Bootstrap for testing the equality of selfsimilarity exponents across multivariate time series

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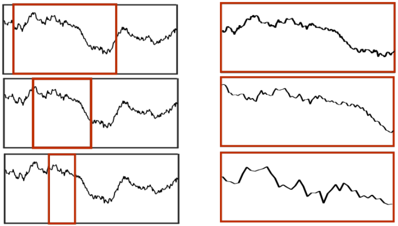
Univariate self-similarity

Scale-free dynamics



$$\{X(t)\}_{t\in\mathbb{R}} \overset{fdd}{=} \{a^H X(t/a)\}_{t\in\mathbb{R}}, \forall a > 0$$

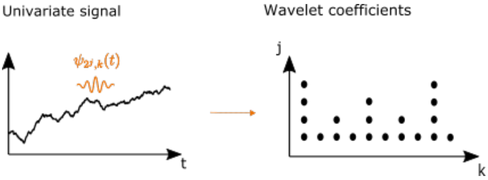
$$0 < H < 1$$

Goal: estimation of H

Univariate estimation of H (Flandrin et al., 1992)

Univariate wavelet transform:

- $D_X(2^j, k) = \langle 2^{-j/2}\psi_0(2^{-j}t - k) | X(t) \rangle$
- ψ_0: mother wavelet



Wavelet spectrum

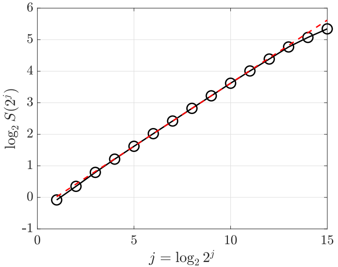
$$S(2^j) = \frac{1}{N_j} \sum_{k=1}^{N_j} D_X(2^j, k)^2 \in \mathbb{R}$$

$$N_j = \frac{N}{2^j},\ N\text{: sample size}$$

X self-similar

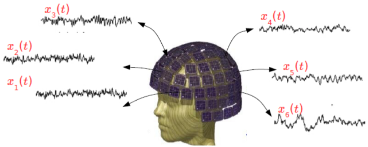
$\Rightarrow$ power law: $S(2^j) \propto 2^{j(2H+1)}$

Linear regression in a log-log diagram



Multivariate self-similarity

Collection of signals

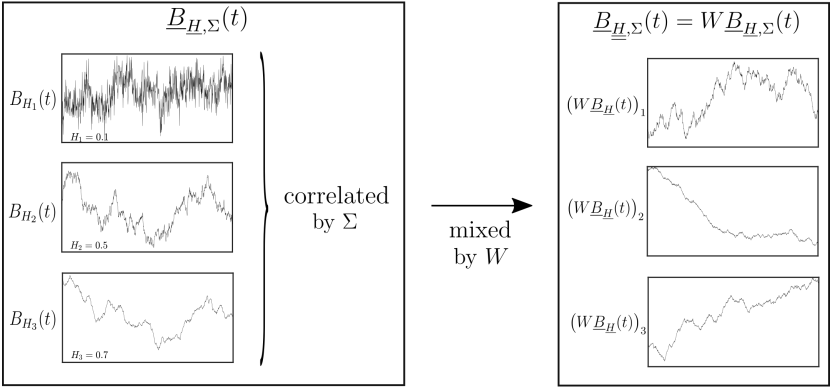


Multivariate setting

Multivariate self-similarity exponent
$\underline{H} = (H_1, \ldots, H_M)$
where $0 < H_1 \leq \ldots \leq H_M < 1$

Goal: testing $H_1 = \ldots = H_M$

Multivariate self-similarity (Didier et al., 2011)



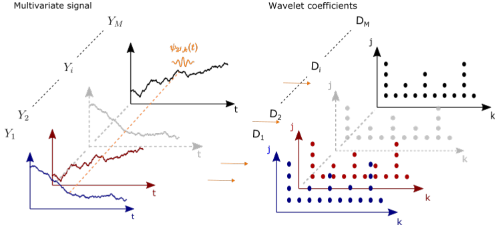
$$\{\underline{B}_{\underline{\underline{H}},\Sigma}(t)\}_{t\in\mathbb{R}} \overset{fdd}{=} \{a^{\underline{\underline{H}}}\underline{B}_{\underline{\underline{H}},\Sigma}(t/a)\}_{t\in\mathbb{R}}, \forall a > 0$$

$$\underline{\underline{H}} = W\mathrm{diag}(\underline{H})W^{-1}$$

Goal: estimation of $\underline{\underline{H}}$

Multivariate estimation

Multivariate wavelet transform of $Y = W\underline{B}_{H,\Sigma}$: $D(2^j,k) = (D_1(2^j,k),\ldots,D_M(2^j,k))$



Wavelet spectrum ($M \times M$ matrix):

$$S_{m_1,m_2}(2^j) = \frac{1}{N_j}\sum_{k=1}^{N_j} D_{m_1}(2^j,k)D_{m_2}(2^j,k)^*$$

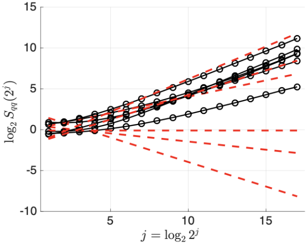
$$N_j = \frac{N}{2^j},\ N\text{: sample size}$$

$Y = W\underline{B}_{H,\Sigma}$ self-similar

$\Rightarrow$ *mixture* of M^2 power laws when $W \neq I$:

$$S_{m_1,m_2}(2^j) = \sum_{k=1}^{M}\sum_{n=1}^{M} A_{k,n}^{(m_1,m_2)} 2^{j(H_k+H_n+1)}$$

Linear regression in a log-log diagram



Estimation of H (Didier and Abry, 2018)

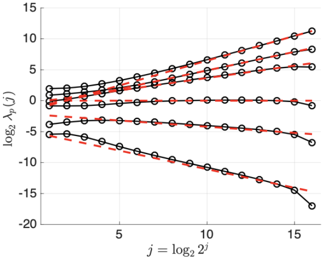
Eigenvalue decomposition:

$$S(2^j) = U(2^j) \begin{bmatrix} \lambda_1(2^j) & 0 & \cdots & \cdots & 0 \\ 0 & \lambda_2(2^j) & \cdots & \cdots & 0 \\ 0 & 0 & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \lambda_M(2^j) \end{bmatrix} U(2^j)^T$$

$Y = W\underline{B}_{\underline{H},\Sigma}$ self-similar
$\Rightarrow$ Asymptotical power law:
$\lambda_m(2^j) \propto 2^{j(2H_m+1)}$

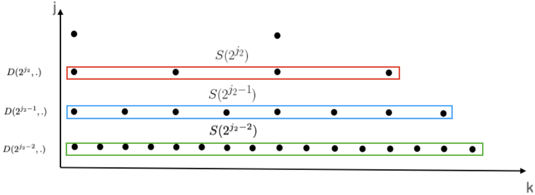
Linear regression on log-eigenvalues $\lambda_m(2^j)$:

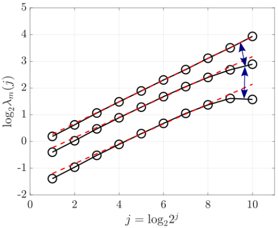
$$\hat{H}_m = \frac{1}{2}\sum_{j=j_1}^{j_2} \omega_j \log_2 \lambda_m(2^j) - \frac{1}{2}$$



Repulsion effect

Gap between eigenvalues larger than expected at each scale





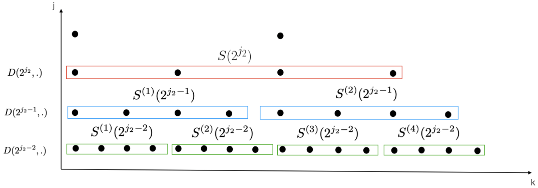
Few coefficients $\Rightarrow$ repulsion effect: important bias when $H_1 = \ldots = H_M$

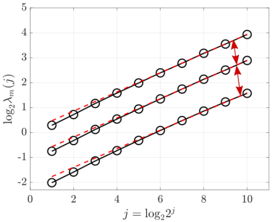
Issue: repulsion effect increases with scale 2^j

Bias corrected estimation

$$S^{(w)}(2^j) \triangleq \frac{1}{n_{j_2}} \sum_{k=1+(w-1)n_{j_2}}^{wn_{j_2}} D(2^j,k)D(2^j,k)^*, \quad w=1,\ldots,2^{j-j_2}, \quad n_{j_2}=\frac{N}{2^{j_2}}$$

Wavelet spectra for same numbers of wavelet coefficients





Eigenvalues of $S^{(w)}(2^j)$: $\{\lambda_1^{(w)}(2^j),\ldots,\lambda_M^{(w)}(2^j)\}$

$\rightarrow$ similar repulsion at all scales $j \in \{j_1,\ldots,j_2\}$

Averaged log-eigenvalues: $\vartheta_m(2^j) \triangleq 2^{j_2-j} \sum_{w=1}^{2^{j-j_2}} \log_2(\lambda_m^{(w)}(2^j))$

Linear regression on averaged log-eigenvalues $\vartheta_m(2^j)$

Testing the equality of $H_1, \ldots, H_M$

Single observation $\underline{H} = (H_1, \ldots, H_M)$

Fluctuation of the estimator: maybe $H_i = H_j$ despite $\hat{H}_i \neq \hat{H}_j$

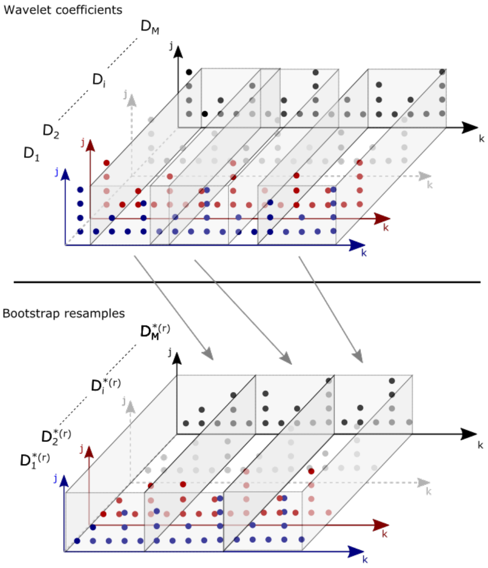
$$\boxed{\text{Testing } H_1 = \ldots = H_M}$$

Asymptotic joint normality of $\underline{\hat{H}} = (\hat{H}_1, \ldots, \hat{H}_M)$
$\rightarrow \chi^2$ statistic:

$$T = (\underline{\hat{H}} - \langle \hat{H}_m \rangle_m)^T \Sigma_{\underline{\hat{H}}}^{-1} (\underline{\hat{H}} - \langle \hat{H}_m \rangle_m).$$

Issue: single observation $\Rightarrow \Sigma_{\underline{\hat{H}}}$ unknown
$\longrightarrow$ estimation of $\Sigma_{\underline{\hat{H}}}$ by Bootstrap resampling

Multivariate wavelet block-bootstrap resamples



R Bootstrap estimates
$$\underline{\hat{H}}^{*(r)} = (\hat{H}_1^{*(r)}, \ldots, \hat{H}_M^{*(r)})$$
computed from
the R wavelet coefficient resamples
$$D^{*(r)} = (D_1^{*(r)}, \ldots, D_M^{*(r)})$$

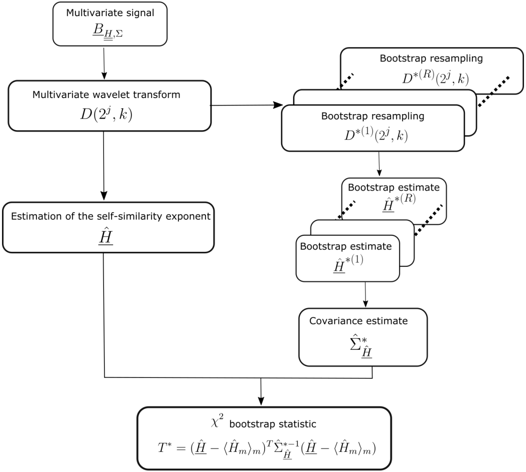
$$\Rightarrow \boxed{\hat{\Sigma}^*_{\underline{\hat{H}}} = cov(\underline{\hat{H}}^*)}$$

Bootstrap test statistic

$$T^* = (\underline{\hat{H}} - \langle \hat{H}_m \rangle_m)^T \hat{\Sigma}^{*\,-1}_{\underline{\hat{H}}} (\underline{\hat{H}} - \langle \hat{H}_m \rangle_m)$$

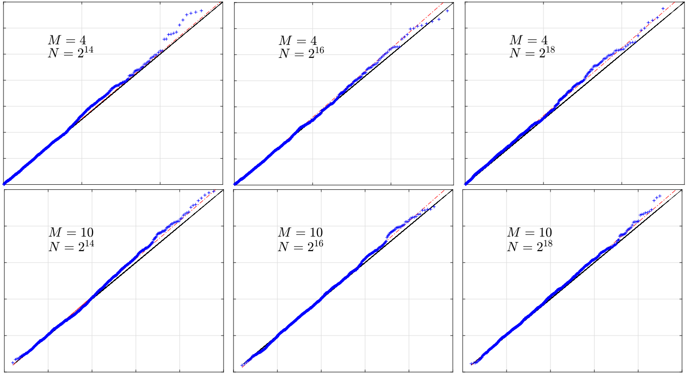
Testing procedure

Algorithm for testing $H_1 = \ldots = H_M$



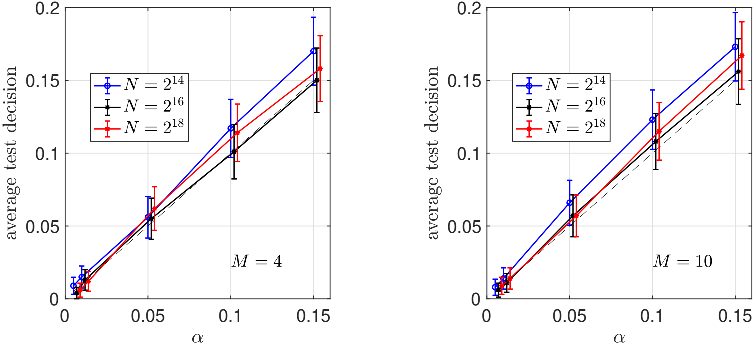
χ^2 statistic under null hypothesis $H_1 = \ldots = H_M$

Monte Carlo simulations



Quantile-quantile plot under $H_1 = \ldots = H_M$
T^* against χ^2 distribution with $M - 1$ degrees of freedom
N: sample size

Significance level under null hypothesis $H_1 = \ldots = H_M$



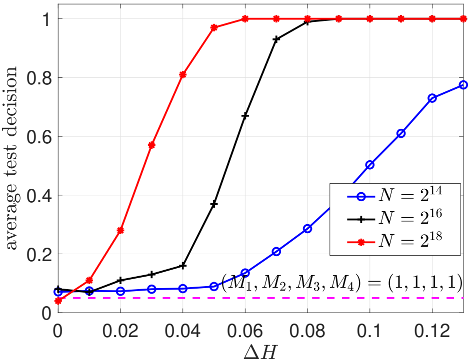
averaged test decisions $\hat{\alpha} \approx$ significance level α

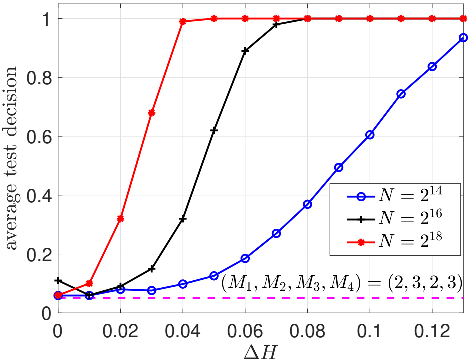
$\Rightarrow$ Significance level well reproduced

Power of the test

$$\underline{H} = (\underbrace{H_1, \ldots, H_1}_{M_1}, \underbrace{H_2, \ldots, H_2}_{M_2}, \underbrace{H_3, \ldots, H_3}_{M_3}, \underbrace{H_4, \ldots, H_4}_{M_4})$$

where $H_m = H_{m-1} + \Delta H$





Conclusion

Achieved:

- bias corrected estimation of multivariate self-similarity exponents
- multivariate wavelet domain bootstrap procedure
- testing procedure for the equality exponents from a single observation

Perspectives:

- how many different values for $\underline{H}$?
- large dimension: number of components M $\approx$ sample size N