A note on Matching-Cut in P_t-free Graphs

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Abstract

A matching-cut of a graph is an edge cut that is a matching. The problem MATCHING-CUT is that of recognizing graphs with a matching-cut and is NP-complete, even if the graph belongs to one of a number of classes. We initiate the study of MATCHING-CUT for graphs without a fixed path as an induced subgraph. We show that MATCHING-CUT is in P for P_5-free graphs, but that there exists an integer $t > 0$ for which it is NP-complete for P_t-free graphs.

For a connected graph G and a subset $E' \subset E(G)$, we say that E' is a *cutset* if $G - E'$ (i.e., the graph obtained by removing the edges in E' but not their endpoints from G) is disconnected.

In 1969, R. L. Graham [8] defined a cutset of edges to be a *matching-cut* if no two edges in the cutset have a vertex in common, and studied those graphs which have no matching-cut, but whose every proper subgraph has a matching-cut. It was Chvátal [6] who initiated the study of MATCHING-CUT, the complexity problem of recognizing graphs admitting a matching-cut, showing that it is NP-complete, even for graphs with maximum degree at most four, yet in P for graphs with maximum degree at most three (unaware of Chvátal's result, Dunbar et al. [7] formulated MATCHING-CUT, leaving its complexity as an open problem that was repopularized in 2016 in [9]). The NP-hardness of MATCHING-CUT has since been shown to also hold for graphs with additional or other structural assumptions; see, for example, [3, 5, 10, 11]. To keep this paper short, we refer the reader to [5, 10] and references therein for a thorough discussion, including real-world applications.

For a positive integer t, we denote by P_t the induced path with t vertices. A graph G is said to be P_t-free if it contains no P_t as an induced subgraph. In this paper, we initiate the study of MATCHING-CUT for graphs without a fixed path as an induced subgraph. In particular, the following theorems are proved.

Theorem 1. MATCHING-CUT *is polynomial-time solvable in P_5-free graphs.*

Theorem 2. *There exists an integer $t > 0$ such that* MATCHING-CUT *is NP-complete in P_t-free graphs.*

Theorem 1 generalizes a result of Bonsma [2], stating that MATCHING-CUT is polynomial-time solvable for cographs. The proof of Theorem 1 is short and simple and inspired by the proof of [4, Theorem 5.4].

The proof of Theorem 2 is also rather short and simple, and involves new arguments.

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1 The proof of Theorem 1

We require the following powerful theorem due to Bacsó and Tuza [1].

Theorem 3. *A connected P_5-free graph G contains a dominating set X that is either a clique or a P_3. Moreover, X can be found in polynomial time.*

We call a graph G *full* if it contains a dominating set that is a clique of size ≥ 3. We also call a coloring of the vertices of G with colors red and blue *good* if every red vertex is adjacent to at most one blue vertex and every blue vertex is adjacent to at most one red vertex. We say that a good coloring is *strong* if it uses both colors. Note that a good coloring defines a matching-cut if and only if it is strong.

We can easily deal with the case when the graph is not full. We abbreviate red and blue by r and b respectively.

Lemma 1. *Enumerating all good colorings of a non-full connected P_5-free graph can be done in polynomial-time.*

Proof. Let G be non-full P_5-free graph. Since G is non-full, by Theorem 3 we can find in polynomial-time a dominating set X of G that is K_1, K_2 or P_3.

For each good coloring c of X and each vertex $x \in X$, how many ways are there of extending c to a good coloring of $X \cup N_{G-X}(x)$? Since at most one vertex adjacent to x can receive the color in $\{r, b\} \setminus c(x)$ at most $|N_{G-X}(x)|+1$ such ways are possible. Therefore, the number of ways there are of extending c to a good coloring of $G = G[X \cup \bigcup_{x \in X} N_{G-X}(x)]$ is polynomial in the number of vertices. This implies the lemma. □

We are now ready to prove the theorem.

Proof of Theorem 1. Let G be a P_5-free graph. We can assume that G is connected (since otherwise we apply our algorithm component-wise). By Theorem 3, we can find in polynomial-time a dominating set X in G that is either K_1, K_2, a clique on at least three vertices or P_3. If X is K_1, K_2 or P_3, then we apply Lemma 1.

Otherwise, since X is a clique with ≥ 3 vertices, we can assume, without loss of generality, that G is precolored by coloring X red. Let C be a component of $G - X$. Since C is dominated by X, any good coloring of C must be monochromatic. Altogether, this implies that if G has a strong coloring, then G has a strong coloring such that at least one component of $G - X$ is blue. As this can clearly be checked in polynomial-time, the proof is complete. □

2 The proof of Theorem 2

An instance $(X, \mathcal{C})$ of *Restricted Positive 1-in-3-SAT* consists of a set of Boolean variables $X = \{x_1, \ldots, x_n\}$ and a collection of clauses $\mathcal{C} = \{C_1, \ldots, C_m\}$, where each clause is a disjunction of exactly three variables, and no variable occurs more than three times in the

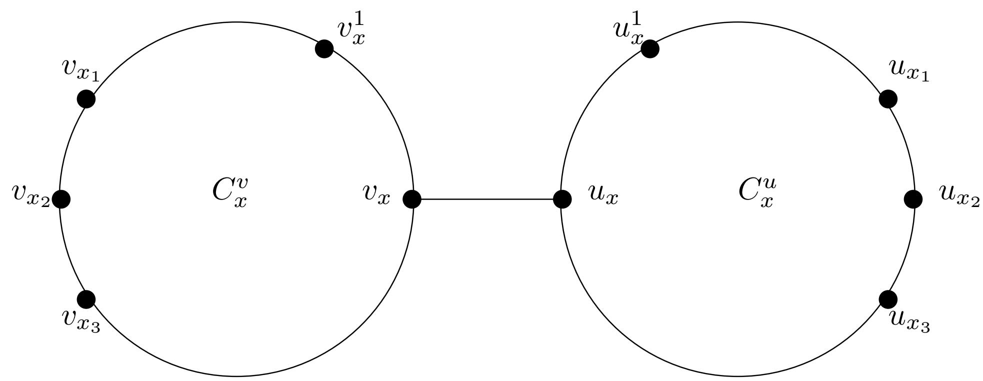

Figure 1: Variable gadget

formula $\theta = C_1 \wedge C_2 \wedge \cdots \wedge C_m$, and the question is to determine whether there exists a satisfying truth assignment of θ so that exactly one variable in each clause is set to true. This problem is NP-complete [12].

Proof of Theorem 2. The problem MATCHING-CUT is clearly in NP. Let $F = (X, \mathcal{C})$ be any instance of Restricted Positive 1-in-3 SAT. We construct, in polynomial time, a graph G that is P_k-free for some positive integer k such that F is satisfiable iff G has a matching-cut.

From now on, we fix a variable $s \in X$. For each variable $x \in X \setminus \{s\}$, we build a variable gadget depicted in Figure 1, where each of the two circles C_x^v and C_x^u depicts a clique on five vertices. Similarly, for s we build a variable gadget as in Figure 1, except that each of the circles C_s^v and C_s^u depicts a clique on $|X| + 3$ vertices, where vertices of the left circle are labelled $v_{s_1}, v_{s_2}, v_{s_3}, v_s, v_s^1, \ldots, v_s^{|X|-1}$ and vertices of the right circle $u_{s_1}, u_{s_2}, u_{s_3}, u_s, u_s^1, \ldots, u_s^{|X|-1}$.

For $i \in \{1, 2, 3\}$, we think of v_{x_i} as *corresponding* to the ith occurrence of x and, as will become evident by the end of the proof, of u_{x_i} as "complementary" to v_{x_i}; we also call v_{x_i} and u_{x_i} *variable vertices*. We connect the set of variable gadgets as follows. Set $V_s = \{v_s^1, \ldots, v_s^{|X|-1}\}$, $U_s = \{u_s^1, \ldots, u_s^{|X|-1}\}$, $W = \{v_x^1 : x \in X \setminus \{s\}\}$ and $Z = \{u_x^1 : x \in X \setminus \{s\}\}$, let $f : V_s \to W$ and $g : U_s \to Z$ be bijective so that, furthermore, $f(v_s^j)$ and $g(u_s^j)$ are members of the the same variable gadget for $j \in \{1, \ldots, |X| - 1\}$, and add the edges $v_s^j f(v_s^j)$, $v_s^j g(u_s^j)$, $u_s^j g(u_s^j)$ and $u_s^j f(v_s^j)$ for $j \in \{1, \ldots, |X| - 1\}$. See Figure 3 for an example illustrating the edges between the variable gadgets corresponding to s, x and y.

For each clause $C = (x \vee y \vee z) \in \mathcal{C}$, we build a clause gadget depicted in Figure 2, where, in this case, this is the first occurence of x, the third of y and the second of z. Note also in the figure that vertices $v_C, u_C^1, u_C^2, u_{x_1,C,1}, u_{x_1,C,2}, u_{z_2,C,1}, u_{z_2,C,2}, u_{y_3,C,1}, u_{y_3,C,2}$ are new vertices, and vertices $v_{x_1}, v_{z_2}, v_{y_3}, u_{x_1}, u_{z_2}, u_{y_3}$ are variable vertices to be found in, respectively, $C_x^v, C_z^v, C_y^v, C_x^u, C_z^u$ and C_y^u. In particular, no two clause gadgets contain any of the same vertices. We call the vertices v_C and u_C^i for $i \in \{1, 2\}$ *special*. We complete the construction of G by adding an edge between every pair of special vertices.

Recall from Section 1 that G has a strong coloring if and only if G has a matching-cut.

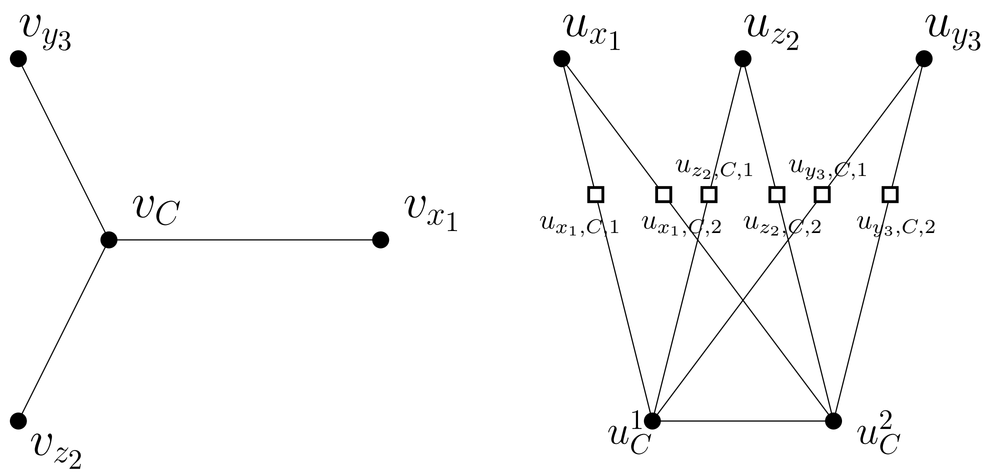

Figure 2: Clause gadget for the clause $C = (x \vee y \vee z)$, where this is the first occurence of x, the third of y and the second of z.

Suppose G has a strong coloring φ.

Claim 1. *The set of special vertices and, for each $x \in X$, the circles C_x^v and C_x^u are each monochromatic.*

Proof. Immediate from the fact that any complete graph on at least three vertices must be monochromatic in a strong coloring. □

Call (the strong coloring) φ *S-splitting* for some $S \subset X$ if $\varphi(C_x^v) \neq \varphi(C_x^u)$ for each $x \in S$.

Claim 2. *Given a clause $C = (x \vee y \vee z)$, if φ is $\{x, y, z\}$-splitting, then $\{v_x, v_y, v_z\}$ is bichromatic.*

Proof. Suppose for a contradiction that $\{v_x, v_y, v_z\}$ is monochromatic and assume, without loss of generality, that its color is red. Then v_C is also red, since otherwise φ is not good.

On the other hand, since φ is $\{x, y, z\}$-splitting, the color of $\{u_x, u_y, u_z\}$ is blue, which in turn implies that at least three of the vertices in $\bigcup_{t \in \{x,y,z\}, i \in \{1,2\}} \{u_{t,C,i}\}$ are blue and so at least one of u_C^1, u_C^2 is also blue. This contradicts Claim 1. □

We abbreviate red and blue by r and b, respectively.

Claim 3. *Given clauses $C = (x \vee y \vee z)$ and $C' = (p \vee q \vee t)$, if φ is $\{x, y, z, p, q, t\}$-splitting, then*

$$|\varphi(\{v_x, v_y, v_z\}) \cap \{r\}| = |\varphi(\{v_p, v_q, v_t\}) \cap \{r\}| \in \{1, 2\}.$$

Proof. By Claim 2, $|\varphi(\{v_x, v_y, v_z\}) \cap \{r\}|, |\varphi(\{v_p, v_q, v_t\}) \cap \{r\}| \in \{1, 2\}$. If for a contradiction $1 = |\varphi(\{v_x, v_y, v_z\}) \cap \{r\}| < |\varphi(\{v_p, v_q, v_t\}) \cap \{r\}| = 2$, then by construction v_C is blue and $v_{C'}$ is red, which contradicts Claim 1. □

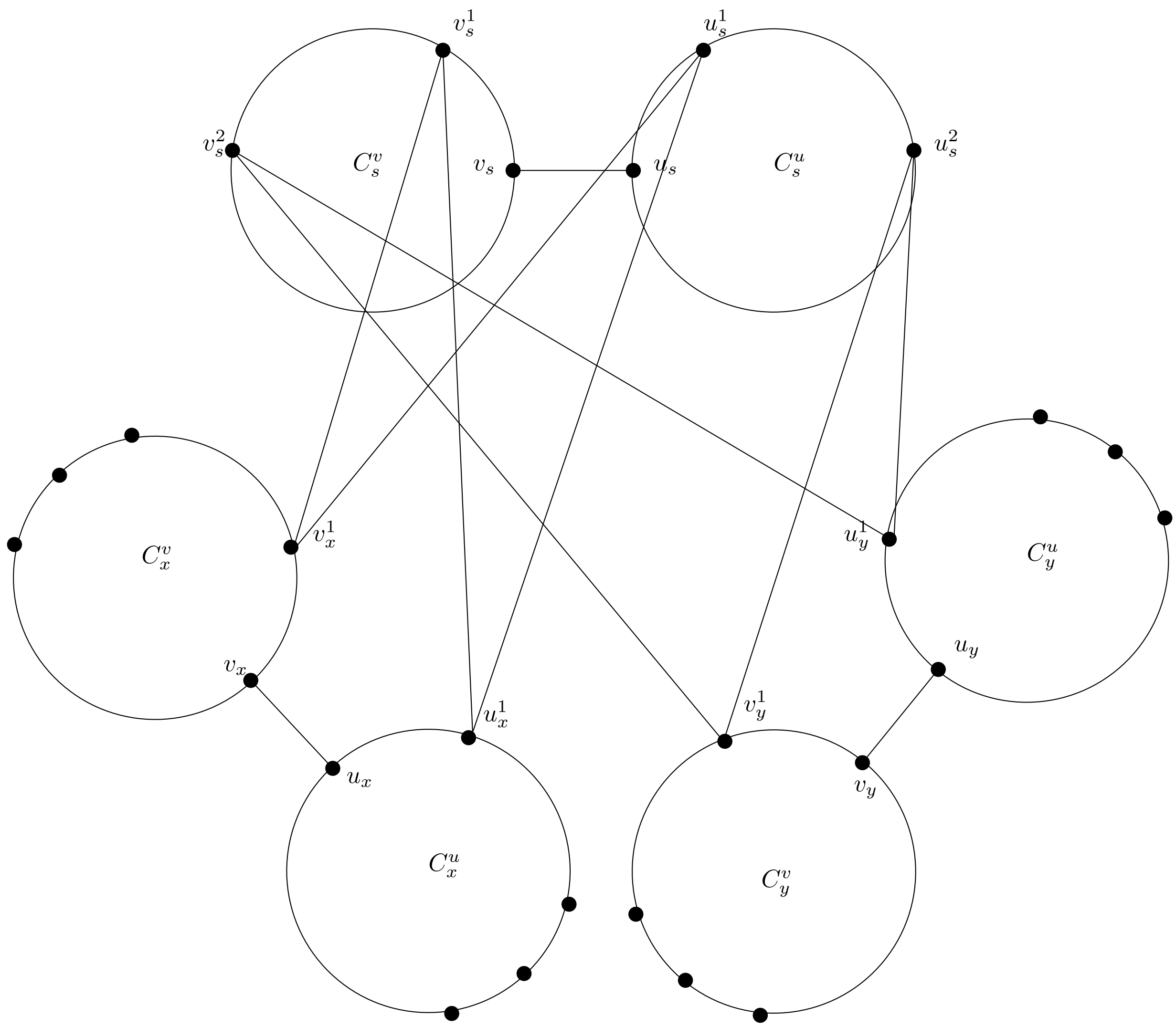

Figure 3: Edges between three variable gadgets corresponding to variables s, x and y.

Claim 4. *If φ is $\{x\}$-splitting for some $x \in X$, then φ is X-splitting.*

Proof. We distinguish two cases.

Suppose first that φ is $\{s\}$-splitting. For each $j \in \{1, \ldots, |X| - 1\}$, since $v_s^j f(v_s^j)$, $v_s^j g(u_s^j)$, $u_s^j g(u_s^j)$ and $u_s^j f(v_s^j)$ are edges and since, by assumption, v_s^j and u_s^j differ in color, $f(v_s^j)$ and $g(u_s^j)$ must also differ in color (else φ is not good). Thus, φ is X-splitting.

In all other cases, φ is $\{x\}$-splitting for some $x \in X \setminus \{s\}$. Then an analogous argument implies φ is s-splitting which in turn implies the claim. □

Claim 5. *φ is X-splitting.*

Proof. Otherwise, by Claim 4 and its proof, the graph induced by the union of the variable gadgets is monochromatic, say has color red. This in turn implies, by construction and Claim 1, that every special vertex is red. On the other hand, by the definition of strong, G has a blue vertex. But this vertex cannot be a non-special vertex of a clause gadget else it would have both neighbors red. Therefore, G is red itself, a contradiction. □

We are now ready to show that F is satisfiable. Since φ is X-splitting by Claim 5, we can assume, by Claim 3 and interchanging the roles of red and blue if necessary, that $|\varphi(\{v_x, v_y, v_z\}) \cap \{r\}| = 1$ for each clause $(x \vee y \vee z) \in \mathcal{C}$. Thus, by setting a variable to true if and only if its corresponding vertices are red, the resulting assignment ensures that exactly one variable per clause is set to true, as needed.

Conversely, suppose F is satisfiable. For each variable $x \in X$, we give the vertices in C_x^v color red if x is set to true and blue otherwise. We extend this partial coloring to an X-splitting coloring of the graph induced by the union of the variable gadgets. We complete this partial coloring to a coloring of G by coloring each special vertex with color blue. Then, for each clause gadget corresponding to a clause, say $C = (x \vee y \vee z)$, where, as in Figure 2, this is the first occurrence of x, the third of y and the second of z, assuming without loss of generality u_{x_1} is blue and u_{y_3} and u_{z_2} are red, color blue $u_{x_1,C,1}, u_{x_1,C,2}, u_{y_3,C,1}, u_{z_2,C,2}$ and red $u_{y_3,C,2}, u_{z_2,C,1}$. To see that the resulting coloring is strong, it suffices to argue that the coloring restricted to C is strong (since, by assumption, the coloring is X-splitting). As v_C is blue, v_{x_1} is red and v_{y_3}, v_{z_2} are blue, each of these vertices has at most one neighbor of the other color and so the coloring restricted to the graph induced by $\{v_C, v_{x_1}, v_{y_3}, v_{z_2}\}$ is strong. Similarly, as u_C^1 and u_C^2 are blue, $u_{x_1,C,1}, u_{x_1,C,2}, u_{y_3,C,1}, u_{z_2,C,2}$ are blue, $u_{y_3,C,2}, u_{z_2,C,1}$ are red, u_{x_1} is blue and u_{y_3} and u_{z_2} are red, each of these vertices has at most one neighbor of the other color and so we are done.

To complete the proof, it remains to show that G is P_k-free for some $k > 0$. Suppose G contains an induced path P with $t \geq 1$ vertices. Since the set of special vertices induces a complete graph, P can contain at most two special vertices and these are consecutive on P. Similarly, by construction, P can contain at most four vertices from each variable gadget. How many variable gadgets can have vertices in common with P?

Any two variable gadgets are connected either via a special vertex or via the variable gadget of s and therefore, by our earlier observations, the number of such variable gadgets is bounded; as P can contain at most two special vertices and at most four vertices from a bounded number of variable gadgets, t is also bounded. This completes the proof. □

We should remark that the same proof works via a reduction from the more well-known Positive 1-in-3-SAT problem, that is, the 1-in-3-SAT problem in which every variable occurs as positive, but may also appear more than three times. We chose the restricted version of this problem for ease of presentation. To reduce instead from Positive 1-in-3-SAT, it suffices to add more vertices in the variable gadgets.

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