Automatic Proofs of Special Functions or Combinatorial Identities

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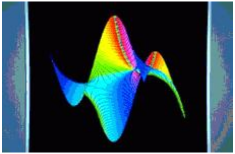

May 11, 2006

I Introduction

Undecidability Issues

Theorem (Richardson 68)

In the class obtained from $\mathbb{Q}(x)$, π, $\log 2$ *by the operations* $+, -, \times$ *and composition with* exp, sin *and* $|\cdot|$, *testing for zero-equivalence is undecidable.*

Consequences:

1. "Simplification" is always difficult;
2. Automatic proofs cannot cover very general classes of identities;
3. Computer algebra isolates classes for which it can provide algorithms.

Algebraic Numbers and Finite Dimension

$$x = \frac{\sin\frac{2\pi}{7}}{\sin^2\frac{3\pi}{7}} - \frac{\sin\frac{\pi}{7}}{\sin^2\frac{2\pi}{7}} + \frac{\sin\frac{3\pi}{7}}{\sin^2\frac{\pi}{7}} = 2\sqrt{7}.$$

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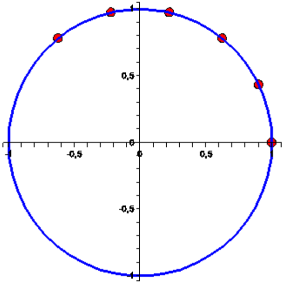
$\mathbb{Q}(\exp(i\pi/7))$ has dim 6 over $\mathbb{Q}$



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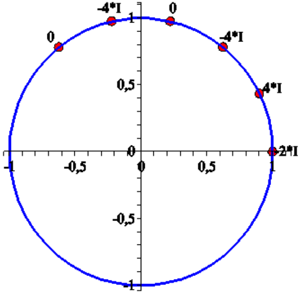
Coordinates of x



Tools: Euclidean division, (extended) Euclidean algorithm, linear algebra.

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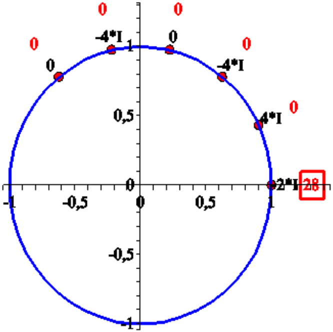

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Algebraic Numbers and Finite Dimension

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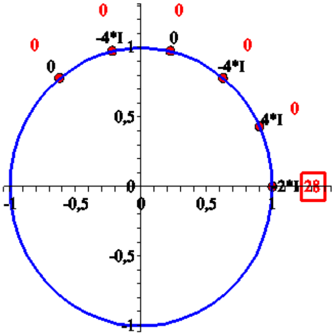


Definition

A number $x \in \mathbb{C}$ is *algebraic* when its powers generate a finite-dimensional vector space over $\mathbb{Q}$.

Tools: Euclidean division, (extended) Euclidean algorithm, linear algebra.

```
> x:=sin(2*Pi/7)/sin(3*Pi/7)^2-sin(Pi/7)/sin(2*Pi/7)^2+sin(3*Pi/7)/sin(Pi/7)^2;
```

$$x := \frac{\sin\left(\frac{2}{7}\pi\right)}{\sin\left(\frac{3}{7}\pi\right)^2} - \frac{\sin\left(\frac{1}{7}\pi\right)}{\sin\left(\frac{2}{7}\pi\right)^2} + \frac{\sin\left(\frac{3}{7}\pi\right)}{\sin\left(\frac{1}{7}\pi\right)^2}$$

```
> convert(x,exp);
```

$$\frac{2\,\mathrm{I}\left(e^{\frac{2}{7}\mathrm{I}\pi} - \frac{1}{e^{\frac{2}{7}\mathrm{I}\pi}}\right)}{\left(e^{\frac{3}{7}\mathrm{I}\pi} - \frac{1}{e^{\frac{3}{7}\mathrm{I}\pi}}\right)^2} - \frac{2\,\mathrm{I}\left(e^{\frac{1}{7}\mathrm{I}\pi} - \frac{1}{e^{\frac{1}{7}\mathrm{I}\pi}}\right)}{\left(e^{\frac{2}{7}\mathrm{I}\pi} - \frac{1}{e^{\frac{2}{7}\mathrm{I}\pi}}\right)^2} + \frac{2\,\mathrm{I}\left(e^{\frac{3}{7}\mathrm{I}\pi} - \frac{1}{e^{\frac{3}{7}\mathrm{I}\pi}}\right)}{\left(e^{\frac{1}{7}\mathrm{I}\pi} - \frac{1}{e^{\frac{1}{7}\mathrm{I}\pi}}\right)^2}$$

```
> R:=subs([seq(exp(I*Pi/7*j)=X^j,j=1..6)],%);
```

$$R := \frac{2\,\mathrm{I}\left(X^2 - \frac{1}{X^2}\right)}{\left(X^3 - \frac{1}{X^3}\right)^2} - \frac{2\,\mathrm{I}\left(X - \frac{1}{X}\right)}{\left(X^2 - \frac{1}{X^2}\right)^2} + \frac{2\,\mathrm{I}\left(X^3 - \frac{1}{X^3}\right)}{\left(X - \frac{1}{X}\right)^2}$$

```
> minpol:=normal((X^7+1)/(X+1));
```

$$minpol := X^6 - X^5 + X^4 - X^3 + X^2 - X + 1$$

```
> gcdex(denom(R),minpol,X,'u','v');
```

$$1$$

```
> rem(numer(R)*u,minpol,X);
```

$$-2\,\mathrm{I} - 4\,\mathrm{I}X^4 - 4\,\mathrm{I}X^2 + 4\,\mathrm{I}X$$

```
> rem(%^2,minpol,X);
```

$$28$$

Automatic Univariate Identities

$$F_nF_{n+2} - F_{n+1}^2 = (-1)^n$$ Cassini

$$\frac{1-\sqrt{1-4z}}{2z} = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{z^n}{n+1}$$ Catalan num

$${}_2F_1\left(\left.\begin{matrix} a, b \\ a+b+1/2 \end{matrix}\right| z\right) = {}_2F_1\left(\left.\begin{matrix} 2a, 2b \\ a+b+1/2 \end{matrix}\right| \frac{1-\sqrt{1-z}}{2}\right)$$ Legendre

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\frac{u^n}{n!} = \frac{\exp\left(\frac{4u(xy-u(x^2+y^2))}{1-4u^2}\right)}{\sqrt{1-4u^2}}$$ Mehler

Automatic Multivariate Identities

$$\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\sum_{j=0}^{k}\binom{k}{j}^3,$$

$$\sum_{n=0}^{+\infty}P_n(x)y^n=\frac{1}{\sqrt{1-2xy+y^2}},\quad P_n(x)=\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}^2x^k$$

$$\int_0^{+\infty}xJ_1(ax)I_1(ax)Y_0(x)K_0(x)\,dx=-\frac{\ln(1-a^4)}{2\pi a^2},$$

$$\oint_0\frac{(1+2xy+4y^2)\exp\left(\frac{4x^2y^2}{1+4y^2}\right)}{y^{n+1}(1+4y^2)^{\frac{3}{2}}}\,dy=\frac{n!H_n(x)}{\lfloor n/2\rfloor!},$$

$$\sum_{k=0}^{n}\frac{q^{k^2}}{(q;q)_k(q;q)_{n-k}}=\sum_{k=-n}^{n}\frac{(-1)^kq^{(5k^2-k)/2}}{(q;q)_{n-k}(q;q)_{n+k}}$$

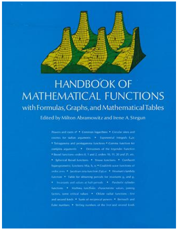
$$\sum_{j=0}^{n}\sum_{i=0}^{n-j}\frac{q^{(i+j)^2+j^2}}{(q;q)_{n-i-j}(q;q)_i(q;q)_j}=\sum_{k=-n}^{n}\frac{(-1)^kq^{7/2k^2+1/2k}}{(q;q)_{n+k}(q;q)_{n-k}}.$$

II D-finiteness in One Variable

D-finite Series & Sequences

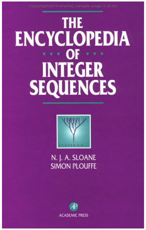
Definition

A *series* $f(x) \in \mathbb{K}[[x]]$ is D-finite over $\mathbb{K}$ when its derivatives generate a finite-dimensional vector space over $\mathbb{K}(x)$. (LDE)
A *sequence* u_n is D-finite over $\mathbb{K}$ when its shifts $(u_n, u_{n+1}, \dots)$ generate a finite-dimensional vector space over $\mathbb{K}(n)$. (LRE)



About 25% of Sloane's encyclopedia,
60% of Abramowitz & Stegun.

eqn+ini. cond.=data structure



Tools: right Euclidean division; right (extended) Euclidean algorithm; linear algebra; equivalence via generating series.
Implemented in `gfun` [SaZi94].

Examples

exp, log, sin, cos, sinh, cosh, arccos, arccosh, arcsin, arctan, arctanh, arccot, arccoth, arccsc, arccsch, arcsec, arcsech, ${}_pF_q$ (includes Bessel J, Y, I and K, Airy Ai and Bi and polylogarithms), Struve, Weber and Anger fcns, the large class of algebraic functions,. . .

First Proof: $\sin^2 + \cos^2 = 1$

```
> series(sin(x)^2+cos(x)^2,x,4);
```

$$1 + O(x^4)$$

Why is this a proof?

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$$1 + O(x^4)$$

Why is this a proof?

1. sin and cos satisfy a 2nd order LDE: $y'' + y = 0$;
2. their squares (and their sum) satisfy a 3rd order LDE;
3. the constant 1 satisfies a 1st order LDE: $y' = 0$;
4. $\rightarrow \sin^2 + \cos^2 - 1$ satisfies a LDE of order at most 4;
5. it is not singular at 0;
6. Cauchy's theorem concludes.

Second proof (same idea): $F_n^2 - F_{n+1}F_{n-1} = (-1)^{n+1}$

```
> for n to 5 do
   fibonacci(n)^2-fibonacci(n+1)*fibonacci(n-1)+(-1)^n od;
```

Third Proof: Mehler's Identity for Hermite Polynomials

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\frac{u^n}{n!} = \frac{\exp\left(\frac{4u(xy-u(x^2+y^2))}{1-4u^2}\right)}{\sqrt{1-4u^2}}$$

1. Definition of Hermite polynomials (D-finite over $\mathbb{Q}(x)$): recurrence of order 2
2. Product by linear algebra: $H_{n+k}(x)H_{n+k}(y)/(n+k)!, k \in \mathbb{N}$ generated over $\mathbb{Q}(x, n)$ by

   $$\frac{H_n(x)H_n(y)}{n!}, \frac{H_{n+1}(x)H_n(y)}{n!}, \frac{H_n(x)H_{n+1}(y)}{n!}, \frac{H_{n+1}(x)H_{n+1}(y)}{n!}$$

   $\longrightarrow$ recurrence of order at most 4;
3. Translation into a differential equation

I. Definition

> $R_1 := \{H(n+2)=(-2n-2)H(n)+2H(n+1)x, H(0)=1, H(1)=2x\}$:

> $R_2 := subs(H=H_2, x=y, R_1)$;

$$R_2 := \{H_2(0)=1, H_2(n+2)=(-2n-2)H_2(n)+2H_2(n+1)y, H_2(1)=2y\}$$

II. Product

> $R_3 := gfun:\text{-}poltorec(H(n)\cdot H_2(n)\cdot v(n), [R_1, R_2, \{v(n+1)\cdot(n+1)=v(n), v(1)=1\}], [H(n), H_2(n), v(n)], c(n))$;

$$R_3 := \Big\{c(0)=1, c(1)=4xy, c(2)=8x^2y^2+2-4y^2-4x^2, c(3)=\frac{32}{3}x^3y^3+24xy-16xy^3-16x^3y, (16n+16)c(n)-16xyc(n+1)+(-8n-20+8y^2+8x^2)c(n+2)-4xc(n+3)y+(n+4)c(n+4)\Big\}$$

III. Differential Equation

> $gfun:\text{-}rectodiffeq(R_3, c(n), f(u))$;

$$\Big\{(16u^3-16u^2yx-4u+8uy^2+8ux^2-4xy)f(u)+(16u^4-8u^2+1)\Big(\frac{d}{du}f(u)\Big), f(0)=1\Big\}$$

> $dsolve(\%, f(u))$;

$$f(u)=\frac{I\, e^{\left(\frac{-4xyu+y^2+x^2}{(2u-1)(2u+1)}\right)}}{e^{(-y^2-x^2)}\sqrt{2u+1}\sqrt{2u-1}}$$

>

Euclidean Division & Finite Dimension

Theorem (XIXth century)

D-finite series and sequences over $\mathbb{K}$ form $\mathbb{K}$-algebras.

Proof.

Linear algebra □

Corollary

D-finite series are closed under Hadamard (termwise) product, Laplace transform, Borel transform (ogf$\leftrightarrow$egf).

Euclidean Division & Finite Dimension

Theorem (Tannery 1874)

D-finite series composed with algebraic power series are D-finite.

Proof.

$$P(x,y)=0 \text{ and } AP+BP_y=1 \Rightarrow y'=-\frac{P_x}{P_y}=-BP_x \bmod P$$

$$\Rightarrow y^{(k)} \in \underbrace{\bigoplus_{\color{red}{i<\deg_y P}} \mathbb{K}(x)y^i}_{\color{red}{\text{finite dim}}}.$$

$(f\circ y)^{(p)}$ linear combination of $(f^{(j)}\circ y)y^k$. □

Also, $\exp\int y$.

Example: Airy Ai at Infinity

$$\mathrm{Ai}(z) = \frac{\sqrt{z}e^{-\xi}}{2\pi}\int_{-\infty}^{\infty} e^{-\xi[(u-1)(4u^2+4u+1)]}\,dv, \quad \xi = \frac{2}{3}z^{3/2}, u = \sqrt{1+\frac{v^2}{3}}$$

$$\sim \frac{1}{2}\pi^{-1/2}z^{-1/4}e^{-\xi}\sum_{n=0}^{\infty}(-1)^n\xi^{-n}\frac{\Gamma(3n+\frac{1}{2})}{54^n n!\Gamma(n+\frac{1}{2})}.$$

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Computation:

1. algebraic change of variables $t^2 = (u-1)(4u^2+4u+1)$;

$$\to \int_{-\infty}^{\infty} e^{-\xi t^2} f(t)\,dt, \quad f(t) = \frac{dv}{dt},$$

2. recurrence satisfied by the coefficients of f (generating series);
3. termwise integration (Hadamard product).

I. Algebraic change of variables

$eq_u := u^2 - \left(1 + \frac{v^2}{3}\right):$

$eq_t := t^2 - (u-1)\cdot\left(4\cdot u^2 + 4\cdot u + 1\right):$

$res := resultant\left(eq_u, eq_t, u\right);$

$$t^4 + 2\,t^2 - 3\,v^2 - \frac{8}{3}\,v^4 - \frac{16}{27}\,v^6$$

$gfun :- algeqtodiffeq\left(res, v(t), \left\{v(0) = 0, D(v)(0) = sqrt\left(\frac{2}{3}\right)\right\}\right);$

$$\left\{-4\,v(t) + 9\,t\left(\frac{d}{dt}\,v(t)\right) + \left(9\,t^2 + 18\right)\left(\frac{d^2}{dt^2}\,v(t)\right), v(0) = 0, (D(v))(0) = \frac{1}{3}\sqrt{6}\right\}$$

II. Recurrence satisfied by the coefficients of f

$gfun :- poltodiffeq(diff(v(t), t), [\%], [v(t)], f(t));$

$$\left\{5\,f(t) + 27\,t\left(\frac{d}{dt}\,f(t)\right) + \left(9\,t^2 + 18\right)\left(\frac{d^2}{dt^2}\,f(t)\right), f(0) = \frac{1}{3}\sqrt{6}, (D(f))(0) = 0\right\}$$

$R_f := gfun :- diffeqtorec(\%, f(t), c(n));$

$$\left\{\left(5 + 18\,n + 9\,n^2\right)c(n) + \left(18\,n^2 + 54\,n + 36\right)c(n+2), c(0) = \frac{1}{3}\sqrt{6}, c(1) = 0\right\}$$

III. Hadamard product

```
assume(ξ > 0); s := Int(exp(-ξ*t^2)*t^n, t=-∞..∞): s=student[intparts](s, exp(-ξ*t^2));
```

$$\int_{-\infty}^{\infty} e^{(-\xi\sim t^2)}\, t^n \, dt = -\int_{-\infty}^{\infty} -\frac{2\,\xi\sim t\, e^{(-\xi\sim t^2)}\, t^{(n+1)}}{n+1}\, dt$$

$$R_i := \left\{ c(n) = \frac{2\cdot\xi}{(n+1)}\cdot c(n+2),\, c(0) = value(eval(s, n=0)),\, c(1) = value(eval(s, n=1)) \right\};$$

$$\left\{ c(n) = \frac{2\,\xi\sim c(n+2)}{n+1},\, c(0) = \frac{\sqrt{\pi}}{\sqrt{\xi\sim}},\, c(1) = 0 \right\}$$

```
> FinalRec:=gfun:-`rec*rec`(R[i],R[f],c(n));
```

$$FinalRec := \left\{ \left(5 + 18\,n + 9\,n^2\right) c(n) + (36\,\xi\sim n + 72\,\xi\sim)\, c(n+2),\, c(1) = 0,\, c(0) = \frac{1}{3}\,\frac{\sqrt{\pi}\,\sqrt{6}}{\sqrt{\xi\sim}} \right\}$$

```
Sol := rsolve(FinalRec, c(n));
```

$$\begin{cases} \dfrac{1}{3}\,\dfrac{(-1)^{\left(\frac{1}{2}n\right)}\, 2^{\left(-1-\frac{1}{2}n\right)}\, \Gamma\left(\frac{1}{2}n+\frac{5}{6}\right)\Gamma\left(\frac{1}{2}n+\frac{1}{6}\right)\xi\sim^{\left(-\frac{1}{2}n\right)}\sqrt{6}}{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}n+1\right)\sqrt{\xi\sim}} & n::even \\ 0 & n::odd \end{cases}$$

Other Uses of the LDE as a Data-Structure

ESF: http://algo.inria.fr/esf [MeSa03]

III Definite Hypergeometric Summation

Creative Telescoping

$$I_n := \sum_{k=0}^{n} \underbrace{\binom{n}{k}}_{u_{n,k}} = 2^n.$$

IF one knows Pascal's triangle:

$$\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k},$$

then summing over k gives

$$I_{n+1} = I_n + I_n = 2I_n.$$

The initial condition $I_0 = 1$ concludes the proof.

Creative Telescoping

$$I_n := \sum_{k=0}^{n} \underbrace{\binom{n}{k}}_{u_{n,k}} = 2^n.$$

Zeilberger's idea: look for rational $r(n,k)$ and $\lambda(n)$ such that

$$v_{n,k+1} - v_{n,k} = u_{n+1,k} + \lambda(n) u_{n,k}, \quad \text{with } v_{n,k} := r(n,k) u_{n,k}$$

When summing over k, the left-hand side telescopes.

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When summing over k, the left-hand side telescopes.

1. Dividing out by $u_{n,k}$ gives a recurrence for r:
$$\frac{n-k}{k+1} r(n,k+1) - r(n,k) = \frac{n+1}{n+1-k} + \lambda(n).$$
2. discussion on poles and degree numerator $\rightarrow r = \frac{a(n)k+b(n)}{n-k}$;
3. normalize and extract coefficients of $k \rightarrow$ linear system
4. solution: $a = -1, b = 0, \lambda = -2$.

The rational function $r(n,k) = k/(k-n)$ is the certificate.

Zeilberger's Algorithm

Input: a hypergeometric term $u_{n,k}$, i.e., $u_{n+1,k}/u_{n,k}$ and $u_{n,k+1}/u_{n,k}$ rational functions;
Output: a linear recurrence satisfied by $\sum_k u_{n,k}$ and a certificate.
For $m = 1, 2, 3, \ldots$

1. Set up the recurrence for $v_{n,k} = r(n,k)u_{n,k}$

$$v_{n,k+1} - v_{n,k} = u_{n+m,k} + \lambda_1(n)u_{n+m-1,k} + \cdots + \lambda_{m-1}(n)u_{n,k}$$

   with unknown r and λ_i;
2. Discuss denominator of r (Gosper's or Abramov's algorithm);
3. Look for numerator → linear system in its coefficients and the λ_i's;
4. If a solution is found, break.

Example: $\zeta(3)$ is Irrational [Apéry78]

$$a_n=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2,\quad b_n=a_n\sum_{k=1}^{n}\frac{1}{k^3}+\sum_{k=1}^{n}\sum_{m=1}^{k}\frac{(-1)^{m+1}\binom{n}{k}^2\binom{n+k}{k}^2}{2m^3\binom{n}{m}\binom{n+m}{m}}.$$

1. $b_n/a_n\to\zeta(3),\ n\to\infty;\ d_n^3 b_n\in\mathbb{Z}$, where $d_n=\operatorname{lcm}(1,\dots,n)$;
2. By creative telescoping, both a_n and b_n satisfy

   $$(n+1)^3u_{n+1}=(34n^3+51n^2+27n+5)u_n-n^3u_{n-1},\qquad n\ge 1;$$

   "Neither Cohen nor I had been able to prove [this] in the intervening two months." [Van der Poorten]
3. $0<\zeta(3)-\dfrac{b_n}{a_n}=\displaystyle\sum_{k\ge n+1}\frac{b_k}{a_k}-\frac{b_{k-1}}{a_{k-1}}$: $b_ka_{k-1}-b_{k-1}a_k=\dfrac{6}{k^3}$;
4. $\lambda a_n+\mu b_n\approx\alpha_\pm^n$, with $\alpha_\pm^2=34\alpha_\pm-1$;
5. Conclusion: $0<\underbrace{a_nd_n^3}_{\in\mathbb{N}}\zeta(3)-\underbrace{d_n^3b_n}_{\in\mathbb{N}}\approx\alpha_-^n e^{3n}\to 0.$

Algolib can be downloaded from http://algo.inria.fr/libraries.

> $libname := \text{"/Users/salvy/lib/maple/Algolib"}, libname:$

> $a := binomial(n, k)^2 \cdot binomial(n + k, k)^2;$

$$a := \mathrm{binomial}(n, k)^2\, \mathrm{binomial}(n + k, k)^2$$

> $Mgfun[creative\_telescoping](a, n :: shift, k :: shift);$

$$\Big[\left(-n^3 - 3 n^2 - 3 n - 1\right) \_f(n, k) + \left(34 n^3 + 153 n^2 + 231 n + 117\right) \_f(n + 1, k) + \left(-n^3 - 6 n^2 - 12 n - 8\right) \_f(n + 2, k), -\frac{4 k^4 \left(4 n^2 + 12 n + 8 + 3 k - 2 k^2\right) (2 n + 3)\, \_f(n, k)}{4 + 12 n - 12 k - 4 n k^3 + 13 n^2 + 13 k^2 + k^4 - 26 n k + n^4 + 6 n^3 - 6 k^3 + 6 n^2 k^2 - 18 n^2 k + 18 n k^2 - 4 n^3 k} \Big]$$

Neither Cohen nor I had been able to prove [this] in the intervening two months. [Van der Poorten]

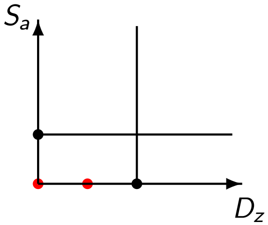
IV D-finiteness in Several Variables

Example: Contiguity of Hypergeometric Series

$$F(a,b;c;z)=\sum_{n=0}^{\infty}\underbrace{\frac{(a)_n(b)_n}{(c)_n n!}}_{u_{a,n}}z^n,\qquad (x)_n:=x(x+1)\cdots(x+n-1).$$

$$\frac{u_{a,n+1}}{u_{a,n}}=\frac{(a+n)(b+n)}{(c+n)(n+1)}\to\ z(1-z)F''+(c-(a+b+1)z)F'-abF=0,$$

$$\frac{u_{a+1,n}}{u_{a,n}}=\frac{n}{a}+1\to S_aF:=F(a+1,b;c;z)=\frac{z}{a}F'+F.$$



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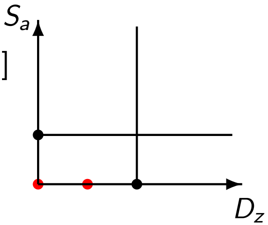

dim=2 $\Rightarrow S_a^2F, S_aF, F$ linearly dependent [Gauss1812]

Also:

- S_a^{-1} in terms of Id, D_z;
- relation between any three polynomials in S_a, S_b, S_c;
  - generalizes to any ${}_pF_q$ and multivariate case [Takayama89].

Ore Polynomials & Ore Algebras

- Skew polynomial ring: $\mathbb{A}[\partial;\sigma,\delta]$, $\mathbb{A}$ integral domain and commutation $\partial P = \sigma(P)\partial + \delta(P)$, $P \in \mathbb{A}$
(ex. $\partial_x P(x) = P(x)\partial_x + P'(x)$, $S_n P(n) = P(n+1)S_n$).
Technical conditions on σ, δ to make product associative.
- Ore algebra: $\mathbb{A}[\partial_1;\sigma_1,\delta_1]\cdots[\partial_n;\sigma_n,\delta_n]$, σ,δ s. t. $\partial_i\partial_j = \partial_j\partial_i$.
Aim [ChSa98]: manipulate (solutions of) systems of mixed linear (q-)differential or (q-)difference operators.

- Main property: the leading term of a product is (up to a cst) the product of leading terms.

Ore Polynomials & Ore Algebras

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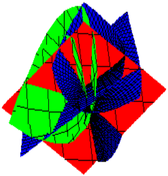
- Ore algebra: $\mathbb{A}[\partial_1;\sigma_1,\delta_1]\cdots[\partial_n;\sigma_n,\delta_n]$, σ,δ s. t. $\partial_i\partial_j=\partial_j\partial_i$.
- Main property: the leading term of a product is (up to a cst) the product of leading terms.
- Consequences:
  1. Univariate: Right Euclidean division and extended Euclidean algorithm [Ore 33];
  2. Multivariate: Buchberger's algorithm for Gröbner bases works in Ore algebras [Kredel93].

0-dimensionality & D-finiteness



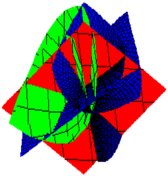
Polynomial algebra	Ore algebra
0-dimensional ideal	D-finite left ideal
$\Updownarrow$	$\Updownarrow$ def

quotient is a finite dimensional vector space

Exs: Orthogonal polynomials, hypergeometric series, their q-analogues,...

system+ini. cond.=data structure

0-dimensionality & D-finiteness



Polynomial algebra	Ore algebra
0-dimensional ideal	D-finite left ideal
$\Updownarrow$	$\Updownarrow$ def

quotient is a finite dimensional vector space

$\Downarrow$	$\Downarrow$
polynomial expressions are algebraic	polynomials and ∂'s are D-finite

Tools: linear algebra, Gröbner bases. Implemented in `Mgfun` [Chyzak98]

Exs: Orthogonal polynomials, hypergeometric series, their q-analogues,. . .

system+ini. cond.=data structure

Example: Binomial Coefficients and Pascal's Triangle

- Algebra: $A = \mathbb{Q}(n,k)[S_n; S_n, 0][S_k; S_k, 0]$
- $\binom{n}{k}$ is annihilated by $S_n - \frac{n+1}{n+1-k}$ and $S_k - \frac{n-k}{k+1}$
- They generate a left ideal $\mathcal{I}$.
- The quotient has dimension 1 ($\equiv$ hypergeometric).
- Pascal's triangle is $P = S_n S_k - S_k - 1 \in \mathcal{I}$.
- Creative telescoping is obtained by left division by $S_k - 1$:

$$P = (S_k - 1)\underbrace{(S_n - 1)}_{\text{certificate}} + \underbrace{S_n - 2}_{\text{result}}$$

General Creative Telescoping [Zeilberger 90]

$$F_n = \sum_k u_{n,k} = ?$$

IF one knows $A(n, S_n)$ and $B(n, k, S_n, S_k)$ such that

$$(A(n, S_n) + \Delta_k B(n, k, S_n, S_k)) \cdot u_{n,k} = 0,$$

then the sum "telescopes", leading to $A(n, S_n) \cdot F_n = 0.$

General Creative Telescoping [Zeilberger 90]

$$I(x) = \int_\Omega u(x,y)\,dy = ?$$

IF one knows $A(x,\partial_x)$ and $B(x,y,\partial_x,\partial_y)$ such that

$$(A(x,\partial_x) + \partial_y B(x,y,\partial_x,\partial_y)) \cdot u(x,y) = 0,$$

then the integral "telescopes", leading to $A(x,\partial_x) \cdot I(x) = 0.$

General Creative Telescoping [Zeilberger 90]

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IF one knows $A(x, \partial_x)$ and $B(x, y, \partial_x, \partial_y)$ such that

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then the integral "telescopes", leading to $A(x, \partial_x) \cdot I(x) = 0$.

Then I come along and try differentating under the integral sign, and often it worked. So I got a great reputation for doing integrals.

Richard P. Feynman 1985

Creative telescoping="differentiation" under integral+"integration" by parts

General Creative Telescoping [Zeilberger 90]

$$I(x) = \int_\Omega u(x,y)\,dy = ?$$

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Creative telescoping="differentiation" under integral+"integration" by parts

- General case: Find annihilators of

$$I(x_1,\dots,x_{n-1}) = \partial_n^{-1}\big|_\Omega f(x_1,\dots,x_n)$$

  knowing generators of Ann_f in
  $\mathbb{O}_n = \mathbb{K}(x_1,\dots,x_n)[\partial_1;\sigma_1,\delta_1]\cdots[\partial_n;\sigma_n,\delta_n]$;
- Crucial step: compute $(\mathbb{O}_n \,\mathrm{Ann}_f + \partial_n \mathbb{O}_n) \cap \mathbb{O}_{n-1}$.

Applications of Creative Telescoping

$$\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\sum_{j=0}^{k}\binom{k}{j}^3 \qquad \text{[Strehl92]}$$

$$\int_0^{+\infty} xJ_1(ax)I_1(ax)Y_0(x)K_0(x)\,dx=-\frac{\ln(1-a^4)}{2\pi a^2} \qquad \text{[GlMo94]}$$

$$\frac{1}{2\pi i}\oint \frac{(1+2xy+4y^2)\exp\left(\frac{4x^2y^2}{1+4y^2}\right)}{y^{n+1}(1+4y^2)^{\frac{3}{2}}}\,dy=\frac{H_n(x)}{\lfloor n/2\rfloor!} \qquad \text{[Doetsch30]}$$

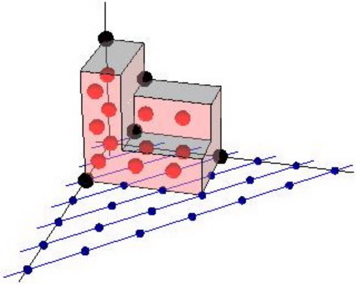
$$\sum_{k=0}^{n}\frac{q^{k^2}}{(q;q)_k(q;q)_{n-k}}=\sum_{k=-n}^{n}\frac{(-1)^k q^{(5k^2-k)/2}}{(q;q)_{n-k}(q;q)_{n+k}} \qquad \text{[Andrews74]}$$

$$\sum_{j=0}^{n}\sum_{i=0}^{n-j}\frac{q^{(i+j)^2+j^2}}{(q;q)_{n-i-j}(q;q)_i(q;q)_j}=\sum_{k=-n}^{n}\frac{(-1)^k q^{7/2k^2+1/2k}}{(q;q)_{n+k}(q;q)_{n-k}} \qquad \text{[Paule85].}$$

(Partial) Algorithms for Creative Telescoping

Aim: $\mathcal{I} = (\mathbb{O}_n \operatorname{Ann}_f + \partial_n \mathbb{O}_n) \cap \mathbb{O}_{n-1}$

- By Gröbner bases, eliminate x_n and set ∂_n to 0 [ChSa98]
  $\rightarrow (\mathbb{O}_n \operatorname{Ann}_f \cap \mathbb{O}_{n-1}[\partial_n] + \partial_n \mathbb{O}_{n-1}) \cap \mathbb{O}_{n-1} \subset \mathcal{I}$
- Differential case: algorithms from $\mathcal{D}$-module theory [SaStTa00,Tsai00], Gröbner bases with negative weights.
- Shift case, $n = 2$, dim 1 ($=$ hypergeometric): [Zeilberger91]
  For increasing k, search for a_i and B
  $$\mathbb{O}_{n-1} \ni \sum_{i=0}^{k} a_i \partial_{n-1}^i f = \partial_n B f$$
  Termination [Abramov03].
- Arbitrary n and $\mathbb{O}_n$: [Chyzak00]
  $$\mathbb{O}_{n-1} \ni \sum_{\lambda} a_\lambda \partial^\lambda = \partial_n B \bmod \operatorname{Ann}_f$$
  B is given by rational solutions of a linear system in σ_n, δ_n.



Open Problems

Efficiency

- Faster Gröbner bases;
- Other elimination techniques (adapt geometric resolution [GiHe93,GiLeSa01] to Ore algebras);
- Structured Padé-Hermite approximants.

Understand non-minimality

- Remove apparent singularities by Ore closure , a generalization of Weyl closure [Tsai00], and of [AbBavH05] ([ChDuLeMaMiSa05] in progress);
- Exploit symmetry (extend [Paule94]).

Easy-to-use Implementations

- Improve `gfun` and `Mgfun`. Make the ESF interactive.

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