Fast Computation of Power Series Solutions of Systems of Differential Equations

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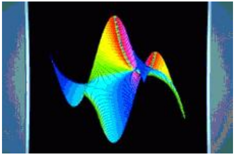

Joint work with
A. Bostan, F. Chyzak, F. Ollivier, É. Schost, A. Sedoglavic

Series Expansions to High Order

Motivations:

- combinatorics (generating functions of ordered structures);
- numerical analysis (e.g., Padé approximants);
- cryptography (e.g., morphisms between curves).

Non-linear differential systems can be reduced to linear ones.

Problem

Input: linear differential equation/system of order r with power series coefficients, positive integer $N \gg r$.
Output: N first terms of one solution or of a basis of solutions.

Complexity measure: number of arithmetic operations.

Aim: quasi-linear wrt N, good wrt r.

$\mathrm{M}(N) = \mathcal{O}(N \log N)$ polynomial multiplication [Schönhage-Strassen71];
$\mathrm{MM}(r, N) = \mathcal{O}(r^2 \mathrm{M}(N) + r^\omega N)$ polynomial matrix product [Bostan-Schost05].

Results

input /output	power series coefficients	polynomial coefficients	constant coefficients	output size
equation /basis	$\mathcal{O}(r^2 N \log N)$ $\star$	$\mathcal{O}(dr^2 N)$	$\mathcal{O}(rN)$ $\star$	rN
equation /1 soln	$\mathcal{O}(rN \log^2 N)$ $\star$	$\mathcal{O}(drN)$	$\mathcal{O}(\log rN)$ $\star$	N
system /basis	$\mathcal{O}(r^2 N \log N)$ $\star$	$\mathcal{O}(dr^\omega N)$	$\mathcal{O}(r^2 \log rN)$ $\star$	$r^2 N$
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Previous results:

- $\mathcal{O}(r^2 N^2)$ undetermined coefficients;
- $\mathcal{O}(r^r N \log N)$ [Brent-Kung78];
- $\mathcal{O}(r^2 N \log^2 N)$ [van der Hoeven02].

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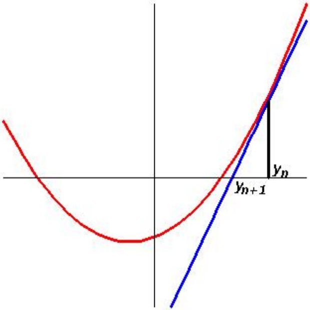
Basic Idea: Newton Iteration has Good Complexity

To solve $\phi(y) = 0$, iterate

$$y_{n+1} = y_n - u_{n+1}, \quad \phi'(y_n)u_{n+1} = \phi(y_n).$$



Example 1: $\phi(y) = 3y^2 + y - 1$

$$
\begin{aligned}
y_0 &= 0.5\\
y_1 &= 0.4375000000000000000000000\\
y_2 &= 0.4342672413793103448275862\\
y_3 &= 0.4342585459735762722344745\\
y_4 &= 0.4342585459106648821898300
\end{aligned}
$$

The cost is dominated by the last iteration.

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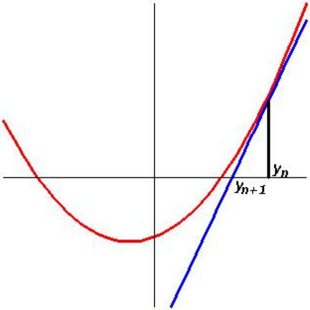

Example 2: $\phi(y) = ty^2 + y - 1$

$$y_0 = 0$$
$$y_1 = 1$$
$$y_2 = 1 - t + 2t^2 - 4t^3 + 8t^4 - 16t^5 + 32t^6 - 64t^7 + \dots$$
$$y_3 = 1 - t + 2t^2 - 5t^3 + 14t^4 - 42t^5 + 132t^6 - 428t^7 + \dots$$

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Newton for Power Series

$$y_{n+1} = y_n - u_{n+1}, \quad \phi'(y_n)u_{n+1} = \phi(y_n).$$

- Reciprocal [Sieveking72, Kung74]: $\phi(y) = a(t) - 1/y$;

$$y_{n+1} = y_n - u_{n+1}, \quad u_{n+1} = y_n \times (y_n \times a(t) - 1).$$

$\rightarrow$ Same complexity $\mathcal{O}(\mathsf{M}(N))$ as polynomial multiplication.

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- Exponential [Brent75]: $\phi(y) = a(t) - \log y$.

$$y_{n+1} = y_n - u_{n+1}, \quad u_{n+1} = y_n \times (\log y_n - a(t)).$$

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- 1st order LODE [Brent & Kung 78]: $y'(t) - a(t)y(t) = b(t)$
    - Homogeneous Case ($b = 0$): $y(t) = y_0(t) := \int \exp(a(t))$;
    - Non-homogeneous Case: variation of constants, $y(t) = y_0(t) \int b(t)/y_0(t)$.

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Natural idea: LDE of order r equivalent to 1st order matrix equation:

$$Y'(t) - A(t)Y(t) = B(t),$$

but $\int \exp A(t)$ is not a solution of $Y'(t) = A(t)Y(t)$.

Newton Iteration for Operators

Newton iteration for $\phi(Y)=0$:

$$Y_{k+1}=Y_k-U_{k+1},\qquad \underbrace{D\phi|_{Y_k}}_{\text{differential}}\cdot U_{k+1}=\phi(Y_k).$$

Special cases:

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Special cases:

- Matrix inversion [Schulz33]: $\phi(Y) = AY - I$, $D\phi \cdot U = AU$.
  $\rightarrow$ equation $AU_{k+1} = AY_k - I$; $\boxed{U_{k+1} = Y_k(AY_k - I).}$
  Complexity: $\mathrm{MM}(r, N) = \mathcal{O}(r^2\mathrm{M}(N) + r^\omega N)$.

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  Complexity: $\mathrm{MM}(r, N) = \mathcal{O}(r^2\mathrm{M}(N) + r^\omega N)$.
- Our problem: $\phi(Y) = Y' - AY$. Note that $D\phi = \phi$!

New Algorithm and its Complexity

To iterate $Y_{k+1} = Y_k - U_{k+1}$, we need to solve

$$U'_{k+1} - A(t)U_{k+1} = Y'_k - A(t)Y_k.$$

Variation of constants:

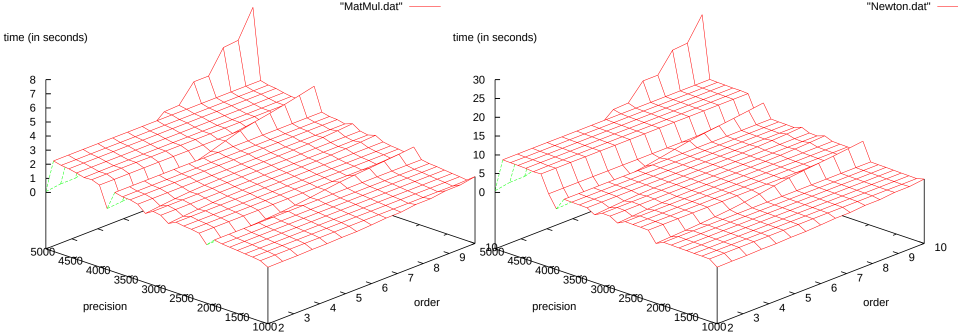
$$U_{k+1} = Y_k Z_{k+1},$$

$$Y_k Z'_{k+1} + \underbrace{Y'_k Z_{k+1} - A(t)Y_k Z_{k+1}}_{0} = Y'_k - A(t)Y_k,$$

$$\boxed{Y_{k+1} = Y_k - Y_k \int Y_k^{-1}(Y'_k - A(t)Y_k) \bmod t^{2^{k+1}}, \qquad \det Y_0 \neq 0}$$

- Quadratic convergence;
- a whole basis of solutions is computed;
- its inverse is computed simultaneously;
- complexity $\mathcal{O}(r^2\mathsf{M}(N) + r^\omega N)$;
- good for $r = 1$ ([Hanrot-Zimmermann02] for $\exp(a(t))$).

Experimental results



Polynomial matrix multiplication vs. solving $Y' = AY$ by our method.

Non-Linear Differential Equations by Linearization

Newton again: $y_{k+1} = y_k - u_{k+1}, \quad \underbrace{D\phi|_{y_k}}_{\text{differential}} \cdot u_{k+1} = \phi(y_k).$

A 1st order example from number theory (related to $\wp$):

$$\phi(y) = (x^3 + ax + b)y'^2 - y^3 - Ay - B = 0.$$

Differential:

$$\phi(y_k+u_{k+1})-\phi(y_k) = \underbrace{(x^3 + ax + b)2y_k' u_{k+1}' - (3y_k^2 - A)u_{k+1}}_{D\phi|_{y_k} \cdot u_{k+1}} + \cdots .$$

$\rightarrow$ Linear differential equation with power series coefficients

$\rightarrow$ fast algorithm for isogenies between elliptic curves [BoMoSaSc06].

Higher order $\rightarrow$ a linear differential system at each step.

Final Comments

Have a look at the paper for

- constant coefficients
  (including nice and fast algorithm for $\exp(tA)$);
- a divide-and-conquer algorithm for only one solution (instead of a basis) in the general case.

Still to be saved:

- a factor r for (1 equation, a basis of solutions);
- a factor $\log N$ for only one solution.