slide (1st 3): Majewicz, Kauers, Chen & Sun;
- last 3: new generalization of previous ones.

Ideas

Confinement in finite dimension + Creative telescoping.

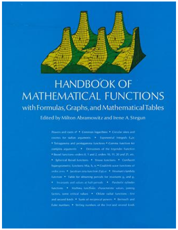
II Confinement in Finite Dimension and Closure Properties



Idea: confine a function and all its derivatives/shifts/...

Finite Dimension and Special Functions

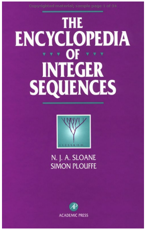
- Classical:
  polynomials represent their roots better than radicals.
  Algorithms: Euclidean division and algorithm, Gröbner bases.
- More Recent:
  same for linear differential or recurrence equations.
  Algorithms: non-commutative analogues & gen. func.



About 25% of Sloane's encyclopedia,
60% of Abramowitz & Stegun.

eqn+ini. cond.=data structure

http://ddmf.msr-inria.inria.fr/



First Proof: Mehler's Identity by Confinement

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\frac{u^n}{n!} = \frac{\exp\left(\frac{4u(xy-u(x^2+y^2))}{1-4u^2}\right)}{\sqrt{1-4u^2}}$$

1. Definition of Hermite polynomials $H_n(t)$: recurrence of order 2 $\leftrightarrow$ vector space of dimension 2 over $\mathbb{Q}(t,n)$;
2. Product: vector space over $\mathbb{Q}(x,y,n)$ generated by

$$\frac{H_n(x)H_n(y)}{n!}, \frac{H_{n+1}(x)H_n(y)}{n!}, \frac{H_n(x)H_{n+1}(y)}{n!}, \frac{H_{n+1}(x)H_{n+1}(y)}{n!}$$

$\rightarrow$ recurrence of order at most 4; (confinement)

3. Translate into differential equation (and solve).

I. Definition

> $R_1 := \{H(n+2) = (-2n-2)H(n) + 2H(n+1)x, H(0)=1, H(1)=2x\}$:

> $R_2 := subs(H=H_2, x=y, R_1)$;

$$R_2 := \{H_2(0)=1, H_2(n+2) = (-2n-2)H_2(n) + 2H_2(n+1)y, H_2(1)=2y\}$$

II. Product

> $R_3 := gfun:-poltorec(H(n)\cdot H_2(n)\cdot v(n), [R_1, R_2, \{v(n+1)\cdot(n+1)=v(n), v(1)=1\}], [H(n), H_2(n), v(n)], c(n))$;

$$R_3 := \Big\{c(0)=1, c(1)=4xy, c(2)=8x^2y^2+2-4y^2-4x^2, c(3)=\frac{32}{3}x^3y^3+24xy-16xy^3-16x^3y, (16n+16)c(n)-16xyc(n+1)+(-8n-20+8y^2+8x^2)c(n+2)-4xc(n+3)y+(n+4)c(n+4)\Big\}$$

III. Differential Equation

> $gfun:-rectodiffeq(R_3, c(n), f(u))$;

$$\Big\{(16u^3-16u^2yx-4u+8uy^2+8ux^2-4xy)f(u)+(16u^4-8u^2+1)\Big(\frac{d}{du}f(u)\Big), f(0)=1\Big\}$$

> $dsolve(\%, f(u))$;

$$f(u) = \frac{\mathrm{I}\,\mathrm{e}^{\left(\frac{-4xyu+y^2+x^2}{(2u-1)(2u+1)}\right)}}{\mathrm{e}^{(-y^2-x^2)}\sqrt{2u+1}\,\sqrt{2u-1}}$$

>

Second Proof: Contiguity of Hypergeometric Series

$$F(a,b;c;z)=\sum_{n=0}^{\infty}\underbrace{\frac{(a)_n(b)_n}{(c)_n n!}}_{u_{a,n}}z^n,\qquad (x)_n:=x(x+1)\cdots(x+n-1).$$

$$\frac{u_{a,n+1}}{u_{a,n}}=\frac{(a+n)(b+n)}{(c+n)(n+1)}\to\ z(1-z)F''+(c-(a+b+1)z)F'-abF=0,$$

$$\frac{u_{a+1,n}}{u_{a,n}}=\frac{n}{a}+1\to S_a\cdot F(a,b,c;z):=F(a+1,b;c;z)=\frac{z}{a}F'+F.$$

Notation: Differential and Shift Operators

$$(a_m(z)D_z^m+\cdots+a_0(z))\cdot F=a_m(z)F^{(m)}(z)+\cdots+a_0(z)F(z)$$
$$\left(b_p(k)S_k^p+\cdots+b_0(k)\right)\cdot u_k=b_p(k)u_{k+p}+\cdots+b_0(k)u_k.$$

∂: any of S_m, D_x, q-shift,...

Second Proof: Contiguity of Hypergeometric Series

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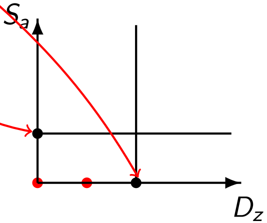
$$\frac{u_{a,n+1}}{u_{a,n}}=\frac{(a+n)(b+n)}{(c+n)(n+1)}\to\ z(1-z)F''+(c-(a+b+1)z)F'-abF=0,$$

$$\frac{u_{a+1,n}}{u_{a,n}}=\frac{n}{a}+1\to S_a\cdot F(a,b,c;z):=F(a+1,b;c;z)=\frac{z}{a}F'+F.$$

Gauss 1812: contiguity relation.

dim=2 $\Rightarrow S_a^2\cdot F, S_a\cdot F, F$ linearly dependent:

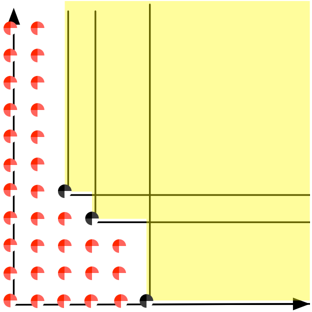
(Coordinates in $\mathbb{Q}(a,b,c,z)$.)



Gröbner Bases: Generalize Euclidean Division and Gcd

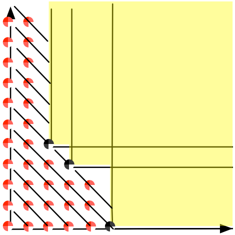
1. Monomial ordering: total order $\prec$ on the monomials, compatible with product, 1 minimal.
2. Gröbner basis of a (left) ideal $\mathcal{I}$ wrt $\prec$: generators of $\mathcal{I}$ at the corners of its stairs.
3. Quotient mod$\mathcal{I}$: vector basis below the stairs $(\mathrm{Vect}\{\partial^\alpha \cdot f\})$.
4. Reduction of P mod $\mathcal{I}$: Unique remainder written on this basis.

$\rightarrow$ An access to (finite dimensional) vector spaces



Hilbert Dimension: a Handle on Infinite Dimension

$M_s(\mathcal{I}) := \text{Vect}\{m \mid m \text{ is below the stairs and of total degree} \leq s\}$



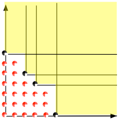
Definition: Hilbert Dimension $\delta(\mathcal{I})$

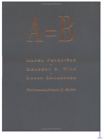
$$\dim M_s(\mathcal{I}) = O\left(s^{\delta(\mathcal{I})}\right).$$

- Finite measure of infinite-dimensional vector-spaces.
- Can be obtained from a Gröbner basis.

Definition (annihilator and ∂-finiteness)

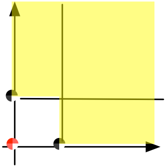
- $\text{Ann} f := \{P \mid P \cdot f = 0\}$
- f is ∂-finite $\Leftrightarrow \delta(\text{Ann}\, f) = 0$
  $\Leftrightarrow$ linear dim. of quotient is finite.





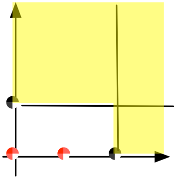
Examples

Binomial coeffs $\binom{n}{k}$ wrt S_n, S_k;
Hypergeometric sequences:



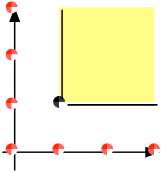
$\delta(\mathcal{I}) = 0$, $\dim S/\mathcal{I} = 1$

Bessel $J_\nu(x)$ wrt S_ν, D_x;
Orthogonal pols wrt S_n, D_x:



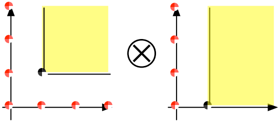
$\delta(\mathcal{I}) = 0$, $\dim S/\mathcal{I} = 2$

Stirling nbs wrt S_n, S_k



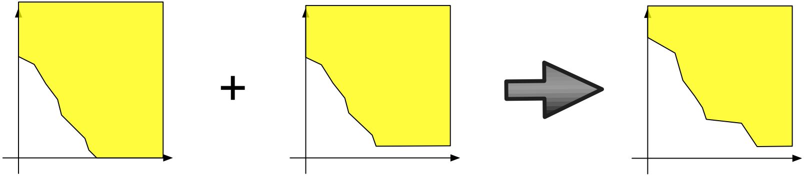
$\delta(\mathcal{I}) = 1$, $\dim S/\mathcal{I} = \infty$

Abel type wrt S_m, S_k, S_r, S_s
$\mathrm{hgm}(m,k)(k+r)^k(m-k+s)^{m-k}\frac{r}{k+r}$:



$\delta(\mathcal{I}) = 2$ in space of dim 4.

Closure Properties



Proposition

$$\delta(\mathrm{Ann}(f+g)) \le \max(\delta(\mathrm{Ann}\, f), \delta(\mathrm{Ann}\, g)),$$
$$\delta(\mathrm{Ann}(fg)) \le \delta(\mathrm{Ann}\, f) + \delta(\mathrm{Ann}\, g),$$
$$\delta(\mathrm{Ann}(\partial \cdot f)) \le \delta(\mathrm{Ann}\, f).$$

Algorithms by linear algebra (Gröbner bases as Input/Output).

III Creative Telescoping — Closure under $\sum$ and $\int$ —

Input: $\mathrm{GB}\,(\mathrm{Ann}\, f)$
Output: $\mathrm{GB}\left(\mathrm{Ann} \int f\right)$ or $\mathrm{GB}\,(\mathrm{Ann} \sum f)$ (or subideals).

Summation by Creative Telescoping

Goal: evaluate $U_n := \sum_{k=0}^{n} \binom{n}{k}$ to 2^n.

GIVEN Pascal's triangle rule:

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} = 2\binom{n}{k} + \binom{n}{k-1} - \binom{n}{k},$$

summing over k gives

$$U_{n+1} = 2U_n.$$

The initial condition $U_0 = 1$ concludes the proof.

Creative Telescoping (Zeilberger 1990)

$$U_n = \sum_{k=a}^{b} u_{n,k} = ?$$

GIVEN $A(n, S_n)$ and $B(n, k, S_n, S_k)$ such that

$$(A(n, S_n) - \Delta_k B(n, k, S_n, S_k)) \cdot u_{n,k} = 0,$$

then the sum “telescopes”, leading to

$$A(n, S_n) \cdot U_n = [B(n, k, S_n, S_k) \cdot u_{n,k}]_{k=a}^{k=b+1} \overset{\text{often}}{=} 0.$$

Adapts easily to $U(z) = \sum_{k=a}^{b} u_k(z)$.

Creative Telescoping (Zeilberger 1990)

$$U(z) = \int_a^b u(z,t)\,dt = ?$$

GIVEN $A(x, D_x)$ and $B(x, y, D_x, D_y)$ such that

$$(A(z, D_z) - D_t B(z, t, D_z, D_t)) \cdot u(z,t) = 0,$$

then the integral "telescopes", leading to

$$A(z, D_z) \cdot U(z) = [B(z, t, D_z, D_t) \cdot u(z,t)]_{t=a}^{t=b} \overset{\text{often}}{=} 0.$$

Then I come along and try differentating under the integral sign, and often it worked. So I got a great reputation for doing integrals.

Richard P. Feynman 1985

Creative telescoping="differentiation" under integral+"integration" by parts

Diff. under $\int$ + Integration by Parts → Algorithm?

Ex.: $\displaystyle\int_0^1 \frac{\cos zt}{\sqrt{1-t^2}}\,dt = \frac{\pi}{2}J_0(z), \quad (\underbrace{zJ_0'' + J_0' + zJ_0}_{A(z,D_z)\cdot J_0} = 0,\ J_0(0) = 1).$

$$\operatorname{Ann}\frac{\cos zt}{\sqrt{1-t^2}} \ni \underbrace{A(z,D_z)}_{\text{no } t,\,D_t} - D_t \underbrace{\frac{t^2-1}{t}D_z}_{\text{anything}}$$

Specification for a Creative Telescoping Algorithm

Input: generators of (a subideal of) $\operatorname{Ann} f$;
Output: A free of t, ∂_t, certificate B, such that $A - \partial_t B \in \operatorname{Ann} f$.

Definition (Telescoping of $\mathcal{I}$ wrt t)

$$T_t(\mathcal{I}) := (\mathcal{I} + \partial_t \mathbb{Q}(z,t)\langle\partial_z,\partial_t\rangle) \cap \mathbb{Q}(z)\langle\partial_z\rangle.$$

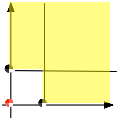
+ variants for multiple sum/int.

IV Algorithms

Example: Rediscovering Pascal's Triangle Rule

1. Gröbner basis for Ann $\binom{n}{k}$:

$$S_n \to \frac{n+1}{n+1-k}1, \quad S_k \to \frac{n-k}{k+1}1$$



over $\mathbb{Q}(n,k)$

2. Reduce all monomials of degree $\leq s = 2$:

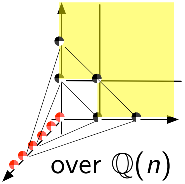
$$S_n^2 \to \frac{(n+2)(n+1)}{(n+2-k)(n+1-k)}1, S_k^2 \to \frac{(n-k-1)(n-k)}{(k+2)(k+1)}1, S_nS_k \to \frac{n+1}{k+1}1.$$

3. Common denominator: $D_2 = (k+1)(k+2)(n+1-k)(n+2-k)$.

$D_2, D_2S_n, D_2S_k, D_2S_n^2, D_2S_k^2, D_2S_nS_k$ confined in

$$\mathrm{Vect}_{\mathbb{Q}(n)}(1, k1, k^21, k^31, k^41)$$

$$\to D_2(S_nS_k - S_k - 1) \in \mathrm{Ann}\binom{n}{k}.$$

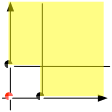


4. This has to happen for some degree: $\deg D_s = O(s)$.

More Examples

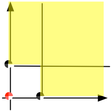
- Proper hypergeometric [Wilf & Zeilberger 1992]:

$$Q(n,k)\xi^k \frac{\prod_{i=1}^{u}(a_i n + b_i k + c_i)!}{\prod_{i=1}^{v}(u_i n + v_i k + w_i)!},$$

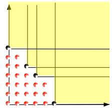


Q polynomial, $\xi \in \mathbb{C}$, a_i, b_i, u_i, v_i integers: essentially the same situation.

- $\dfrac{1}{n^2+k^2}$: confinement in a space of dimension $O(s^2)$, no elimination of k succeeds.



- $f = \dfrac{a(z,t_1,\dots,t_r)}{b(z,t_1,\dots,t_r)}$: $D_s = b^s$, confinement in a space of dimension $O(s^1)$ over $\mathbb{Q}(z)$, elimination of $t_1,\dots,t_r$ has to succeed.



Base case of the proof that D-finite functions are "holonomic".

Polynomial Growth and Creative Telescoping when $\delta > 0$

Definition (Polynomial Growth p)

There exists a sequence of polynomials $P_s(z_1, \dots, z_k, t)$, s.t.

$$|a| + b \leq s \Rightarrow P_s \partial_{z_1}^{a_1} \cdots \partial_{z_k}^{a_k} \partial_t^b \rightarrow \text{pol of degree } O(s^p) \text{ in } t.$$

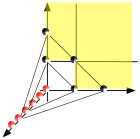
Theorem (Chyzak, Kauers & Salvy 2009)

$$\delta(T_t(\mathcal{I})) \leq \max(\delta(\mathcal{I}) + p - 1, 0).$$

Corollary (Sufficient Condition for Creative Telescoping)

$\delta(\mathcal{I}) + p - 1 < k \Rightarrow$ *identities exist for the sum/int wrt t.*

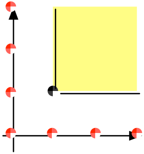
Proof. Same as above. Also an algorithm.



Non-∂-Finite Examples (both with $p=1$)

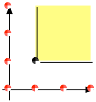
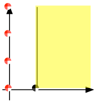
- Stirling: $\delta = 1 \rightarrow$ for ≥ 3 vars, e.g., Frobenius:

$$\sum_{k=0}^{n} (-1)^{m-k} k! \binom{n-k}{m-k} \left\{ {n+1 \atop k+1} \right\} = \left\langle {n \atop m} \right\rangle .$$



- Abel type: $\delta = 2 \rightarrow$ for ≥ 4 vars, e.g., Abel:

$$\sum_{k=0}^{n} \binom{n}{k} i(k+i)^{k-1}(n-k+j)^{n-k} = (n+i+j)^n.$$

 $\otimes$ 

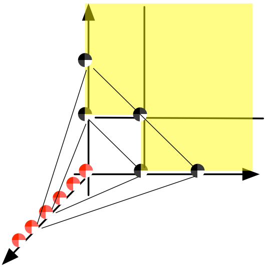
Algorithm I. Sister Celine Style

Polynomial growth + linear algebra $\to \mathcal{J} := \mathcal{I} \cap \mathbb{Q}(z)\langle\partial_z, \partial_t\rangle$.

Algorithm: Eliminate t

For increasing s until $\delta(\mathcal{J}) \leq$ bound,

1. Reduce all $\partial_z^a \partial_t^b$ with $|a| + b \leq s$;
2. Normalize to a common denominator;
3. Set up a linear system to cancel the positive powers of t;
4. If a solution is found, it has the form $A(z, \partial_z) + \partial_t B(z, \partial_z, \partial_t)$. Return it.



This computes in

$$((\mathcal{I} \cap \mathbb{Q}(z)\langle\partial_z, \partial_t\rangle) + \partial_t \mathbb{Q}(z)\langle\partial_z, \partial_t\rangle) \cap \mathbb{Q}(z)\langle\partial_z\rangle,$$

not in

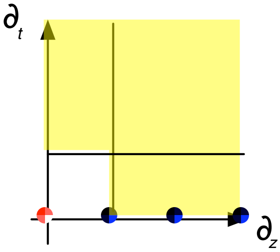
$$T_t(\mathcal{I}) := (\mathcal{I} + \partial_t \mathbb{Q}(z,t)\langle\partial_z, \partial_t\rangle) \cap \mathbb{Q}(z)\langle\partial_z\rangle.$$

Algorithm II. Zeilberger Style Extended to $\delta > 0$

Compute in $T_t(\mathcal{I}) := (\mathcal{I} + \partial_t \mathbb{Q}(z,t)\langle \partial_z, \partial_t \rangle) \cap \mathbb{Q}(z)\langle \partial_z \rangle$

Faster, more precise.

1. Hypergeometric case: Zeilberger 1990;



Algorithm (Zeilberger & variants)

for $s = 0, 1, 2, \ldots,$ until found:

1. reduce $A - \partial_t B$ with

$$A := \sum_{\alpha \leq s} \eta_\alpha(z) \partial^\alpha, \; B := \phi(z,t),$$

for **undetermined** rational $\eta_\alpha(z)$, $\phi(z,t)$.

3. solve by an extended Gosper algorithm

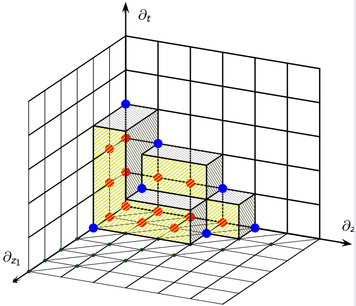
return the pairs (A, B).

Algorithm II. Zeilberger Style Extended to $\delta > 0$

Compute in $T_t(\mathcal{I}) := (\mathcal{I} + \partial_t \mathbb{Q}(z,t)\langle\partial_z,\partial_t\rangle) \cap \mathbb{Q}(z)\langle\partial_z\rangle$

Faster, more precise.

1. Hypergeometric case: Zeilberger 1990;
2. ∂-finite case ($\delta = 0$): Chyzak 2000;



Algorithm (Chyzak)

for $s = 0, 1, 2, \ldots,$ until $\delta(\mathcal{J}) = 0$:

1. reduce $A - \partial_t B$ with

$$A := \sum_{\alpha \mid |\alpha| \le s} \eta_\alpha(z)\partial^\alpha, \quad B := \sum_{\beta \in \bullet} \phi_\beta(z,t)\partial^\beta,$$

for **undetermined** rational $\eta_\alpha(z)$, $\phi_\beta(z,t)$.

2. extract coeffs of $\bullet$ to form a linear system of first order w.r.t. ∂_t
3. solve and set $\mathcal{J}$ to the ideal of the A's

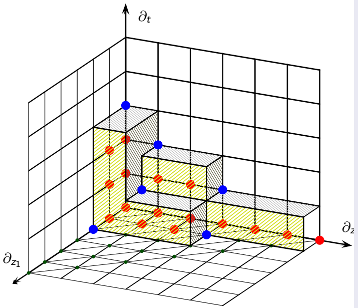
return the pairs (A, B).

Algorithm II. Zeilberger Style Extended to $\delta > 0$

Compute in $T_t(\mathcal{I}) := (\mathcal{I} + \partial_t \mathbb{Q}(z,t)\langle\partial_z,\partial_t\rangle) \cap \mathbb{Q}(z)\langle\partial_z\rangle$

Faster, more precise.

1. Hypergeometric case: Zeilberger 1990;
2. ∂-finite case ($\delta = 0$): Chyzak 2000;
3. Non-∂-finite:



Algorithm (new)

for $s = 0, 1, 2, \dots,$ until $\delta(\mathcal{J}) \leq$ bound:

1. reduce $A - \partial_t B$ with

$$A := \sum_{\alpha | |\alpha| \leq s} \eta_\alpha(z)\partial^\alpha,\ B := \sum_{\beta \in M_s(\mathcal{I})} \phi_\beta(z,t)\partial^\beta,$$

for **undetermined** rational $\eta_\alpha(z)$, $\phi_\beta(z,t)$.

2. extract coeffs of $M_{s+1}(\mathcal{I})$ to form a linear system of first order w.r.t. ∂_t
3. solve and set $\mathcal{J}$ to the ideal of the A's

return the pairs (A, B).

Final Example

$$\sum_k \binom{n}{k} \left\{ {k \atop \ell} \right\} \left\{ {n-k \atop m} \right\} = \binom{\ell+m}{\ell} \left\{ {n \atop m+\ell} \right\}$$

1. Gröbner bases for $\binom{n}{k}$, $\left\{ {k \atop \ell} \right\}$, $\left\{ {n-k \atop m} \right\}$:

   $\{(k-n-1)S_n+n+1, (k+1)S_k+k-n, S_m-1, S_\ell-1\}, \{S_kS_\ell-(\ell+1)S_\ell-1, S_n-1, S_m$

2. Product by closure:

   $\{(k+1)(m+1)S_kS_\ell S_m+(k-n)S_m+(1+k)S_kS_\ell+(1+\ell)(k-n)S_\ell S_m, (1+k)S_kS_\ell S_n$

3. Creative telescoping: $s=1,2 \rightarrow$ nothing; $s=3$: system 14×28

   $A = S_\ell S_m S_n-(\ell+m+2)S_\ell S_m-S_m-S_\ell$, $B = \frac{k(k+1)}{k^2-1-n-kn}S_\ell+\frac{(m+1)k}{k-n-1}S_mS_\ell$

4. $\delta = 2 \rightarrow$ stop

V Conclusion

Conclusion

- Summary:
  - Linear differential/recurrence equations as a data structure;
  - Confinement in vector spaces + creative telescoping $\rightarrow$ identities;
  - Input dimension + polynomial growth $\rightarrow$ output dimension.
- Also:
  - Multiple summation/integration;
  - Bounds $\rightarrow$ identities + their size + complexity.
- Open questions:
  - Replace polynomial growth by something intrinsic;
  - Exploit symmetries;
  - Compute all of $T_t(\mathcal{I})$;
  - Structured Padé-Hermite approximants;
  - Understand non-minimality.