

# The Dynamic Dictionary of Mathematical Functions

Bruno Salvy  
Bruno.Salvy@inria.fr



ICMS 2010. September 14, 2010

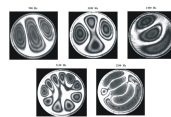
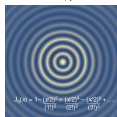
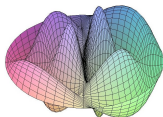
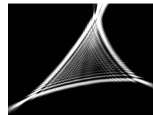
Joint work with: Alexandre Benoit, Frédéric Chyzak,  
Alexis Darrasse, Stefan Gerhold, Marc Mezzarobba

# I Motivation

# Mathematical Functions



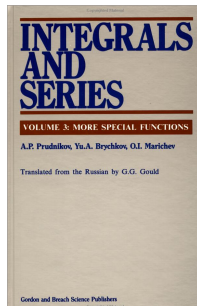
exp, log, sin, cos, tan,  
sinh, cosh, tanh, arcsin, arccos, arctan,  
arcsinh, arccosh, arctanh,  
Bessel  $J_\nu, I_\nu, Y_\nu, K_\nu$ , Airy Ai and Bi,  
hypergeometric, generalized hypergeo-  
metric, classical orthogonal polynomials,  
Struve  $\mathbf{H}_\nu, \mathbf{L}_\nu, \mathbf{M}_\nu, \mathbf{K}_\nu$ , Lommel  $s_{\mu,\nu}$  and  $S_{\mu,\nu}$ ,  
Anger  $\mathbf{J}_\nu$ , Weber  $\mathbf{E}_\nu$ , Whittaker  $W_{\kappa,\mu}, M_{\kappa,\mu} \dots$



## Special functions

Functions that have been met sufficiently often to deserve a name.

Scientists need help with these functions

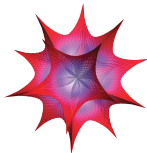
[illegible]

Started between 60 and 30 years ago.

# Progress in the Past 30 Years

Two important changes in the way we work:

- **Symbolic Computation.** Several million users.



Mathematical functions implemented from these dictionaries.

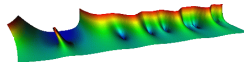
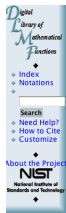
- **The Web**



New kinds of interaction with documents.

# First Step: the NIST DLMF

## Navigation, Exports, Search Engine



### NIST Digital Library of Mathematical Functions companion to the [NIST Handbook of Mathematical Functions](#)

#### Project News

- 2010-05-11 [Handbook published and DLMF goes public](#)
- 2010-05-06 [Firefox 3.6 shown on Windows](#)
- [More news](#)

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7 Error Functions, Dawson's and Fresnel Integrals	27 Functions of Number Theory
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9 Airy and Related Functions	29 Lamé Functions
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11 Struve and Related Functions	31 Heun Functions
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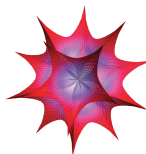
Still computed, compiled and edited by hand.

# The Dynamic Dictionary of Mathematical Functions

## Aim of the project

**DDMF** = Mathematical Handbooks + Computer Algebra + Web

- 1 Computer algebra algorithms to **generate** the formulas;
- 2 Web-like interaction with the document **and the computation**.



## Building Blocks:

- 1 Linear differential equations as a data-structure;
- 2 New language for maths on the web.  
(compatible with browsers!).

## II Demonstration



<http://ddmf.msr-inria.inria.fr>

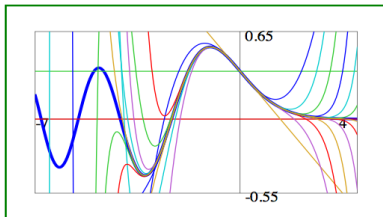
## Dynamic Dictionary of Mathematical Functions

Welcome to this interactive site on [Mathematical Functions](#), with properties, truncated expansions, numerical evaluations, plots, and more. The functions currently presented are elementary functions and special functions of a single variable. More functions — special functions with parameters, orthogonal polynomials, sequences — will be added with the project advances.

Select a special function from the list

- [Help](#) on selecting and configuring the mathematical rendering
- DDMF [developers](#) list
- [Motivation](#) of the project
- List of [related projects](#)
- Release [history](#)

The DDMF project (2008–2010) is hosted and supported by the [Microsoft Research – INRIA Joint Centre](#).



Generated on 2010-06-25 17:03:34 using commit b9253b3..., Fri Jun 25 16:21:12 2010 +0200 (modified).  
Powered by [DynaMoW](#)

Contents

[rendering](#) [link](#)

- The [inverse cosecant](#)  $\operatorname{arccsc}(x)$
- The [inverse cosine](#)  $\arccos(x)$
- The [inverse cotangent](#)  $\operatorname{arccot}(x)$
- The [inverse hyperbolic cosecant](#)  $\operatorname{arccsch}(x)$
- The [Airy function of the first kind](#)  $\operatorname{Ai}(x)$
- The [inverse secant](#)  $\operatorname{arcsec}(x)$
- The [inverse sine](#)  $\arcsin(x)$
- The [inverse tangent](#)  $\arctan(x)$
- The [Airy function \(of the second kind\)](#)  $\operatorname{Bi}(x)$
- The [hyperbolic cosine integral](#)  $\operatorname{Chi}(x)$
- The [cosine integral](#)  $\operatorname{Ci}(x)$
- The [cosine](#)  $\cos(x)$
- The [exponential integral](#)  $\operatorname{Ei}(x)$
- The [error function](#)  $\operatorname{erf}(x)$
- The [complementary error function](#)  $\operatorname{erfc}(x)$
- The [imaginary error function](#)  $\operatorname{erfi}(x)$
- The [inverse hyperbolic cosine](#)  $\operatorname{arcosh}(x)$
- The [inverse hyperbolic cotangent](#)  $\operatorname{arcoth}(x)$
- The [inverse hyperbolic secant](#)  $\operatorname{arsech}(x)$
- The [inverse hyperbolic sine](#)  $\operatorname{arsinh}(x)$
- The [inverse hyperbolic tangent](#)  $\operatorname{artanh}(x)$
- The [hyperbolic cosine](#)  $\cosh(x)$
- The [hyperbolic sine](#)  $\sinh(x)$
- The [dilogarithm](#)  $\operatorname{dilog}(x)$
- The [hyperbolic sine integral](#)  $\operatorname{Shi}(x)$
- The [sine integral](#)  $\operatorname{Si}(x)$
- The [sine](#)  $\sin(x)$
- The [exponential](#)  $e^x$
- The [logarithm](#)  $\ln(x)$

## III DynaMoW: Dynamic Mathematics on the Web

# DynaMoW

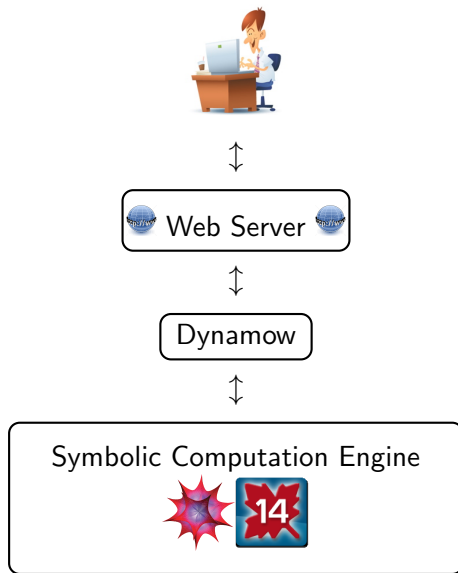
## Principle

The document being generated by the computer algebra system is an object of the language.

Consequences:

- ① the structure of the document may depend on values that have been computed;
- ② intermediate steps can be turned into a mathematical proof in natural language;
- ③ easy to write demo code.

# Architecture



# DynaMoW, an OCaml Library

DynaMoW = ocaml + quotations + antiquotations

Symbolic result, converted to LaTeX, put into a paragraph

```
let res = <:symp< symbolic expression >> in
  <:par< some text <:imath< some latex $(symp:res) >>
>>
```

Using ocaml values in symbolic computations

```
let n = 23 and s = "foo" in
  <:symp< f($(int:n), $(str:s)) >>
```

Symbolic objects cast to ocaml types

```
let n = 23 + <:int< symbolic expression >> in ...
if <:bool< symbolic expression >> then ... else ...
<:unit< f := symbolic expression >>
```

# Example

The proof that a function is odd



equation into a homogeneous one.

The homogeneous differential equation and the fact that

$$\frac{d^i}{dx^i} y(-x) = (-1)^i \frac{d^i}{dx^i} y(x), \quad \text{for } i \in \mathbb{N},$$

imply that the function  $y_1(x) = -y(-x)$  [metadata](#) satisfies the differential equation

$$(-1 - x^2) \frac{d^2}{dx^2} y_1(x) - 2x \frac{d}{dx} y_1(x) = 0$$

with initial conditions [metadata](#)

$$y_1(0) = 0, (y_1'(0) = 1.$$

The functions  $\arctan(x)$  and  $-\arctan(-x)$  [metadata](#) thus satisfy the same differential equation, and their derivatives at  $x=0$  agree up to order 1. Since  $x=0$  is an ordinary point of the equation, these functions are analytic and equal in a neighborhood of 0:

$$\arctan(x) = -\arctan(-x).$$

This identity extends to the whole common [metadata](#) domain of definition of these functions by uniqueness of the analytic continuation.

jsMath

```
let ending = Wording.ending_of_seq <:symp< inicondsAlt >> in
let body =
  <:par< The  $\text{\textit{homStr}}$  differential equation and the fact that
    <:dmath< <:symp< diff(y(-x),[x$i]) = (-1)^i*diff(y(x),[x$i])
      \quad \text{\textit{for } i\in\mathbb{N}}, >>
    imply that the function <:imath<  $y_1(x) = \text{\textit{sign}}y(-x)$  >>
    satisfies the differential equation
    <:dsymb< add(op(i+2,altDiffEq) * diff(y[1](x),[x$i]),
      i=seq(nops(altDiffEq)-2..0,-1)) = -op(1,altDiffEq)
    with initial condition $\text{\textit{ending}}$ 
    <:dmath< <:symp< op(inicondsAlt) >> . >> >> :: body in

let sf_name_as_math =
  if sf_name = "" then "y" else <:isymb<  $\text{\textit{sf\_name}}$  >> in

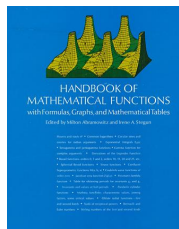
let body =
  let ordinary_point = DC.Text (Glossary.g "ordinary point") in
  <:par< The functions <:imath<  $\text{\textit{sf\_name\_as\_math}}(x)$  >>
    and <:imath<  $\text{\textit{sign}}\text{\textit{sf\_name\_as\_math}}(-x)$  >>
    thus satisfy the same differential
    equation, and their derivatives at <:imath<  $x=0$  >> agree up
    order <:isymb< nops(iniconds)-1 >>. Since <:imath<  $x=0$  >>
    is an  $\text{\textit{ordinary\_point}}$  of the equation, these functions
    are analytic and equal in a neighborhood of 0:
    <:dmath<  $\text{\textit{sf\_name\_as\_math}}(x) =$ 
       $\text{\textit{sign}}\text{\textit{sf\_name\_as\_math}}(-x)$  . >>
    This identity extends to the whole common domain of definition
    of these functions by uniqueness of the analytic continuation
  >> :: body
in
((title req_params, List.rev body), ret)
```

## IV Symbolic Computation



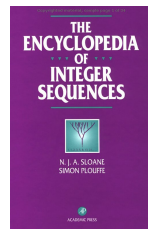
# Defining a Mathematical Function by an Equation

- Classical:  
polynomials represent their roots better than radicals.  
**Algorithms:** Euclidean division and algorithm, Gröbner bases.
- More Recent:  
same for **linear differential or recurrence equations**.  
**Algorithms:** non-commutative analogues & gen. func.



About **25%** of Sloane's encyclopedia,  
**60%** of Abramowitz & Stegun.

eqn+ini. cond.=data structure



# Recent Progress

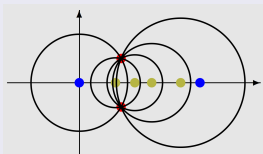
- Automatic computation of bounds, fast evaluation at high precision; [MezzarobbaSalvy2010]
- Chebyshev expansions; [BenoitSalvy2009]
- More formulas accessible to computer algebra [ChyzakKauersSalvy2009].

# Guaranteed Numerical Evaluation

## 1. Inside the disk of convergence

- ① Effective majorant-series analysis;
- ② Efficient evaluation of truncated series;
- ③ Time complexity quasi-linear wrt precision.

## 2. Effective analytic continuation



Path:  $0 \rightarrow 1.5 \rightarrow 2.3 \rightarrow 3 \rightarrow 4.22 \rightarrow 5.$

# Computation of Identities by Confinement

$$\int_0^\infty x^{k-1} \zeta(n, \alpha + \beta x) dx = \beta^{-k} B(k, n-k) \zeta(n-k, \alpha),$$

$$\int_0^\infty x^{\alpha-1} \operatorname{Li}_n(-xy) dx = \frac{\pi(-\alpha)^n y^{-\alpha}}{\sin(\alpha\pi)},$$

$$\int_0^\infty x^{k-1} \exp(xy) \Gamma(n, xy) dx = \frac{\pi y^{-k}}{\sin((n+k)\pi)} \frac{\Gamma(k)}{\Gamma(1-n)},$$

$$\int_0^\infty \frac{x}{(x^2 + y^2) \sin(xz)} dx = \frac{\pi}{2 \sinh(yz)},$$

not accessible to computer algebra before.

- How can we compute them?
- Why do they exist?

**Algorithm:**

- 1 Creative telescoping;
- 2 Confinement in finite-dim vector spaces

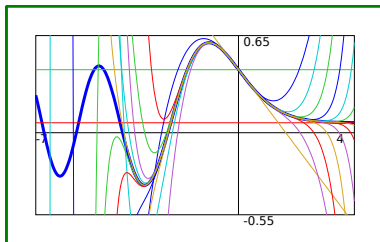


# Future Work

- More expansions on bases of functions;
- more integral transforms;
- families of functions or functions with parameters;
- automatic generation of numerical code;
- information on the zeros of functions;
- user-defined functions.

Summary: you want to bookmark

<http://ddmf.msr-inria.inria.fr>



# THE END