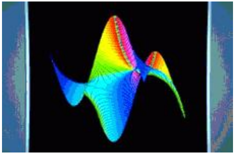
Automatic Proofs of Identities

Bruno Salvy
Bruno.Salvy@inria.fr

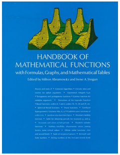
Algorithms Project, Inria

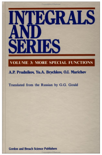


Vancouver, May 17, 2011

I Introduction

Dynamic Dictionary of Mathematical Functions





Aim of the project

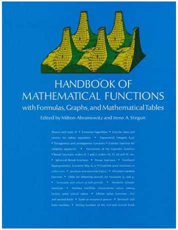
DDMF = Mathematical Handbooks + Computer Algebra + Web

1. Develop and use computer algebra algorithms to generate the formulas;
2. Provide web-like interaction with the document and the computation.

http://ddmf.msr-inria.inria.fr/

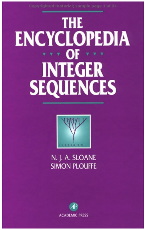
Equations Are a Good Data Structure

- Classical:
  polynomials represent their roots better than radicals.
  Algorithms: Euclidean division and algorithm, Gröbner bases.
- Recent:
  same for linear differential or recurrence equations.
  Algorithms: non-commutative analogues.



About 25% of Sloane's encyclopedia,
60% of Abramowitz & Stegun.

$\boxed{\text{eqn+ini. cond.=data structure}}$



Examples of Identities

$$\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\sum_{j=0}^{k}\binom{k}{j}^3 \qquad \text{[Strehl92]}$$

$$\int_0^{+\infty} xJ_1(ax)I_1(ax)Y_0(x)K_0(x)\,dx=-\frac{\ln(1-a^4)}{2\pi a^2} \qquad \text{[GlMo94]}$$

$$\frac{1}{2\pi i}\oint\frac{(1+2xy+4y^2)\exp\left(\frac{4x^2y^2}{1+4y^2}\right)}{y^{n+1}(1+4y^2)^{\frac{3}{2}}}\,dy=\frac{H_n(x)}{\lfloor n/2\rfloor!} \qquad \text{[Doetsch30]}$$

$$\sum_{k=0}^{n}\frac{q^{k^2}}{(q;q)_k(q;q)_{n-k}}=\sum_{k=-n}^{n}\frac{(-1)^kq^{(5k^2-k)/2}}{(q;q)_{n-k}(q;q)_{n+k}} \qquad \text{[Andrews74]}$$

$$\sum_{j=0}^{n}\sum_{i=0}^{n-j}\frac{q^{(i+j)^2+j^2}}{(q;q)_{n-i-j}(q;q)_i(q;q)_j}=\sum_{k=-n}^{n}\frac{(-1)^kq^{7/2k^2+1/2k}}{(q;q)_{n+k}(q;q)_{n-k}} \qquad \text{[Paule85].}$$

More Identities

$$\sum_{k=0}^{n} \binom{n}{k} i(k+i)^{k-1}(n-k+j)^{n-k} = (n+i+j)^n \quad \text{[Abel1826]}$$

$$\sum_{k=0}^{n} (-1)^{m-k} k! \binom{n-k}{m-k} \left\{ {n+1 \atop k+1} \right\} = \left\langle {n \atop m} \right\rangle, \quad \text{[Frobenius1910}$$

$$\sum_{k=0}^{m} \binom{m}{k} B_{n+k} = (-1)^{m+n} \sum_{k=0}^{n} \binom{n}{k} B_{m+k}, \quad \text{[Gessel03]}$$

$$\int_0^\infty x^{k-1} \zeta(n, \alpha + \beta x)\, dx = \beta^{-k} B(k, n-k) \zeta(n-k, \alpha),$$

$$\int_0^\infty x^{\alpha-1} \operatorname{Li}_n(-xy)\, dx = \frac{\pi(-\alpha)^n y^{-\alpha}}{\sin(\alpha\pi)},$$

$$\int_0^\infty x^{s-1} \exp(xy) \Gamma(a, xy)\, dx = \frac{\pi y^{-s}}{\sin((a+s)\pi)} \frac{\Gamma(s)}{\Gamma(1-a)}$$

Computer Algebra Algorithms

Aim

- Prove these identities automatically (fast?);
- Compute the rhs given the lhs;
- Explain why these identities exist.

Examples:

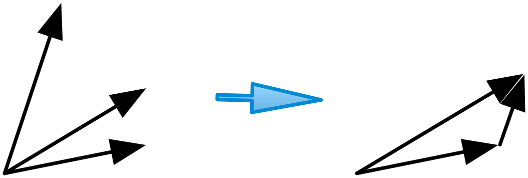
- 1st slide: Zeilberger's algorithm and variants;
- 2nd slide (1st 3): Majewicz, Kauers, Chen & Sun;
- last 3: recent generalization of previous ones (with Chyzak & Kauers).

Ideas

Confinement in finite dimension + Creative telescoping.

II Confinement in Finite Dimension

Confinement Provokes Identities



Obvious

$k+1$ vectors in dimension $k \rightarrow$ an identity.

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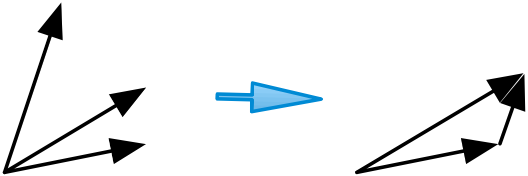

Obvious

$k + 1$ vectors in dimension $k \to$ an identity.

Idea: confine a function and all its derivatives.

First Algorithmic Proof: $\sin^2 + \cos^2 = 1$

```
> series(sin(x)^2+cos(x)^2-1,x,4);
```

$$O(x^4)$$

Why is this a proof?

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Why is this a proof?

1. sin and cos satisfy a 2nd order LDE: $y'' + y = 0$;
2. their squares (and their sum) satisfy a 3rd order LDE;
3. the constant 1 satisfies a 1st order LDE: $y' = 0$;
4. $\rightarrow \sin^2 + \cos^2 - 1$ satisfies a LDE of order at most 4;
5. Cauchy's theorem concludes.

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What about $\sin' = \cos$?

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Second algorithmic proof (same idea): $F_n^2 - F_{n+1}F_{n-1} = (-1)^{n+1}$

```
> for n to 5 do
   fibonacci(n)^2-fibonacci(n+1)*fibonacci(n-1)+(-1)^n od;
```

Third Proof: Contiguity of Hypergeometric Series

$$F(a,b;c;z)=\sum_{n=0}^{\infty}\underbrace{\frac{(a)_n(b)_n}{(c)_n n!}}_{u_{a,n}}z^n,\qquad (x)_n:=x(x+1)\cdots(x+n-1).$$

$$\frac{u_{a,n+1}}{u_{a,n}}=\frac{(a+n)(b+n)}{(c+n)(n+1)}\rightarrow\ z(1-z)F''+(c-(a+b+1)z)F'-abF=0,$$

$$\frac{u_{a+1,n}}{u_{a,n}}=\frac{n}{a}+1\rightarrow S_aF:=F(a+1,b;c;z)=\frac{z}{a}F'+F.$$

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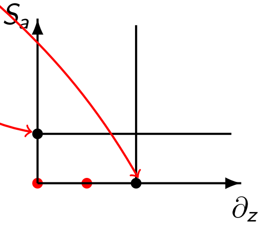

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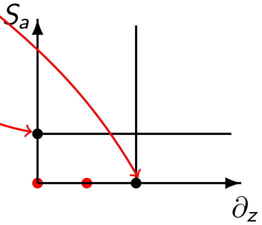
Gauss 1812: contiguity relation.

dim=2 $\Rightarrow S_a^2F, S_aF, F$ linearly dependent:

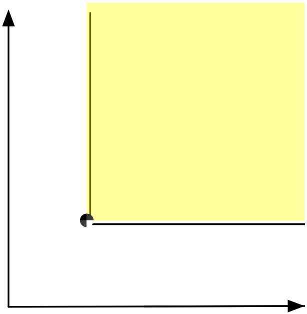
(Coordinates in $\mathbb{Q}(a,b,c,z)$.)



$$(a+1)(z-1)S_a^2F+((b-a-1)z+2-c+2a)S_aF+(c-a-1)F=0.$$

Gröbner Basis: Euclidean Division in Several Variables

1. Monomial ordering: order on $\mathbb{N}^k$, compatible with $+$, 0 minimal.



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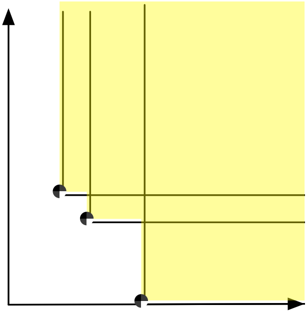

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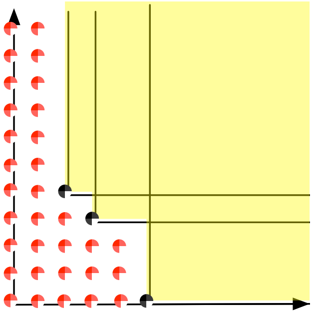

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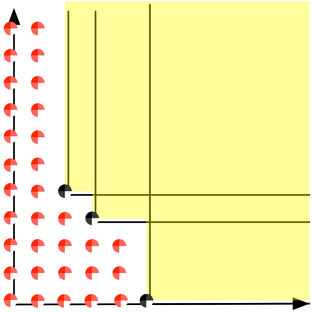

$\rightarrow$ An access to (finite dimensional) vector spaces

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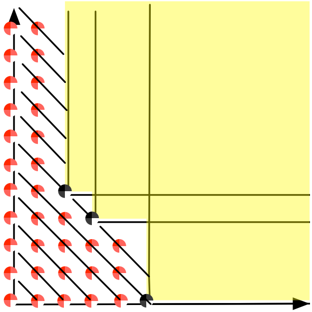

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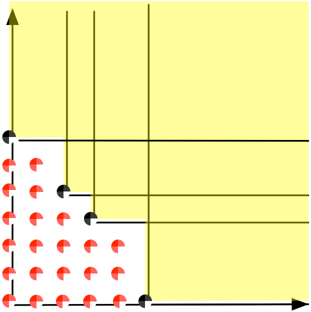
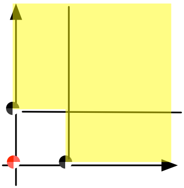
6. D-finiteness: dim $= 0$.



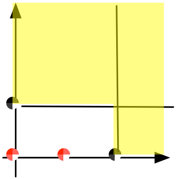
$\rightarrow$ An access to (finite dimensional) vector spaces

Examples

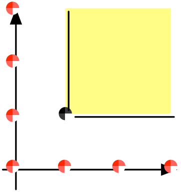
Binomial coeffs $\binom{n}{k}$ wrt S_n, S_k
Hypergeometric sequences



Bessel $J_\nu(x)$ wrt S_ν, ∂_x
Orthogonal pols wrt S_n, ∂_x

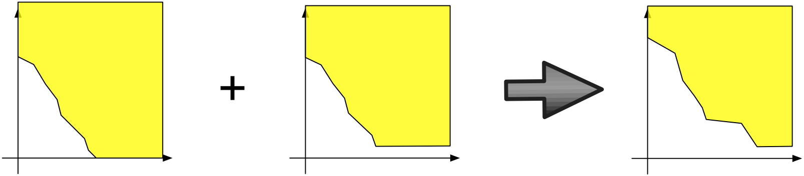


Stirling nbs wrt S_n, S_k



Abel type wrt S_m, S_r, S_k, S_s
$\mathrm{hgm}(m,k)(k+r)^k(m-k+s)^{m-k}\frac{r}{k+r}$
dim $= 2$ in space of dim 4.

Closure Properties



Proposition

$$\dim \operatorname{ann}(f+g) \le \max(\dim \operatorname{ann} f, \dim \operatorname{ann} g),$$
$$\dim \operatorname{ann}(fg) \le \dim \operatorname{ann} f + \dim \operatorname{ann} g,$$
$$\dim \operatorname{ann} \partial f \le \dim \operatorname{ann} f.$$

Algorithms by linear algebra.

Fourth Algorithmic Proof: Mehler's Identity for Hermite Polynomials

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\frac{u^n}{n!} = \frac{\exp\left(\frac{4u(xy-u(x^2+y^2))}{1-4u^2}\right)}{\sqrt{1-4u^2}}$$

1. Definition of Hermite polynomials (D-finite over $\mathbb{Q}(x)$): recurrence of order 2;
2. Product by linear algebra: $H_{n+k}(x)H_{n+k}(y)/(n+k)!, k \in \mathbb{N}$ generated over $\mathbb{Q}(x, n)$ by

   $$\frac{H_n(x)H_n(y)}{n!}, \frac{H_{n+1}(x)H_n(y)}{n!}, \frac{H_n(x)H_{n+1}(y)}{n!}, \frac{H_{n+1}(x)H_{n+1}(y)}{n!}$$

   $\rightarrow$ recurrence of order at most 4;
3. Translate into differential equation.

Fifth Algorithmic Proof: Moments of K_0^n

$$c_{n,k} = \int_0^{+\infty} t^k K_0(t)^n \, dt$$

1. Bessel K_0 satisfies a LDE of order 2;
2. K_0^n satisfies one of order $n+1$;
3. multiply by t^k and translate into a recurrence.

Theorem (BorweinSalvy2008)

$$(k+1)^{n+1} c_{n,k} + \sum_{\substack{2 \le j < n \\ j \text{ even}}} P_{n,j}(k) c_{n,k+j} = 0 \quad \textit{with} \quad \deg P_{n,j} \le n+1-j.$$

Also, a fast algorithm.
Initially a conjecture by Bailey, Borwein^2, Crandall.

III Creative Telescoping

Summation by Creative Telescoping

$$I_n := \sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

IF one knows Pascal's triangle:

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} = 2\binom{n}{k} + \binom{n}{k-1} - \binom{n}{k},$$

then summing over k gives

$$I_{n+1} = 2I_n.$$

The initial condition $I_0 = 1$ concludes the proof.

Creative Telescoping (Zeilberger 90)

$$F_n = \sum_k u_{n,k} =?$$

IF one knows $A(n, S_n)$ and $B(n, k, S_n, S_k)$ such that

$$(A(n, S_n) + \Delta_k B(n, k, S_n, S_k)) \cdot u_{n,k} = 0,$$

then the sum "telescopes", leading to $A(n, S_n) \cdot F_n = 0$.

Creative Telescoping (Zeilberger 90)

$$I(x) = \int_\Omega u(x,y)\,dy = ?$$

IF one knows $A(x,\partial_x)$ and $B(x,y,\partial_x,\partial_y)$ such that

$$(A(x,\partial_x) + \partial_y B(x,y,\partial_x,\partial_y)) \cdot u(x,y) = 0,$$

then the integral "telescopes", leading to $A(x,\partial_x) \cdot I(x) = 0.$

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IF one knows $A(x,\partial_x)$ and $B(x,y,\partial_x,\partial_y)$ such that

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then the integral "telescopes", leading to $A(x,\partial_x) \cdot I(x) = 0$.

Then I come along and try differentating under the integral sign, and often it worked. So I got a great reputation for doing integrals.

Richard P. Feynman 1985

Creative telescoping="differentiation" under integral+"integration" by parts

Diff. under $\int$ + Integration by Parts $\to$ Algorithm?

Ex.: $\displaystyle\int_0^1 \frac{\cos zt}{\sqrt{1-t^2}}\,dt = \frac{\pi}{2}J_0(z), \quad (\underbrace{zJ_0'' + J_0' + zJ_0}_{A(z,\partial_z)\cdot J_0} = 0,\ J_0(0) = 1).$

Diff. under $\int$ + Integration by Parts → Algorithm?

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$$I(z) = \int_0^1 \frac{\cos zt}{\sqrt{1-t^2}}\,dt, \quad I'(z) = \int_0^1 -t\frac{\sin zt}{\sqrt{1-t^2}}\,dt,$$

$$I''(z) = \int_0^1 -t^2\frac{\cos zt}{\sqrt{1-t^2}}\,dt = -I(z) + \int_0^1 \sqrt{1-t^2}\cos zt\,dt,$$

$$I''(z) + I(z) = \underbrace{\left[\sqrt{1-t^2}\frac{\sin zt}{z}\right]_0^1}_{0} + \int_0^1 \frac{t}{\sqrt{1-t^2}}\frac{\sin zt}{z}\,dt = -\frac{I'(z)}{z}.$$

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$$\operatorname{ann}\frac{\cos zt}{\sqrt{1-t^2}} \ni \underbrace{A(z,\partial_z)}_{\text{no } t,\partial_t} - \partial_t \underbrace{\frac{t^2-1}{t}\partial_z}_{\text{anything}}$$

Diff. under $\int$ + Integration by Parts → Algorithm?

Ex.: $$\int_0^1 \frac{\cos zt}{\sqrt{1-t^2}}\,dt = \frac{\pi}{2} J_0(z), \quad (\underbrace{zJ_0'' + J_0' + zJ_0}_{A(z,\partial_z)\cdot J_0} = 0,\ J_0(0) = 1).$$

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Creative Telescoping

Input: generators of (a subideal of) ann f;
Output: A, B such that $A - \partial_t B \in \operatorname{ann} f$, A free of t, ∂_t.
Algorithm: sometimes. (Why would they exist?)

Telescoping of $\mathcal{I}$ wrt t:

$$T_t(\mathcal{I}) := (\mathcal{I} + \partial_t \mathbb{Q}(z,t)\langle \partial_z, \partial_t\rangle) \cap \mathbb{Q}(z)\langle \partial_z\rangle.$$

Example: Pascal's Triangle Again

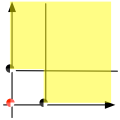
$$(S_nS_k - S_k - 1)\cdot\binom{n}{k} = 0 = (\underbrace{S_n - 2}_{\text{no } k,\, S_k} + (S_k - 1)(S_n - 1))\cdot\binom{n}{k}.$$

Sum over $k \Rightarrow (S_n - 2)\sum_k \binom{n}{k} = 0.$

Example: Pascal's Triangle Again

$$(S_nS_k - S_k - 1)\cdot\binom{n}{k} = 0 = (S_n - 2 + (S_k - 1)(S_n - 1))\cdot\binom{n}{k}.$$

Reduce all monomials of degree $\leq s = 2$:



$$1 \to 1, \quad S_n \to \frac{n+1}{n+1-k}1, \quad S_k \to \frac{n-k}{k+1}1$$

$$S_n^2 \to \frac{(n+2)(n+1)}{(n+2-k)(n+1-k)}1, \quad S_k^2 \to \frac{(n-k-1)(n-k)}{(k+2)(k+1)}1,$$

$$S_nS_k \to \frac{n+1}{k+1}1.$$

Common denominator: $D_2 = (k+1)(k+2)(n+1-k)(n+2-k)$.

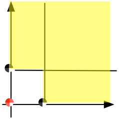
$D_2, D_2S_n, D_2S_k, D_2S_n^2, D_2S_k^2, D_2S_nS_k$ confined in

$$\mathrm{Vect}_{\mathbb{Q}(n)}(1, k1, k^21, k^31, k^41).$$

Example: Pascal's Triangle Again

$$(S_nS_k - S_k - 1)\cdot\binom{n}{k} = 0 = (S_n - 2 + (S_k-1)(S_n-1))\cdot\binom{n}{k}.$$

Reduce all monomials of degree $\leq s = 2$:



$$1 \to 1, \quad S_n \to \frac{n+1}{n+1-k}1, \quad S_k \to \frac{n-k}{k+1}1$$

$$S_n^2 \to \frac{(n+2)(n+1)}{(n+2-k)(n+1-k)}1, \quad S_k^2 \to \frac{(n-k-1)(n-k)}{(k+2)(k+1)}1,$$

$$S_nS_k \to \frac{n+1}{k+1}1.$$

Common denominator: $D_2 = (k+1)(k+2)(n+1-k)(n+2-k)$.

$D_2, D_2S_n, D_2S_k, D_2S_n^2, D_2S_k^2, D_2S_nS_k$ confined in

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This has to happen for some degree: $\deg D_s = O(s)$.

Polynomial Growth

Definition (Polynomial Growth p)

There exists a sequence of polynomials P_s, s.t. for all $(a_1,\dots,a_k)$ with $a_1+\cdots+a_k\le s$, $P_s\partial_1^{a_1}\cdots\partial_k^{a_k}$ reduces to a combination of elements below the stairs with polynomial coefficients of degree $O(s^p)$.

Theorem (ChyzakKauersSalvy2009)

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$$\dim T_t(\mathcal{I}) \leq \max(\dim \mathcal{I} + p - 1, 0).$$

Proof. Same as above. Set $q := \dim \mathcal{I} + p$.

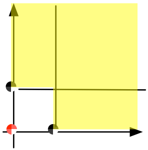
- In degree s, dim $O(s^q)$ below stairs.
- Number of monomials in $\partial_t, \partial_{i_1}, \dots, \partial_{i_q}$: $O(s^{q+1})$;

$\Rightarrow$ any q variables linearly dependent $\Rightarrow$ dim $\leq q - 1$.

This proof gives an algorithm. Also, bounds available.

Examples (all with $p = 1$)

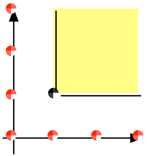
- Proper hypergeometric [Wilf & Zeilberger 1992]:



$$Q(n,k)\xi^k \frac{\prod_{i=1}^{u}(a_i n + b_i k + c_i)!}{\prod_{i=1}^{v}(u_i n + v_i k + w_i)!},$$

Q polynomial, $\xi \in \mathbb{C}$, a_i, b_i, u_i, v_i integers.

- Differential D-finite (definite integration);
- Stirling: ok for $n \geq 3$, e.g., Frobenius:



$$\sum_{k=0}^{n}(-1)^{m-k}k!\binom{n-k}{m-k}\left\{ {n+1 \atop k+1} \right\} = \left\langle {n \atop m} \right\rangle.$$

  - Abel type: $\dim = 2 \to$ ok for $n \geq 4$, e.g., Abel:

$$\sum_{k=0}^{n}\binom{n}{k} i(k+i)^{k-1}(n-k+j)^{n-k} = (n+i+j)^n.$$

IV Conclusion

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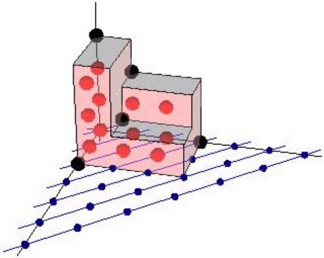
  - Confinement in vector spaces + creative telescoping $\rightarrow$ identities.



- Also:
  - Fast algorithms: Zeilberger 1990 (hypergeom); Chyzak 2000 (D-finite) Us 2009 (non-D-finite).
  - Bounds $\rightarrow$ identities;
  - Fast algorithms for special classes;
  - Efficient numerical evaluation.
- Open questions:
  - Replace polynomial growth by something intrinsic;
  - Exploit symmetries;
  - Structured Padé-Hermite approximants;
  - Understand non-minimality.

Conclusion

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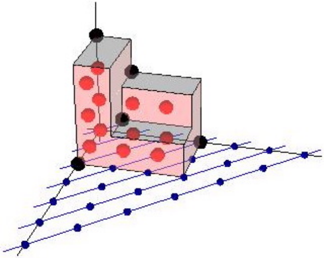

THE END

- Open questions:
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