Linear Differential Equations as a Data-Structure

Bruno Salvy

Inria & ENS de Lyon

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Computer Algebra

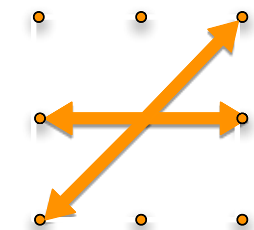
Effective mathematics: what can we compute exactly?
And complexity: how fast? (also, how big is the result?)

Systems with several million users

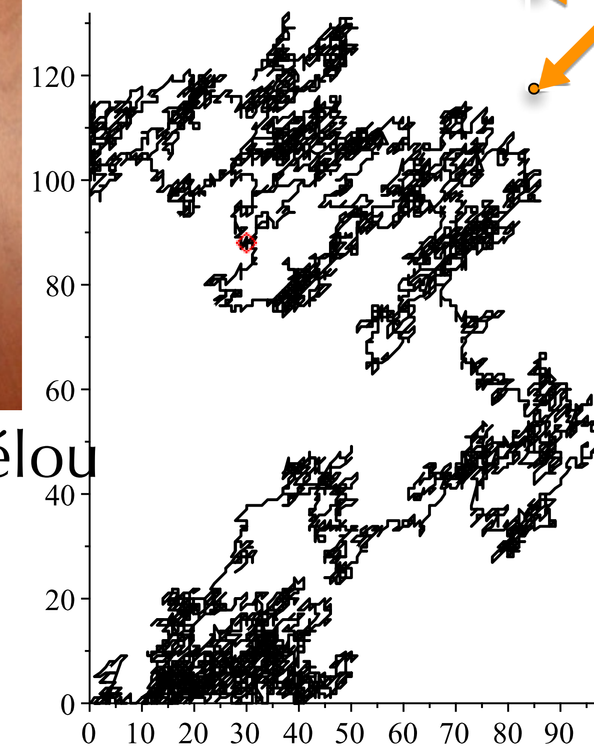
50+ years of algorithmic progress
in computational mathematics!

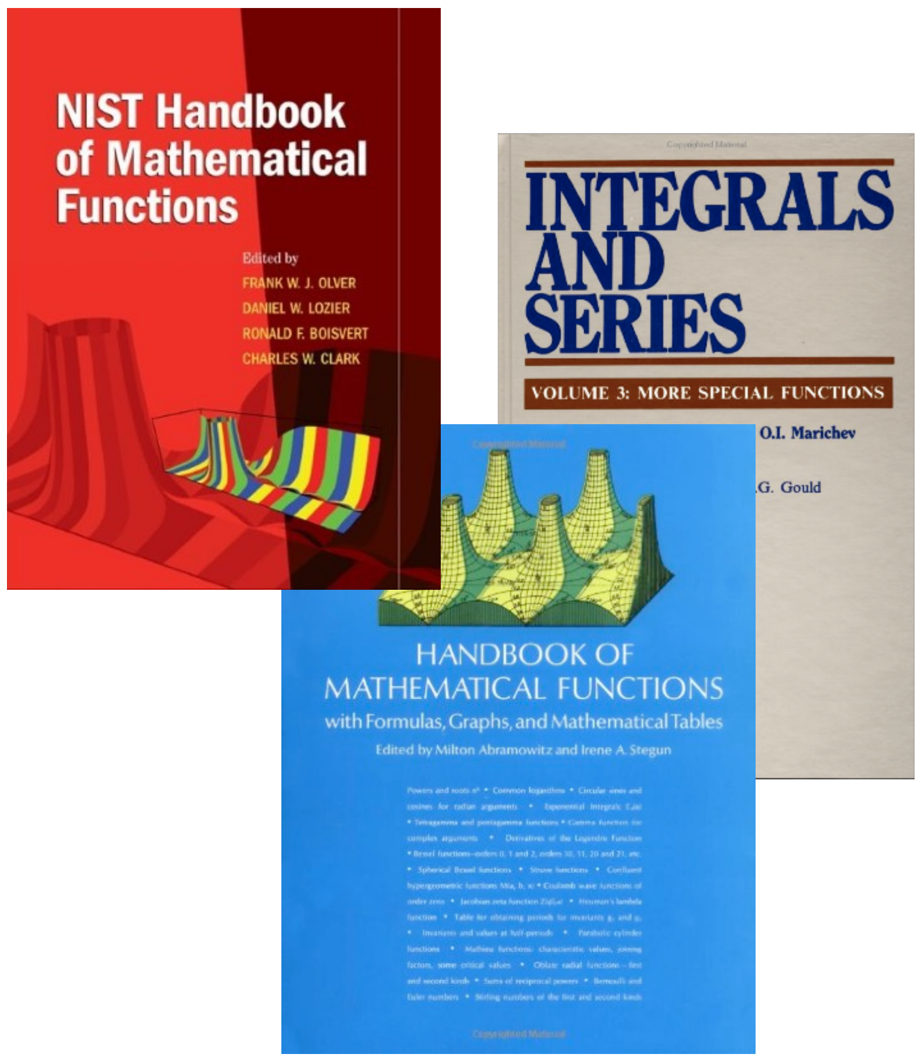
Sources of Linear Differential Equations

Generating functions
in combinatorics



M. Bousquet-Mélou





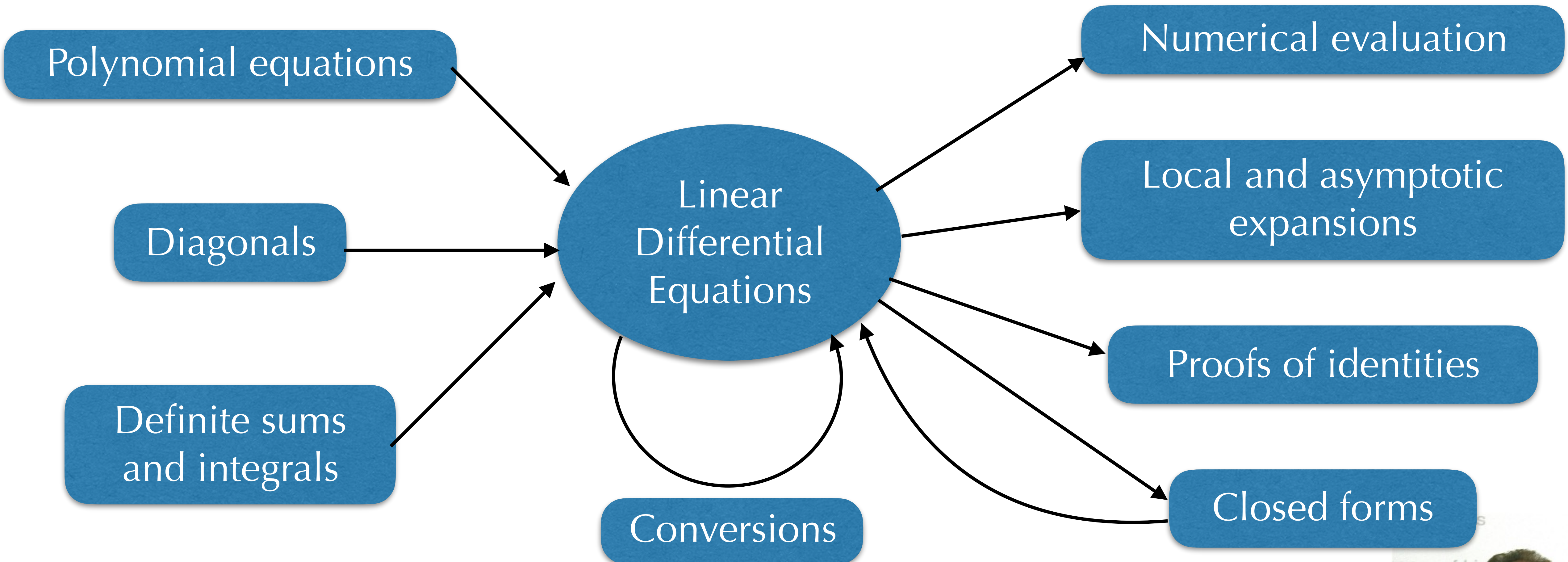
Classical elementary
and special functions
(small order)

Periods

P. Lairez

A. Bostan

LDEs as a Data-Structure



M. Singer


Solutions called differentially finite (abbrev. D-finite)

A. Using Linear Differential Equations Exactly

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I. Numerical Values

Fast Computation with Linear Recurrences (70's and 80's)

1. Multiplication of integers is fast (Fast Fourier Transform): millions of digits ≪ 1sec.
2. n! in complexity $\tilde{O}(n)$ by divide-and-conquer

$$n! := \underbrace{n \times \cdots \times \lceil n/2 \rceil}_{\text{size } O(n \log n)} \times \underbrace{(\lceil n/2 \rceil - 1) \times \cdots \times 1}_{\text{size } O(n \log n)}$$

Notation: $\tilde{O}(n)$ means $O(n \log^k n)$ for some k

3. Linear recurrence: convert into 1st order recurrence on vectors and apply the same idea.

Ex: $e_n := \sum_{k=0}^{n} \frac{1}{k!}$ satisfies a 2nd order rec, computed via

$$\begin{pmatrix} e_n \\ e_{n-1} \end{pmatrix} = \frac{1}{n} \underbrace{\begin{pmatrix} n+1 & -1 \\ n & 0 \end{pmatrix}}_{A(n)} \begin{pmatrix} e_{n-1} \\ e_{n-2} \end{pmatrix} = \frac{1}{n!} A!(n) \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

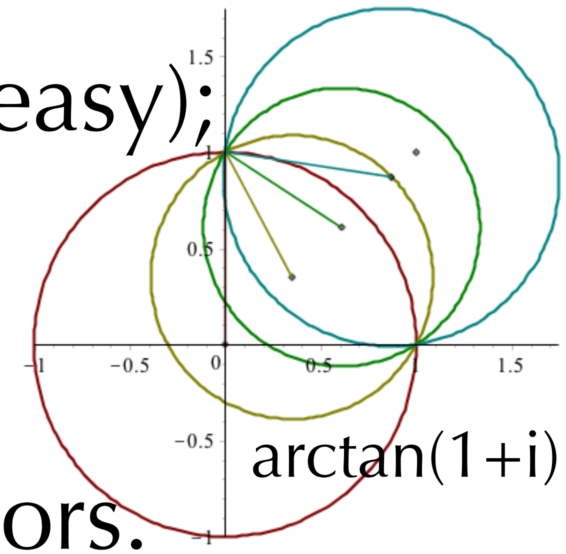
Conclusion: Nth element in $\tilde{O}(N)$ ops.

Numerical evaluation of solutions of LDEs

Principle: $$f(x) = \underbrace{\sum_{n=0}^{N} a_n x^n}_{\text{fast evaluation}} + \underbrace{\sum_{n=N+1}^{\infty} a_n x^n}_{\text{good bounds}}$$

f solution of a LDE with coeffs in $\mathbb{Q}(x)$

1. linear recurrence in N for the first sum (easy);
2. tight bounds on the tail (technical);
3. extend to $\mathbb{C}$ by analytic continuation.



Computation on integers. No roundoff errors.

Conclusion: value anywhere with N digits in $\tilde{O}(N)$ ops.

Sage code available

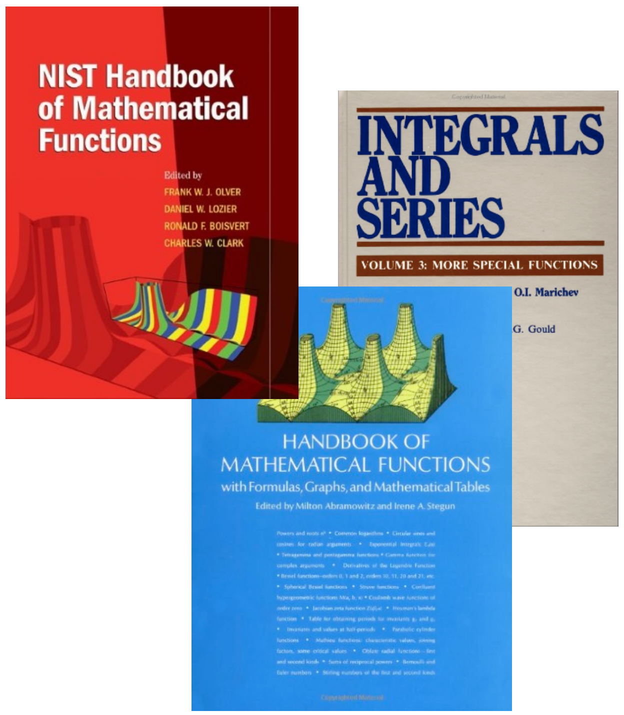

M. Mezzarobba

[Chudnovsky-Chudnovsky87;van der Hoeven99;Mezzarobba-S.10;Mezzarobba16]

A. Using Linear Differential Equations Exactly

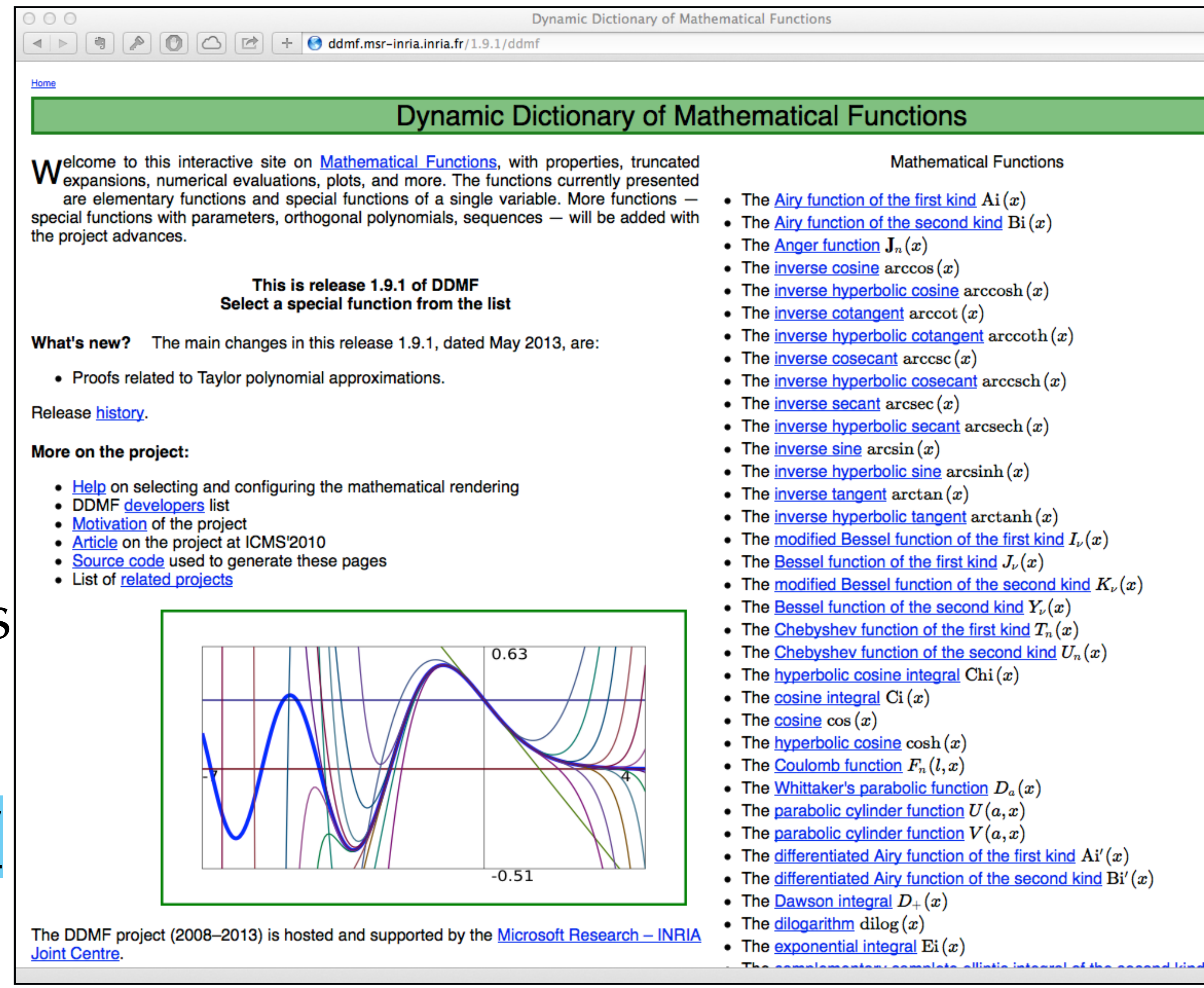
II. Local and Asymptotic Expansions

Dynamic Dictionary of Mathematical Functions



- User need
- Recent algorithmic progress
- Maths on the web

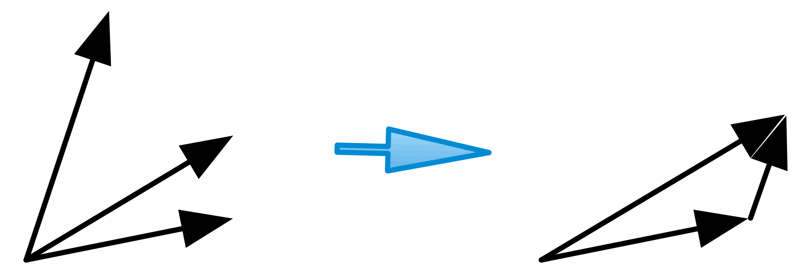
http://ddmf.msr-inria.inria.fr/



A. Using Linear Differential Equations Exactly

III. Proofs of Identities

Proof technique



```
> series(sin(x)^2+cos(x)^2-1,x,4);
```

$O(x^4)$

f satisfies a LDE
$\Longleftrightarrow$
f,f′,f″,… live in a finite-dim. vector space

Why is this a proof?

1. sin and cos satisfy a 2nd order LDE: $y''+y=0$;
2. their squares and their sum satisfy a 3rd order LDE;
3. the constant -1 satisfies $y'=0$;
4. thus $\sin^2+\cos^2-1$ satisfies a LDE of order at most 4;
5. the Cauchy-Lipschitz theorem concludes.

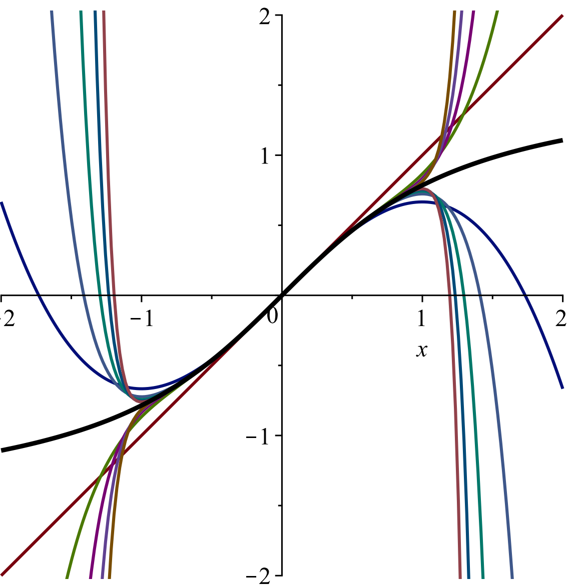
Proofs of non-linear identities by linear algebra!

Mehler's identity for Hermite polynomials

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\frac{u^n}{n!} = \frac{\exp\left(\frac{4u(xy-u(x^2+y^2))}{1-4u^2}\right)}{\sqrt{1-4u^2}}$$

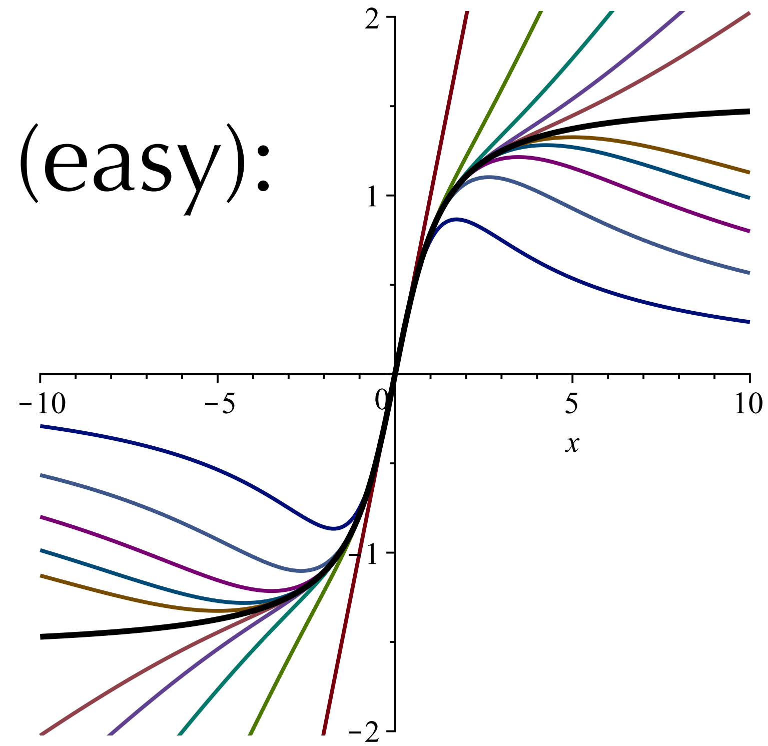
1. Definition of Hermite polynomials:
   recurrence of order 2;
2. Product by linear algebra: $H_{n+k}(x)H_{n+k}(y)/(n+k)!$, $k\in\mathbb{N}$ generated over $\mathbb{Q}(x,n)$ by
   $$\frac{H_n(x)H_n(y)}{n!}, \frac{H_{n+1}(x)H_n(y)}{n!}, \frac{H_n(x)H_{n+1}(y)}{n!}, \frac{H_{n+1}(x)H_{n+1}(y)}{n!}$$
   → recurrence of order at most 4;
3. Translate into differential equation.

Guess & Prove Continued Fractions



1. Taylor expansion produces first terms (easy):

$$\arctan x = \cfrac{x}{1+\cfrac{\frac{1}{3}x^2}{1+\cfrac{\frac{4}{15}x^2}{1+\cfrac{\frac{9}{35}x^2}{1+\cdots}}}}$$



2. Guess a formula (easy): $a_n = \dfrac{n^2}{4n^2-1}$

3. Prove that the CF with these a_n converges to arctan:

show that $H_n := Q_n^2\left((x^2+1)(P_n/Q_n)' - 1\right) = O(x^n)$

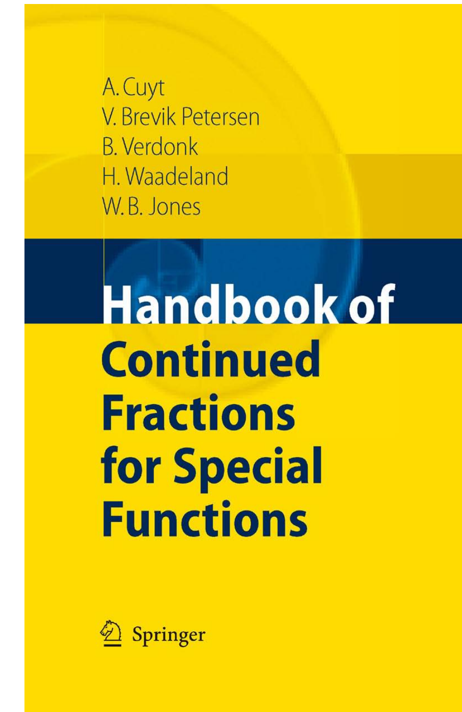
where P_n/Q_n is the nth convergent.

`gfun[ContFrac]`

Algo ≈ compute a LRE for H_n and simplify it.

No human intervention needed.

It Works!



- This method has been applied to all explicit C-fractions in Cuyt *et alii*, starting from either:
  a Riccati equation:
  $$y' = A(z) + B(z)y + C(z)y^2$$
  a q-Riccati equation:
  $$y(qz) = A(z) + B(z)y(z) + C(z)y(z)y(qz)$$
  a difference Riccati equation:
  $$y(s+1) = A(s) + B(s)y(s) + C(s)y(s)y(s+1)$$
- It works in all cases, including Gauss's CF, Heine's q-analogue and Brouncker's CF for Gamma.
- In all cases, H_n satisfies a recurrence of small order.

In progress: 1. explain why this method works so well,
2. classify the formulas it yields.

[Maulat-S. 15,*17*]

B. Conversions (LDE → LDE)

From equations to operators

$D_x \leftrightarrow d/dx$	$S_n \leftrightarrow (n \mapsto n+1)$
$x \leftrightarrow$ mult by x	$n \leftrightarrow$ mult by n
product $\leftrightarrow$ composition	product $\leftrightarrow$ composition
$D_x x = x D_x + 1$	$S_n n = (n+1) S_n$

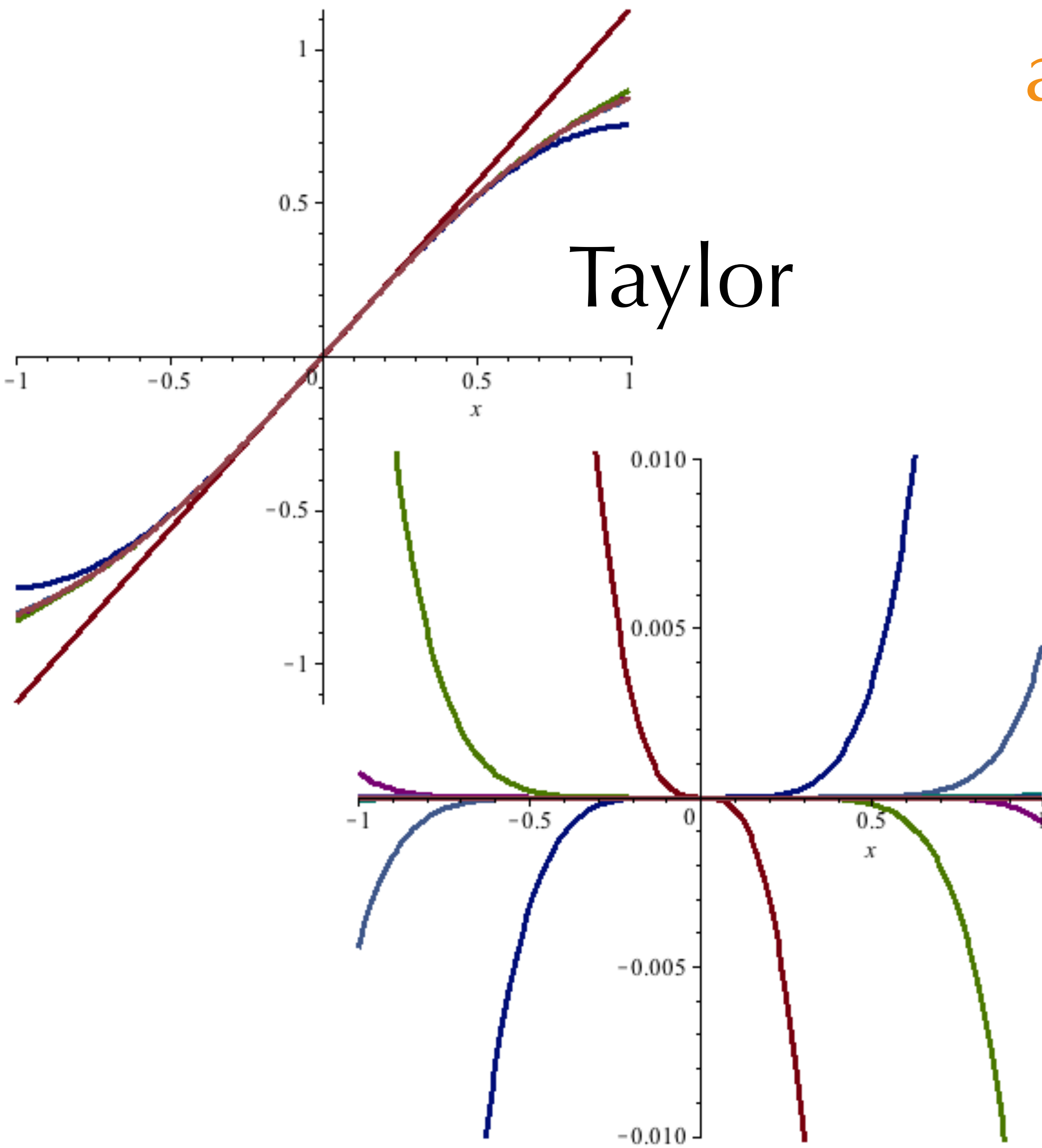
Taylor morphism: $D_x \mapsto (n+1)S_n$; $x \mapsto S_n^{-1}$
produces linear recurrence from LDE

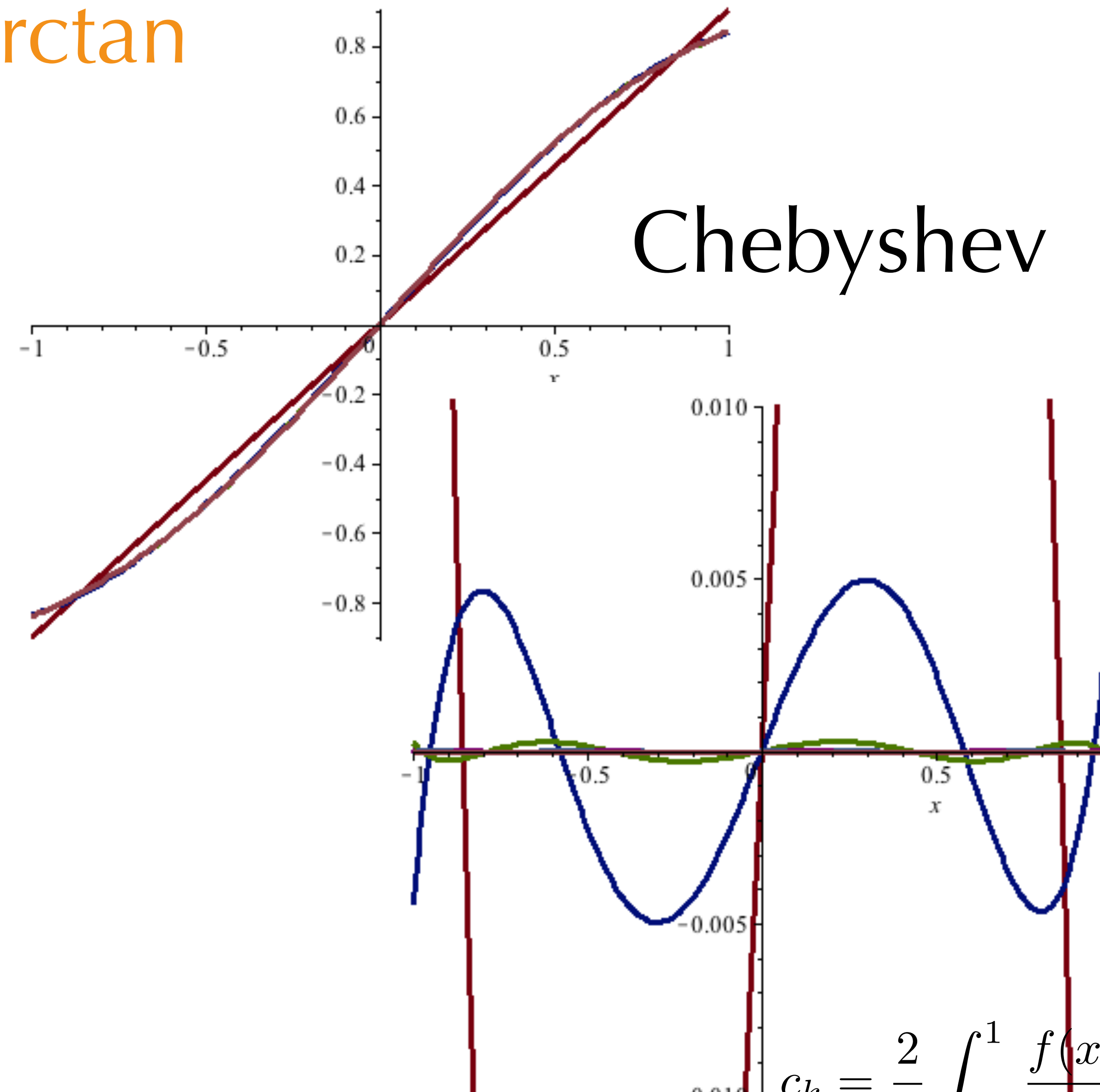
Ex. (erf):

$$D_x^2 + 2xD_x \mapsto (n+1)S_n(n+1)S_n + 2S_n^{-1}(n+1)S_n = (n+1)(n+2)S_n^2 + 2n$$

Chebyshev expansions

arctan





$$c_k = \frac{2}{\pi}\int_{-1}^{1} \frac{f(x)T_k(x)}{\sqrt{1-x^2}}\,dx$$

$$z - \frac{1}{3}z^3 + \frac{1}{5}z^5 + \cdots$$

$$2(\sqrt{2}+1)\left(\frac{T_1(x)}{(2\sqrt{2}+3)} - \frac{T_3(x)}{3(2\sqrt{2}+3)^2} + \frac{T_5(x)}{5(2\sqrt{2}+3)^3} + \cdots\right)$$

Ore fractions

Generalize commutative case:

$R = Q^{-1}P$ with P & Q operators.

$B^{-1}A = D^{-1}C$ when $bA = dC$ with $bB = dD = \mathrm{LCLM}(B,D)$.

Algorithms for sum and product:

$B^{-1}A + D^{-1}C = \mathrm{LCLM}(B,D)^{-1}(bA + dC)$, with $bB = dD = \mathrm{LCLM}(B,D)$

$B^{-1}AD^{-1}C = (aB)^{-1}dC$, with $aA = dD = \mathrm{LCLM}(A,D)$.

[Ore1933]

Application: Chebyshev expansions

Taylor

$x^{n+1}=x\cdot x^n \leftrightarrow x \mapsto X:=S^{-1}$

$(x^n)'=nx^{n-1} \leftrightarrow d/dx \mapsto D:=(n+1)S$

Chebyshev

$2xT_n(x)=T_{n+1}(x)+T_{n-1}(x)$

$\leftrightarrow x \mapsto X:=(S_n+S_n^{-1})/2$

$2(1-x^2)T_n'(x)=-nT_{n+1}(x)+nT_{n-1}(x)$

$\leftrightarrow d/dx \mapsto D:=(1-X^2)^{-1}n(S_n-S_n^{-1})/2.$

Prop. If y is a solution of L(x,d/dx), then its Chebyshev coefficients annihilate the numerator of L(X,D).

```
> deqarctan:=(x^2+1)*diff(y(x),x)-1:
> diffeqToGFSRec(deqarctan,y(x),u(n),functions=ChebyshevT(n,x));
```

$$nu(n)+6(n+2)u(n+2)+(n+4)u(n+4)$$

Applications to Validated Numerical Approximation

M. Joldes

[Benoit-S.09;Benoit12;BenoitJoldesMezzarobba17]

C. Computing Linear Differential Equations (Efficiently)

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I. Algebraic Series and Questions of Size

Algebraic Series can be Computed Fast

$$P(X, Y(X)) = 0 \qquad P \text{ irreducible}$$

Wanted: the first *N* Taylor coefficients of *Y*.

$$P_x(X, Y(X)) + P_y(X, Y(X)) \cdot Y'(X) = 0$$

$$Y'(X) = (-P_x P_y^{-1} \bmod P)(X, Y(X))$$

a polynomial

Note:
F sol LDE
⇒ *F(Y(X))* sol LDE
(same argument)

$$Y(X), Y'(X), Y''(X), \ldots \text{ in } \mathrm{Vect}_{\mathbb{Q}(X)}(1, Y, Y^2, \ldots)$$

finite dimension (deg P)

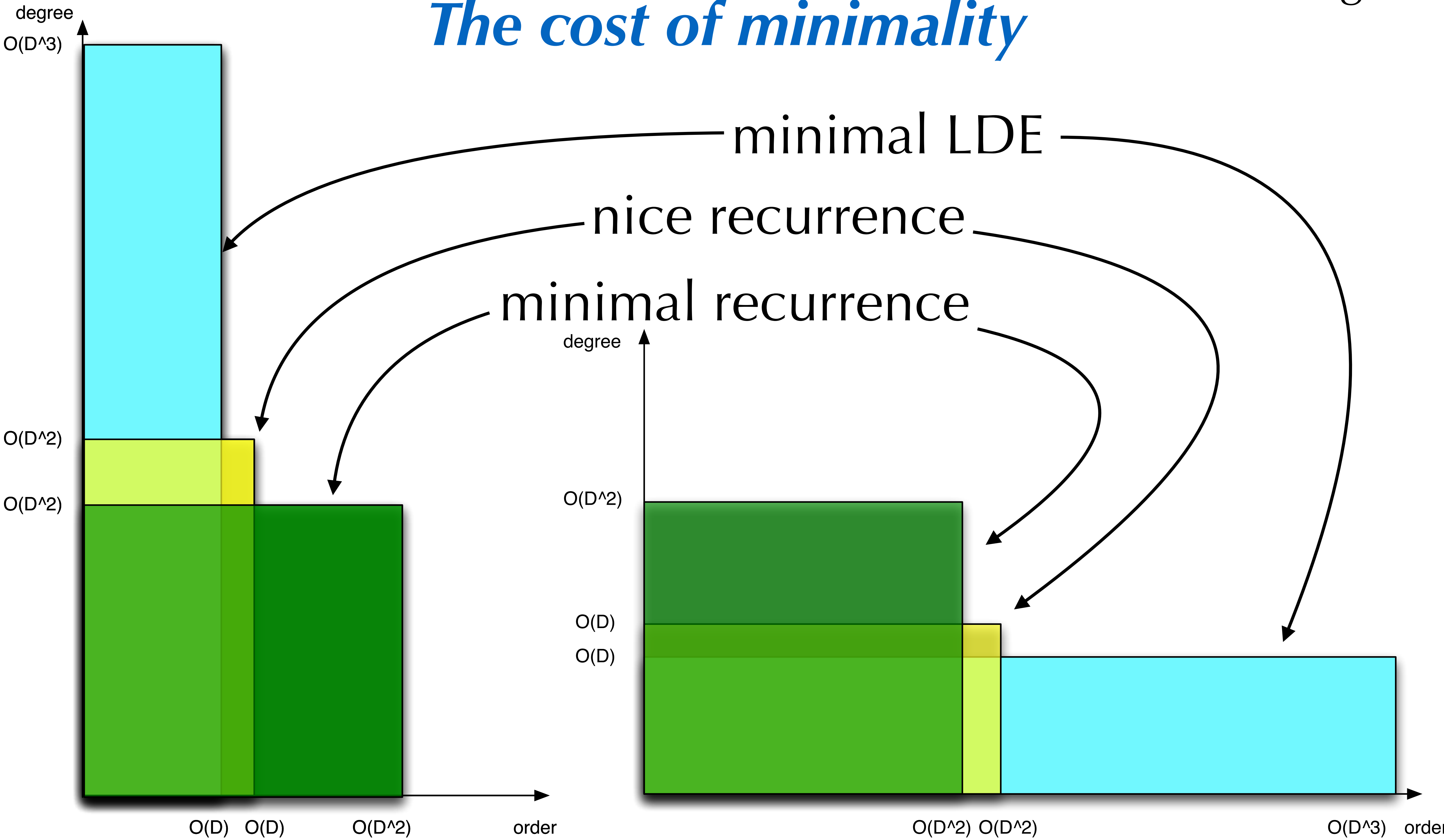
→ a LDE by linear algebra

[Abel1827;Cockle1861;Harley1862;Tannery1875]

Order-Degree Curve

The cost of minimality

D=deg P



differential equations corresponding recurrences

C. Computing Linear Differential Equations (Efficiently)

II. Creative Telescoping

Examples

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \sum_{j=0}^{k} \binom{k}{j}^3$$

$$\sum_{j,k} (-1)^{j+k} \binom{j+k}{k+l} \binom{r}{j} \binom{n}{k} \binom{s+n-j-k}{m-j} = (-1)^l \binom{n+r}{n+l} \binom{s-r}{m-n-l}$$

$$\int_0^{+\infty} x J_1(ax) I_1(ax) Y_0(x) K_0(x)\, dx = -\frac{\ln(1-a^4)}{2\pi a^2}$$

$$\frac{1}{2\pi i} \oint \frac{(1+2xy+4y^2)\exp\left(\frac{4x^2y^2}{1+4y^2}\right)}{y^{n+1}(1+4y^2)^{\frac{3}{2}}}\, dy = \frac{H_n(x)}{\lfloor n/2 \rfloor !}$$

$$\sum_{k=0}^{n} \frac{q^{k^2}}{(q;q)_k (q;q)_{n-k}} = \sum_{k=-n}^{n} \frac{(-1)^k q^{(5k^2-k)/2}}{(q;q)_{n-k}(q;q)_{n+k}}$$

Aims:
1. Prove them automatically
2. Find the rhs given the lhs

Note: at least one free variable

First: **find a LDE** (or LRE)

Creative telescoping

$$\mathsf{I}(x) = \int \mathsf{f}(x,t)\,dt = ? \quad \text{or} \quad \mathsf{U}(n) = \sum_k \mathsf{u}(n,k) = ?$$

Input: equations (differential for *f* or recurrence for *u*).
Output: equations for the sum or the integral.

Ex.: $U_n := \sum_k \binom{n}{k}$

$$U_{n+1} - 2U_n = \sum_k \binom{n+1}{k} - 2\binom{n}{k} = \sum_k \underbrace{\binom{n+1}{k} - \binom{n+1}{k+1}}_{\text{telescopes}} + \underbrace{\binom{n}{k+1} - \binom{n}{k}}_{\text{telescopes}}$$

Aim: find $A(n,S_n)$ and $B(n,k,S_n,S_k)$ such that

Def: $\Delta_k := S_k - 1$.

$$(A(n,S_n) + \Delta_k B(n,k,S_n,S_k)) \cdot u(n,k) = 0,$$

certificate

then the sum telescopes, leading to $A(n,S_n) \cdot U(n) = 0$.

Integrals: differentiate under the $\int$ sign and integrate by parts.

Telescoping Ideal

$$\mathrm{T_t(f)} := \Big(\operatorname{Ann}\mathrm{f} + \underbrace{\partial_t \mathbb{Q}(\boldsymbol{x},t)\langle \boldsymbol{\partial_x}, \partial_t\rangle}_{\substack{\text{int. by parts} \\ \text{(certificate)}}}\Big) \cap \underbrace{\mathbb{Q}(\boldsymbol{x})\langle \boldsymbol{\partial_x}\rangle}_{\text{diff. under } \int} .$$

First generation of algorithms relying on holonomy

Restrict int. by parts to $\mathbb{Q}(\boldsymbol{x})\langle \boldsymbol{\partial_x}, \partial_{\mathrm{t}}\rangle$ and use elimination.

Second generation: faster using better certificates & algorithms

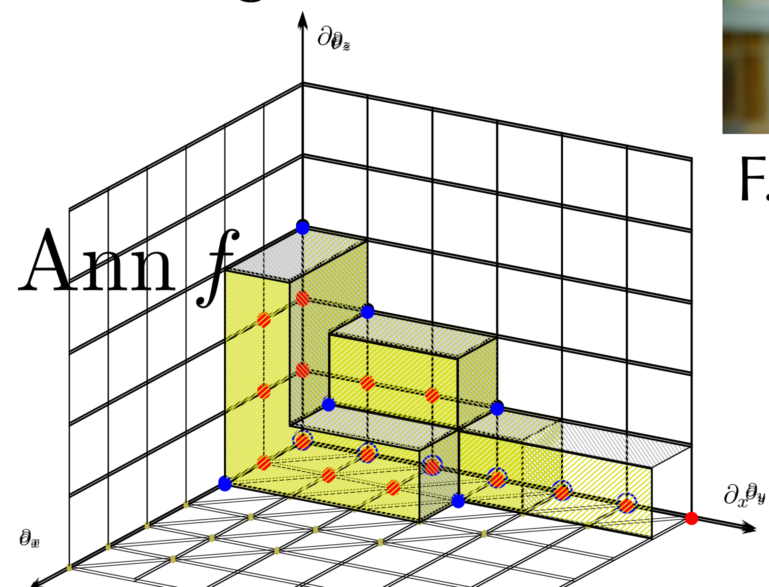
Hypergeometric summation: dim=1 + param. Gosper.

Undetermined coefficients in finite dim, Ore algebras & GB.

Idem in infinite dim.

F. Chyzak

$$\sum_k c_k(x)\partial_x^k - \partial_t \sum_{i,j\in\mathcal{S}} a_{i,j}(x,t)\partial_x^i\partial_t^j \in \operatorname{Ann} f$$



[Zeilberger *et alii* 90,91,92;Chyzak00;Chyzak-Kauers-S.09]

C. Computing Linear Differential Equations (Efficiently)

III. 3rd Generation Creative Telescoping

Certificates are big

$$C_n := \sum_{r,s} \underbrace{(-1)^{n+r+s} \binom{n}{r} \binom{n}{s} \binom{n+s}{s} \binom{n+r}{r} \binom{2n-r-s}{n}}_{f_{n,r,s}}$$

$(n+2)^3 C_{n+2} - 2(2n+3)(3n^2+9n+7)C_{n+1} - (4n+3)(4n+4)(4n+5)C_n =$ 180 kB $\simeq$ 2 pages

$$I(z) = \oint \frac{(1+t_3)^2 dt_1 dt_2 dt_3}{t_1 t_2 t_3 (1+t_3(1+t_1))(1+t_3(1+t_2)) + z(1+t_1)(1+t_2)(1+t_3)^4}$$

$z^2(4z+1)(16z-1)I'''(z) + 3z(128z^2+18z-1)I''(z) + (444z^2+40z-1)I'(z) + 2(30z+1)I(z) =$ 1 080 kB

$\simeq$ 12 pages

3rd-generation algorithms: avoid computing the certificate

Periods

$$I(t) = \oint \underbrace{\frac{P(t,\underline{x})}{Q^m(t,\underline{x})}}_{\in \mathbb{Q}(t,\underline{x})} d\underline{x}$$

Q square-free

Int. over a cycle where Q≠0.

$\underline{x} = (x_1, \ldots, x_n)$

$N := \deg_{\underline{x}} Q, \quad d_t := \max(\deg_t Q, \deg_t P)$

$\deg_x P$ not too big

Thm. A linear differential equation for I(t) can be computed in $O(e^{3n}N^{8n}d_t)$ operations in $\mathbb{Q}$.
It has order $\leq N^n$ and degree $O(e^nN^{3n}d_t)$.

tight

Note: generically, the certificate has at least $N^{n^2/2}$ monomials.

Applications to diagonals
& to multiple binomial sums.

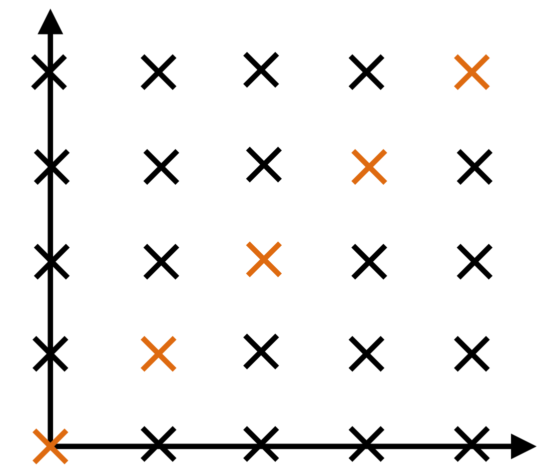
[Picard1902;Dwork62;Griffiths69;Christol85;Bostan-Lairez-Salvy13;Lairez16]

Diagonals

in this talk

If $F(\boldsymbol{z}) = \dfrac{G(\boldsymbol{z})}{H(\boldsymbol{z})}$ is a multivariate rational function with Taylor expansion

$$F(\boldsymbol{z}) = \sum_{\boldsymbol{i} \in \mathbb{N}^n} c_{\boldsymbol{i}} \boldsymbol{z}^{\boldsymbol{i}},$$



its **diagonal** is $\Delta F(t) = \sum_{k \in \mathbb{N}} c_{k,k,\dots,k} t^k.$

$\binom{2k}{k}$: $\qquad \dfrac{1}{1-x-y} = 1 + x + y + 2xy + x^2 + y^2 + \cdots + 6x^2y^2 + \cdots$

$\dfrac{1}{k+1}\binom{2k}{k}$: $\qquad \dfrac{1-2x}{(1-x-y)(1-x)} = 1 + y + 1xy - x^2 + y^2 + \cdots + 2x^2y^2 + \cdots$

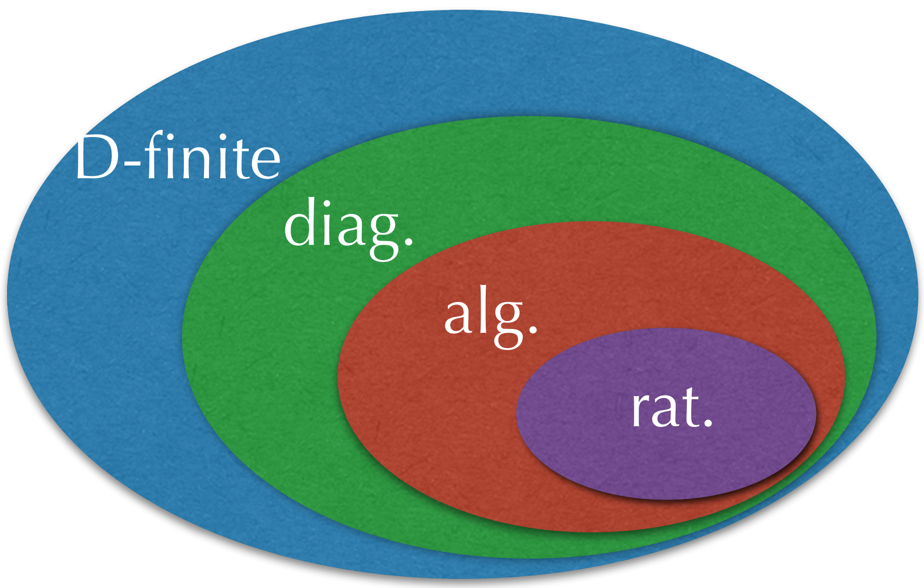
Apéry's a_k : $\qquad \dfrac{1}{1-t(1+x)(1+y)(1+z)(1+y+z+yz+xyz)} = 1 + \cdots + 5xyzt + \cdots$

Christol's conjecture: All differentially finite power series with integer coefficients and radius of convergence in (0,∞) are diagonals.

Diagonals are Differentially Finite

[Christol84,Lipshitz88]

$$\Delta F(z_1,\dots,z_d) = \left(\frac{1}{2\pi i}\right)^{d-1} \oint F\left(\frac{t}{z_2\cdots z_d}, z_2,\dots,z_d\right)\frac{dz_2}{z_2}\cdots\frac{dz_d}{z_d}$$



Univariate power series

Thm. If F has degree *d* in *n* variables, ΔF satisfies a LDE with order $\approx d^n$, coeffs of degree $d^{O(n)}$.

+ algo in $\tilde{O}(d^{8n})$ ops.

[Bostan-Lairez-S.13;Lairez16]

Multiple Binomial Sums

Ex. $S_n = \sum_{r\geq 0}\sum_{s\geq 0}(-1)^{n+r+s}\binom{n}{r}\binom{n}{s}\binom{n+s}{s}\binom{n+r}{r}\binom{2n-r-s}{n}$

Thm. Diagonals ≡ binomial sums with 1 free index.

defined properly

```
> BinomSums[sumtores](S,u): (…)
```

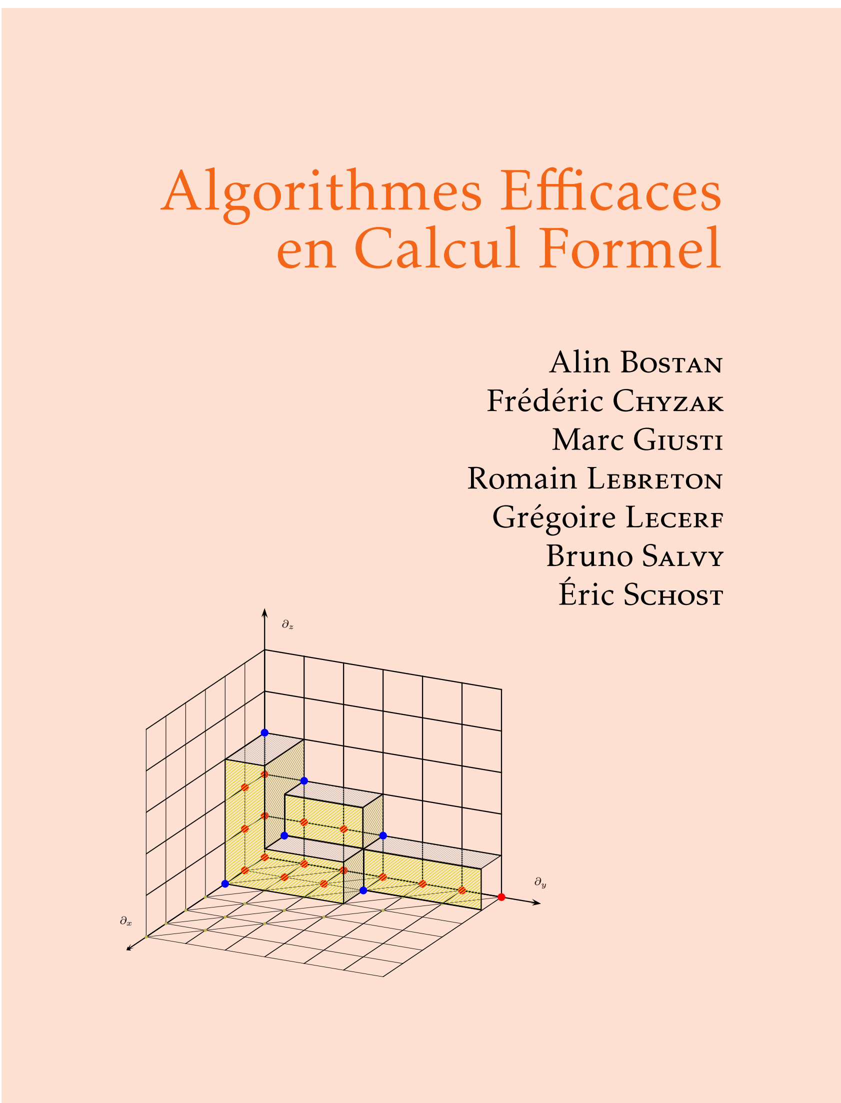
$$\frac{1}{1-t(1+u_1)(1+u_2)(1-u_1u_3)(1-u_2u_3)}$$

has for diagonal the generating function of S_n

→LDE→LRE

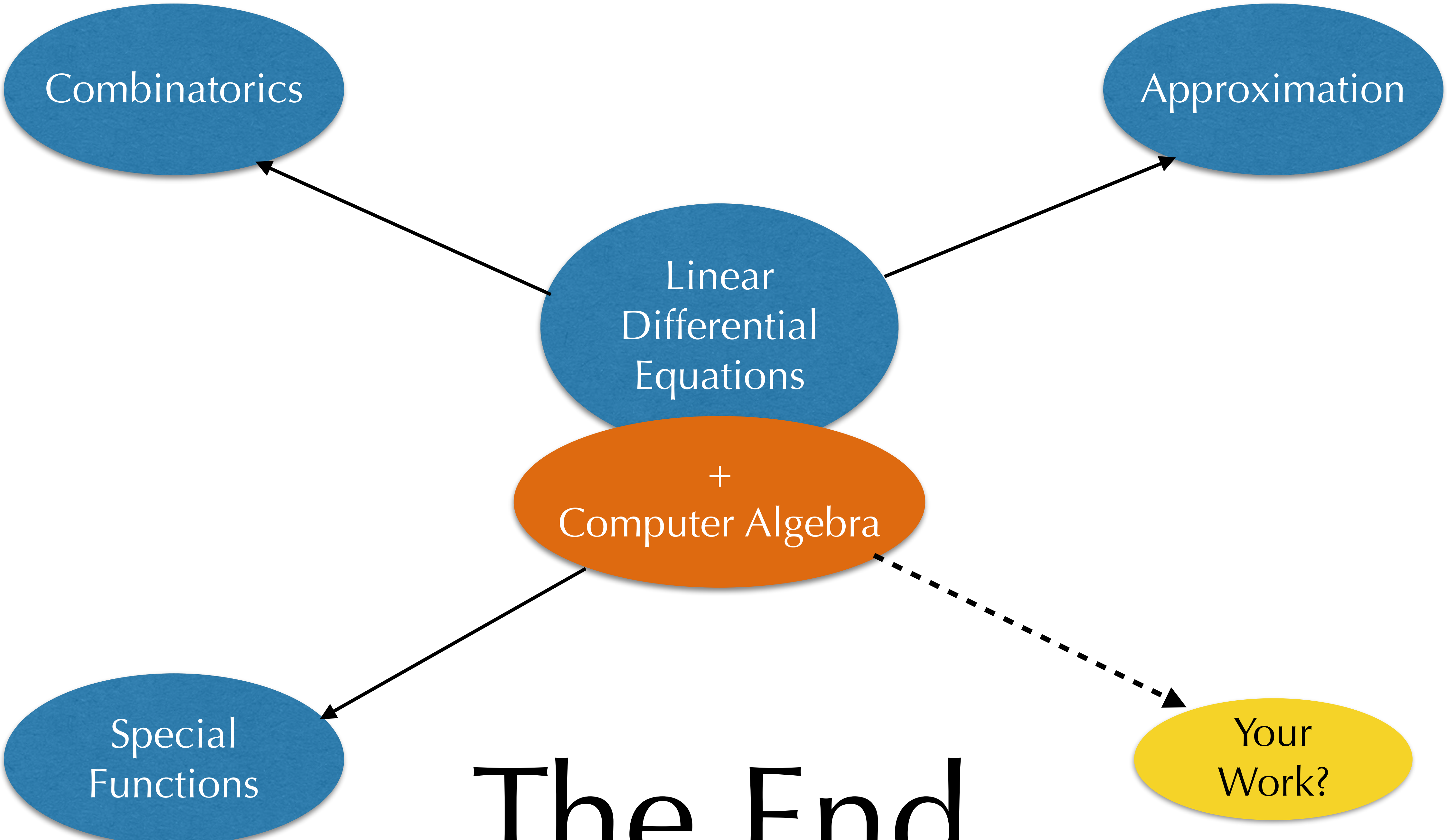
[Bostan-Lairez-S.17]

(Non-)Commercial



New book (≈700p.), based on our course.
Freely available from our web pages, forever.
Paper version before the end of 2017.

Conclusion



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