Approximations in the polyhedral model

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Outline

1. The polyhedral model
2. Scheduling, SURES, and approximated loops
3. Data mapping & communication optimizations

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1. The polyhedral model
   - Paul Feautrier's static control programs
   - Analyses, optimizations, and tools
   - The polyhedral model is. . . a model
2. Scheduling, SURES, and approximated loops
3. Data mapping & communication optimizations

Affine bounds and affine array access functions

Fortran DO loops:

```
DO i=1, N
  DO j=1, N
    a(i,j) = c(i,j-1)
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- Nested loops, static control.
- Iteration domain and vector.
- Loop increment $= 1$.
- Affine bounds of surrounding counters & parameters.
- Multi-dimensional arrays, same restriction for access functions.

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Polyhedral model: the "all-affine" world, with exact analysis

- Iteration domain $=$ polytope.
- Sequential order $\leq_{seq}$.
- Data $=$ images of polytopes by affine functions.

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Typical criticism: such codes do not exist.

(Parametric) analysis, transformations, optimizations

Data-flow array analysis

- Array expansion.
- Single assignment.
- Liveness array analysis.
- Data reuse.

Mapping computations & data

- Systolic arrays design.
- Data distribution.
- Communication opt.

Loop transformations

- Automatic parallelization.
- Transformations framework.
- Code generation (with loops or with automaton).

Counting & Ehrhart polynomials

- Cache misses.
- Memory size computations.
- Latency computations.

And many more. . .

Many languages fit in the polyhedral model

C for loops:

```
for (i=1, i<=N, i++) {
  for (j=1, j<=N, j++) {
    a[i][j] = c[i][j-1];
    c[i][j] = a[i][j] + a[i-1][N];
  }
}
```

C while loops:

```
y = 0; x = 0;
while (x <= N && y <= N) {
  if (?) {
    x=x+1;
    while (y >= 0 && ?) y=y-1;
  }
  y=y+1;
}
```

Uniform recurrence equations

$\forall(i,j)$ such that $1 \leq i,j \leq N$

$$\begin{cases} a(i,j) = c(i,j-1) \\ b(i,j) = a(i-1,j) + b(i,j+1) \\ c(i,j) = a(i,j) + b(i,j) \end{cases}$$

FAUST: audio processing

```
random = +(12345) ~ *(1103515);
noise = random/2147483.0;
process = random/2 : @(10);
```

and more: Matlab, Fortran90, StreamIt, HPF, C for HLS, . . .

Many tools and a recent revival

PIP Parametric integer programming.

POLYLIB Polyhedra manipulations.

FADALIB Fuzzy array data-flow analysis.

CLOOG Code generation, from polytopes to loops.

EHRHART & BARVINOK Counting tools.

CL@K Critical and admissible lattices.

PIPS Automatic parallelizer & code transformation framework.

PLUTO Automatic parallelizer & locality optimizer for multicores.

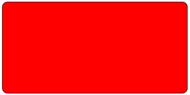
GRAPHITE High-level memory optimizations framework in GCC.

R-STREAM High-level compiler of Reservoir Labs.

. . .

But still, how to deal with non-static control programs?

Polyhedral model.



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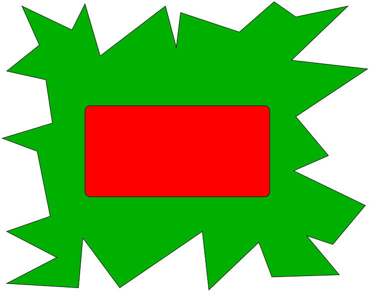
Polyhedral model.

Real life.



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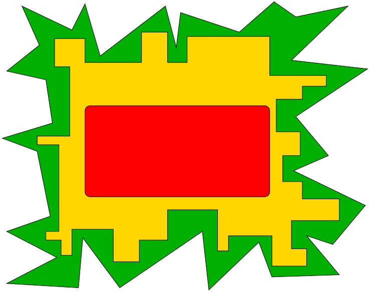

Extensions.

- Non-affine constraints.
- Handling of while loops.
- Recursive programs.
- Beyond induction variables.

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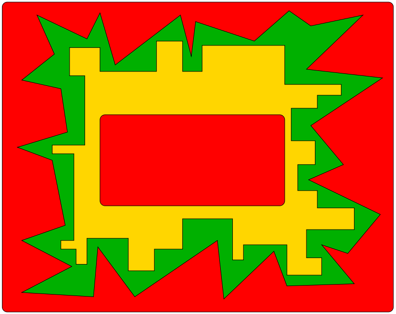

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Approximations.

- Dependences, lifetime, data & iteration domains, etc.
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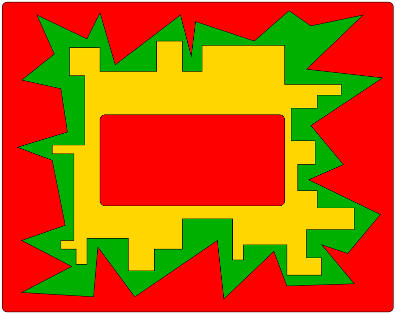

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Think conservative!

Apparent dependence graph and parallelism detection

Is there some loop parallelism (i.e., parallel loop iterations) in the following two codes? What is their degree of parallelism?

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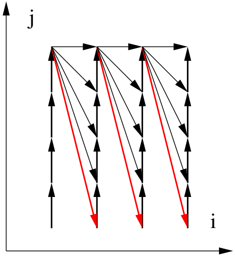


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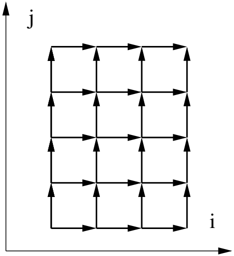

Apparent evolution of variables and program termination

Does this program terminate?
If yes, how many steps in the worst case? Useful for WCET.

```
y = 0; x = 0;
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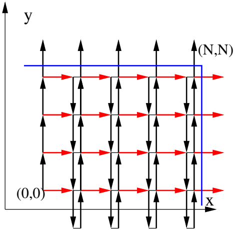

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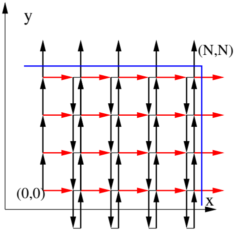


➽ Terminates in at most $N^2 + 3N + 2 = O(N^2)$ steps.

Note: a single while loop can generate quadratic (or more) WCCC.
Surprisingly, similar to parallel detection in Fortran DO loops.

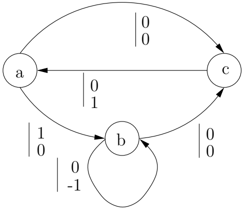
Outline

SURE: system of uniform recurrence equations (1967)

Karp, Miller, Winograd: "The organization of computations for uniform recurrence equations" (J. ACM, 14(3), pp. 563-590).

$$\forall p \in \mathcal{P} = \{p = (i,j) \mid 1 \le i,j \le N\}$$

$$\begin{cases} a(i,j) = c(i,j-1) \\ b(i,j) = a(i-1,j) + b(i,j+1) \\ c(i,j) = a(i,j) + b(i,j) \end{cases}$$



Semantics:

- RDG (reduced dependence graph) $G = (V, E, w)$.
- Explicit dependences & iteration domain $\mathcal{P}$, implicit schedule.
- $e = (u, v) \Leftrightarrow v(p)$ depends on $u(p - w(e))$, i.e., must be computed after. If $p - w(e) \notin \mathcal{P}$, it is an input.
- EDG (expanded dep. graph): vertices $V \times \mathcal{P}$ = unrolled RDG.

Looking for zero-weight cycles

Computability: Can we compute $a(p)$ in a finite number of steps?
Scheduling: If yes, how to find an explicit and "good" schedule?

Lemma 1

A SURE is computable for all bounded domains $\mathcal{P}$ if and only if the RDG has no cycle C with $w(C) = 0$.

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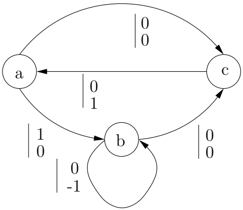
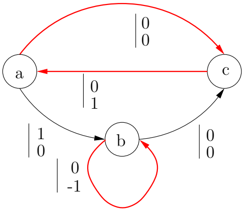
Key structure: the subgraph G' induced by all edges that belong to a multi-cycle (i.e., union of cycles) of zero weight.



Graph G



Graphs G and G'

Key properties

Three elementary key lemmas.

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A zero-weight cycle is a zero-weight multi-cycle.

➽ *Look in G' only.*

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Lemma 4

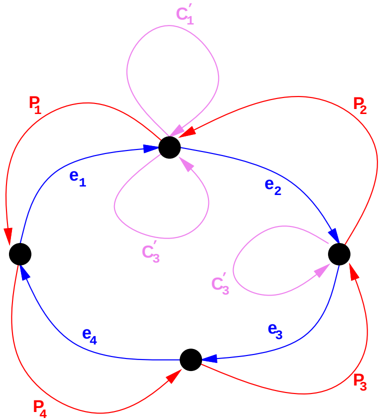
If G' is strongly connected, there is a zero-weight cycle.

➠ *Terminating case.*

Key properties

Lemma 5

If G' is strongly connected, there is a zero-weight cycle.



- $\sum_i e_i$ cycle that visits all vertices.
- e_i in multi-cycle C_i, with $w(C_i) = 0$.
- $C_i = e_i + P_i + C_i'$.
- Follow the e_i, then the P_i and, on the way, plug the C_i'.

Karp, Miller, and Winograd's decomposition

Boolean KMW(G):

- Build G' the subgraph of zero-weight multicycles of G.
- Compute $G'_1, \ldots, G'_s$, the s SCCs of G'.
    - If $s = 0$, G' is empty, return TRUE.
    - If $s = 1$, G' is strongly connected, return FALSE.
    - Otherwise return $\wedge_i$KMW(G'_i) (logical AND).

Then, G is computable iff KMW(G) returns TRUE.

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Depth d of the decomposition

$d = 0$ if G is acyclic, $d = 1$ if all SCCs have an empty G', etc.

Theorem 1 (Depth of the decomposition)

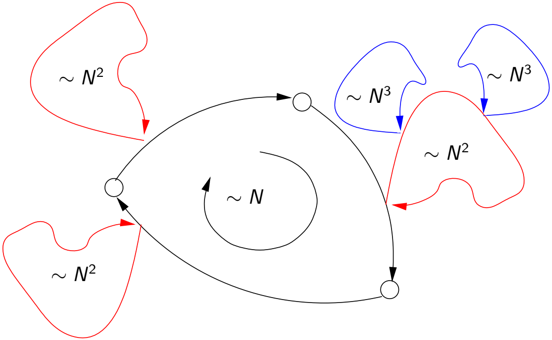
If G is computable, $d \leq n$, otherwise, $d \leq n + 1$.

(n is the dimension of the problem, i.e., the dimension of $\mathcal{P}$.)

Length of longest dependence path in the EDG

Theorem 2 (Longest dependence path)

If $\mathcal{P}$ contains a n-dimensional cube of size $\Omega(N)$, there exists a dependence path of length $\Omega(N^d)$.



Subtlety: needs to make sure that the path stays in the EDG.

But how to compute G'? Primal and dual programs.

$e \in G'$ iff $v_e = 0$ in any optimal solution of the linear program:

$$\min \left\{ \textstyle\sum_e v_e \;\middle|\; q \geq 0,\; v \geq 0,\; q + v \geq 1,\; Cq = 0,\; Wq = 0 \right\}$$

☛ A single (rational) linear program.

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Always interesting to take a look at the dual program:

$$\max \left\{ \sum_e z_e \;\middle|\; 0 \leq z \leq 1,\; X.w(e) + \rho_v - \rho_u \geq z_e,\; \forall e = (u, v) \in E \right\}$$

Additional property, for any optimal solution:

- $e \in G' \Leftrightarrow X.w(e) + \rho_v - \rho_u = 0.$
- $e \notin G' \Leftrightarrow X.w(e) + \rho_v - \rho_u \geq 1.$

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Schedule $\sigma : V \times \mathcal{P} \to \mathbb{N}$, with $\sigma(u, p) = X.p + \rho_u$, is valid if:

$$\begin{aligned} & \sigma(v, p) \geq \sigma(u, p - w(e)) + 1 \\ \Leftrightarrow\ & X.p + \rho_v \geq X.(p - w(e)) + \rho_u + 1 \\ \Leftrightarrow\ & X.w(e) + \rho_v - \rho_u \geq 1 \end{aligned}$$

Scheduling: dual of computability.

- $e \in G' \Leftrightarrow X.w(e) + \rho_v - \rho_u = 0.$
- $e \notin G' \Leftrightarrow X.w(e) + \rho_v - \rho_u \geq 1.$

Multi-dimensional scheduling: hours, minutes, seconds, etc.

$e \notin G'$: u & v computed at different hours.
Different iterations of the outer loop = loop-carried.

$e \in G'$: u & v same hour, constraints pushed to inner dimensions.
Same iteration of outer loop = loop-independent.

Special form of schedule: affine, same linear part in a SCC of G'.

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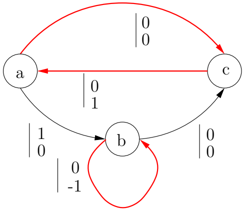

$$\left.\begin{array}{l} X_1.(0,1) = 0 \\ X_1.(1,1) \geq 2 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} X_1 = (2,0),\ \rho_a = 1 \\ \rho_b = 0,\ \rho_c = 1 \end{array}\right.$$

$$\text{Final schedule} \left\{\begin{array}{l} \sigma_a(i,j) = (2i+1, 2j) \\ \sigma_b(i,j) = (2i, -j) \\ \sigma_c(i,j) = (2i+1, 2j+1) \end{array}\right.$$

Performances of schedules for computable equations

Theorem 3 (Optimality of multi-dimensional schedules)

If P contains a n-dim. cube of size $\theta(N)$, there is a dependence path of length $\Omega(N^d)$ and a schedule of latency $O(N^d)$.

Theorem 4 (Case of one-dimensional schedules)

If $d = 1$, the best affine schedule is $\sim \lambda N$, for some $\lambda > 0$, and so is the maximal dependence length.

Theorem 5 (Case of a single equation)

For one equation, $d = 0$ or $d = 1$. Moreover, if $d = 1$, the best linear schedule is optimal up to a constant.

Theorem 6 (Link with tiling)

The maximal number of permutable loops is linked to the dimension of the vector space $Vect(\{w(C) \mid C \text{ cycle of } G'\})$.

Outline

Loop terminology

Fortran DO loops:

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```

- Nested loops, static control.
- Iteration domain and vector.
- Sequential order $\leq_{seq}$.
- Dependences:
  - R/W, W/R, W/R.

$$S(I) <_{seq} T(J) \Leftrightarrow (I|_d <_{lex} J|_d) \text{ or } (I|_d = J|_d \text{ and } S <_{txt} J)$$

- EDG: dependence graph between operations $S(I) \Rightarrow T(J)$.
- RDG: dependence graph between statements $S \rightarrow T$.
- ADG: over-approximation, if $S(I) \Rightarrow T(J)$, then $S \rightarrow T$.

Representation of dependences

- Pair set (exact dependences): $R_{S,T} = \{(I,J) \mid S(I) \Rightarrow T(J)\}$, in particular affine dependence $I = f(J)$ if possible.
- Distance set: $E_{S,T} = \{(J-I) \mid S(I) \Rightarrow T(J)\}$.
- Over-approximations $E'_{S,T}$ such that $E_{S,T} \subseteq E'_{S,T}$.

```
DO i=1, N
  DO j=1, N
    a(i,j) = a(j,i) + 1
  ENDDO
ENDDO
```

Distance set:

$$E = \left\{ \begin{pmatrix} i-j \\ j-i \end{pmatrix} \;\middle|\; i-j \geq 1,\ 1 \leq i,\ j \leq N \right\}$$

Polyhedral approximation:

$$E' = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \end{pmatrix} \;\middle|\; \lambda \geq 0 \right\}$$

Direction vectors:

$$E' = \begin{pmatrix} + \\ - \end{pmatrix} = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ -1 \end{pmatrix} \;\middle|\; \lambda,\ \mu \geq 0 \right\}$$

Level:

$$E' = ① = \begin{pmatrix} + \\ * \end{pmatrix} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \end{pmatrix} \;\middle|\; \lambda \geq 0 \right\}$$

Uniformization of dependences: example

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a(i,j) $\Rightarrow$ a(i-1,N)
Dep. distance $(1, j - N)$.

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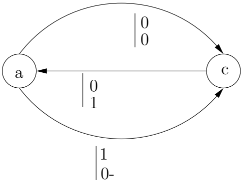
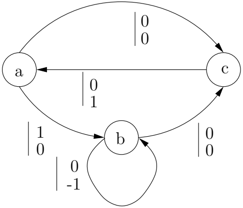
Dep. distance $(1, j-N)$.

$$\left.\begin{array}{l} \text{Direction vector } (1, 0-) = (1,0) + k(0,-1),\ k \geq 0. \\ \text{Also } X.(1,0-) \geq 1 \Rightarrow X.(1,0) \geq 1 \text{ and } X.(0,-1) \geq 0. \end{array}\right\} \text{☛ SURE!}$$





No parallelism ($d = 2$). Code appears (here it is) purely sequential.

Emulation of dependence polyhedra

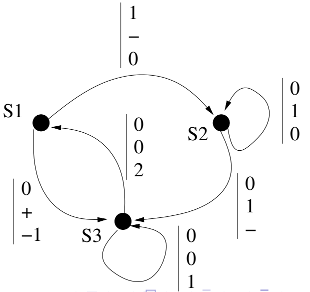
For a (self) dependence polyhedron $\mathcal{P}$, with vertex v and ray r:

$$\forall p \in \mathcal{P}\, X.p \geq 1 \Leftrightarrow \forall \lambda \geq 0\, X.(v+\lambda r) \geq 1 \Leftrightarrow X.v \geq 1 \text{ and } X.r \geq 0$$

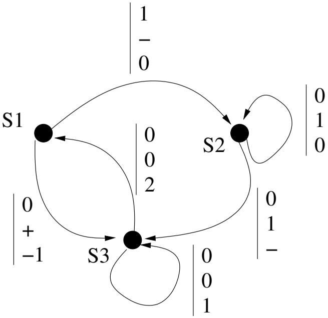
☛ Emulate vertices, rays, and lines.

Example with direction vectors:

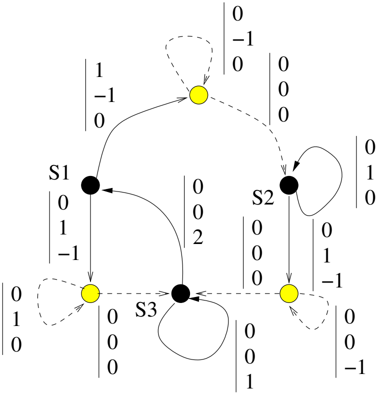
```
DO i= 1, N
  DO j = 1, N
    DO k = 1, j
      a(i,j,k) = c(i,j,k-1) + 1
      b(i,j,k) = a(i-1,j+i,k) + b(i,j-1,k)
      c(i,j,k+1) = c(i,j,k) + b(i,j-1,k+i)
                       + a(i,j-k,k+1)
    ENDDO
  ENDDO
ENDDO
```



Second example: dependence graphs

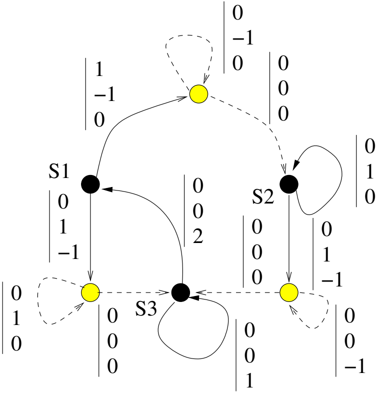


Initial RDG.

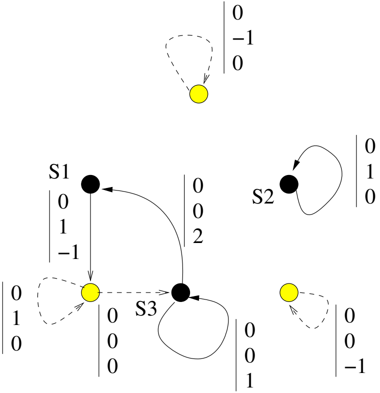


Uniformized RDG.

Second example: G and G'



Uniformized RDG.



G': zero-weight multi-cycles.

$\boxed{(2i, j) \text{ for } S_2,\ (2i+1, 2k) \text{ for } S_1, \text{ and } (2i+1, 2k+3) \text{ for } S_3.}$

Second exemple: parallel code generation

```
DOSEQ i=1, n
  DOSEQ j=1, n /* scheduling (2i, j) */
    DOPAR k=1, j
      b(i,j,k) = a(i-1,j+i,k) + b(i,j-1,k)
    ENDDOPAR
  ENDDOSEQ
  DOSEQ k = 1, n+1
    IF (k ≤ n) THEN /* scheduling (2i+1, 2k) */
      DOPAR j=k, n
        a(i,j,k) = c(i,j,k-1) + 1
      ENDDOPAR
    IF (k ≥ 2) THEN /* scheduling (2i+1, 2k+3) */
      DOPAR j=k-1, n
        c(i,j,k) = c(i,j,k-1) + b(i,j-1,k+i-1) + a(i,j-k+1,k)
      ENDDOPAR
  ENDDOSEQ
ENDDOSEQ
```

Loop parallelization: optimality w.r.t. dep. abstraction

- Lamport (1974): hyperplane method = skew + interchange.
- Allen-Kennedy (1987): loop distribution, optimal for levels.
- Wolf-Lam (1991): unimodular, optimal for direction vectors and one statement. Based on finding permutable loops.
- Darte-Vivien (1997): unimodular + shifting + distribution, optimal for polyhedral abstraction and perfectly nested loops. Finds permutable loops, too.
- Feautrier (1992): general affine scheduling, complete for affine dependences and affine transformations, but not optimal.
- Lim-Lam (1998): extension to coarse-grain parallelism, vague.
- Ramanujam-Sadayappan (2009): second (more sound) extension to permutable loops, with locality optimization.

Outline

Yet another application of SUREs: understand "iterations"

Fortran DO loops:

```
DO i=1, N
  DO j=1, N
    a(i,j) = c(i,j-1)
    c(i,j) = a(i,j) + a(i-1,N)
  ENDDO
ENDDO
```

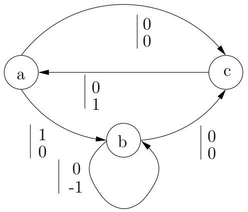
Uniform recurrence equations:

$$\forall p \in \{p = (i,j) \mid 1 \leq i,j \leq N\}$$

$$\begin{cases} a(i,j) = c(i,j-1) \\ b(i,j) = a(i-1,j) + b(i,j+1) \\ c(i,j) = a(i,j) + b(i,j) \end{cases}$$

C for and while loops:

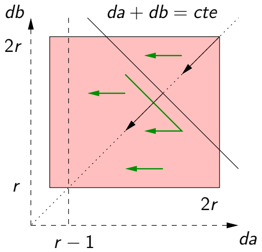
```
y = 0; x = 0;
while (x <= N && y <= N) {
  if (?) {
    x=x+1;
    while (y >= 0 && ?) y=y-1;
  }
  y=y+1;
}
```



Context: transforming WHILE loops into DO loops

Example of GCD of 2 polynomials

```
// expression expr, array A, r>0 integer.
da = 2r; db = 2r;
while (da >= r) {
  cond = (da >= db || A[expr] == 0);
  if (!cond) {
    tmp = db; db = da; da = tmp - 1;
  } else da = da - 1;
}
```



Hard to optimize for HLS tools:

- No loop unrolling possible.
- Limited software pipelining.
- No nested-loops optimization.
- No information for coarse-grain scheduling/pipelining.

☛ Need to bound the number of iterations. When feasible, proves program termination as by-product.

Phase 1: build an integer interpreted automaton

Identify relevant variables:

- vector $\vec{x} \in \mathbb{Z}^n$, $n =$ problem dimension.

Build RDG:

- control-flow graph and conditional transitions.
- express evolution of $\vec{x}$ with affine relations, a bit more general than affine dependences.

Refine automaton (if desired):

- analysis of Booleans: better accuracy, higher complexity.
- simple-path compression: reduces complexity.
- multiple-paths summary: better accuracy, impacts complexity.

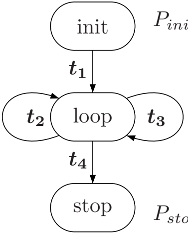
Sequential automaton similar to affine recurrence equations, with a different semantics: different relations express non-determinism.

Phase 2: abstract interpretation to get "invariants"

Explicit dependences and schedule, but implicit iteration domains!

Here, we need to prove $db \geq r$. ☛ Use abstract interpretation.

```
// expression expr, array A,
// r>0 integer.
da = 2r; db = 2r;
while (da >= r) {
  cond = (da >= db
    || A[expr] == 0);
  if (!cond) {
    tmp = db; db = da;
    da = tmp - 1;
  } else da = da - 1;
}
```



$P_{init} = 1 \leqslant r$

$$P_{loop} = \begin{cases} r - 1 \leqslant da \leqslant 2r \\ r \leqslant db \leqslant 2r \\ 1 \leqslant r \end{cases}$$

$$P_{stop} = \begin{cases} 1 \leqslant r \\ r \leqslant db \leqslant 2r \\ r - 1 \leqslant da < r \end{cases}$$

- Invariant = integer points in a polyhedron $\mathcal{P}_k$: conservative approximation of reachable values for each control point k.
- Possibly infinite, parameterized by program inputs.

Phase 3: ranking function to prove termination

Ranking function Mapping $\sigma : \mathcal{K} \times \mathbb{Z}^n \to (\mathcal{W}, \preceq)$, decreasing on each transition, where $(\mathcal{W}, \preceq)$ is a well-founded set.

Multi-dimensional rankings $W = \mathbb{N}^p$ with lexicographic order.

Affine ranking $\sigma(k, \vec{x}) = A_k.\vec{x} + \vec{b_k}$ ➠ Farkas lemma.

☛ Similar to multi-dimensional scheduling for loops, except:

- Higher dimension n (number of relevant variables).
- Flow not always lexico-positive ➠ recurrence equations.
- Hidden "counters" (number p of dimension of the ranking).

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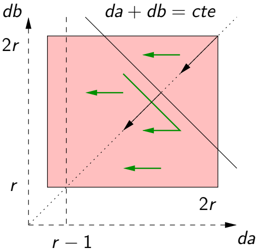
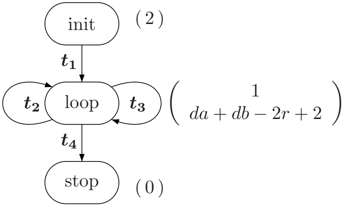




Phase 4: bound on the number of program steps

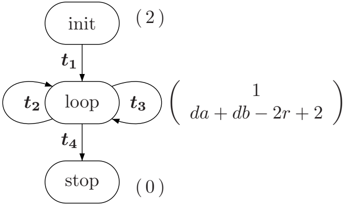
Worst-case computational complexity (WCCC): maximum number of transitions fired by the automaton:

$$WCCC \leq \# \bigcup \sigma(k, \mathcal{P}_k) \leq \sum_k \# \sigma(k, \mathcal{P}_k)$$

Counting points in (images of) polyhedra: Ehrhart polynomials, projections, Smith form, union of polyhedra, etc.

$$\begin{aligned} WCCC \leq\ & \#\sigma(\text{init}, \mathcal{P}_{\text{init}}) \\ & + \#\sigma(\text{loop}, \mathcal{P}_{\text{loop}}) \\ & + \#\sigma(\text{end}, \mathcal{P}_{\text{end}}) \end{aligned}$$

$$= 2 + \#\{(1, i) \mid 1 \leq i \leq 2r + 2\}$$
$$= 2r + 4$$



Alias-Darte-Feautrier-Gonnord (2010)

Greedy algorithm

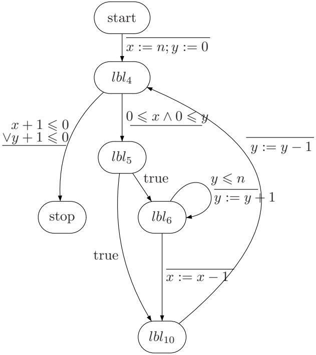
- $i = 0$; $T = \mathcal{T}$, set of all transitions.
- While T is not empty do
  - Find a 1D affine function (X, ρ_S), not increasing for any transitions, and decreasing for as many transitions as possible.
  - Let $\sigma_i = X$; $i = i + 1$;
  - If no transition is decreasing, return FALSE.
  - Remove from T all decreasing transitions.
- $d = i$, return TRUE.

Theorem 7 (Completeness of greedy algorithm w.r.t. invariants)

If an affine interpreted automaton, with associated invariants, has a multi-dimensional affine ranking function, then the greedy algorithm generates one such ranking. Moreover, the dimension of the generated ranking is minimal.

Yet another example

```
y = 0;
x = m;
while(x>=0 && y>=0){
  if(indet()){
    while(y <= m && indet())
      y++;
    x--;
  }
  y--;
}
```



$start$	$m \geq 0$	$2m + 4$
lbl_4	$m \geq x > 0, m \geq y > 0$	$(2x + 3, 3y + 3)$
lbl_5	$m \geq x \geq 0, m \geq y \geq 0$	$(2x + 3, 3y + 2)$
lbl_6	$m \geq x \geq 0, m + 1 \geq y \geq 0$	$(2x + 2, m - y + 1)$
lbl_{10}	$\begin{cases} m \geq x \geq -1, m + 1 \geq y \geq 0 \\ 2m \geq x + y \end{cases}$	$(2x + 3, 3y + 1)$

$\text{WCCC} = 5 + 7m + 4m^2$

Link with Karp, Miller, Winograd's decomposition

Podelski-Rybalchenko (2004) $\sim$ URE $\sim$ Lamport (1974).
Bradley-Manna-Sipma (2005) $\sim$ Wolf-Lam (1991).
Colón-Sipma (2002) between Wolf-Lam & Darte-Vivien (1997).
Alias-Darte-Feautrier-Gonnord (2010) $\sim$ Feautrier (1992).

Gulwani (2009) very different but similar theoretical power.

- Iteration domains $\Leftrightarrow$ Invariants.
- Loop counters $\Leftrightarrow$ Integer variables involved in the control.
- Dependences: partial order $\Leftrightarrow$ Evolution of variables.
- Scheduling functions $\Leftrightarrow$ Ranking functions.
- Latency $\Leftrightarrow$ Worst-case execution time (ideal).
- Parallelism $\Leftrightarrow$ Non determinism.
- In both cases, algorithm depth = measure of sequentiality.

Outline

1. The polyhedral model
2. Scheduling, SURES, and approximated loops
3. Data mapping & communication optimizations
   - Lattice-based memory reduction
   - Communication optimizations for remote data
   - Conclusion

Example of intermediate buffer: DCT-like example

Two synchronized, pipelined (ASAP) processes, communicating through a shared buffer A.

```
DO b_r = 0, 63
  DO b_c = 0, 63
    DO r = 0, 7
      S: A(b_r, b_c, r, *) = ...
    ENDDO
  ENDDO
ENDDO
```

```
DO b_r = 0, 63
  DO b_c = 0, 63
    DO c = 0, 7
      T: ... = A(b_r, b_c, *, c)
    ENDDO
  ENDDO
ENDDO
```

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```

```
DO b_r = 0, 63
  DO b_c = 0, 63
    DO c = 0, 7
      T: ... = A(b_r, b_c, *, c)
    ENDDO
  ENDDO
ENDDO
```

Full array (no reuse) $64 \times 64 \times 8 \times 8 = 2^{18} = 256K$.

"Intuitive solution" write in $A(b_r \bmod 2, b_c \bmod 2, r, c)$ (4 blocks)

Best linear allocation 112 with $\sigma = \begin{cases} r \bmod 4 \\ 16(b_r + b_c) + 2r + c \bmod 28 \end{cases}$

Memory reuse for scheduled programs

Given

- An array A with multiple reads and writes.
- Scheduled program or communicating processes, thanks to θ.

Goal

- reduction of the allocation size (size of buffer);
- simplicity of the addressing functions.

Solutions

- Optimal size with Ehrhart counting ☛ approximations?
- Approximation of maximal number of live values ☛ mapping?
- Modular mapping $\vec{i} \mapsto A\vec{i} \bmod b$ ☛ simple and quite efficient.

Modular mapping and admissible lattice

Definition 1 (Modular mapping)

A modular mapping $(M, \vec{b})$, with $M \in \mathcal{M}_{p,n}(\mathbb{Z})$ and $\vec{b} \in \mathbb{N}^p$, maps index $\vec{i}$ to $\sigma(\vec{i}) = M\vec{i} \bmod \vec{b}$ in p-dimensional array with shape $\vec{b}$.

Definition 2 (Lifetime analysis)

Two indices $\vec{i}$ and $\vec{j}$ of $\mathbb{Z}^n$ are conflicting $(\vec{i} \bowtie \vec{j})$ if they correspond to two simultaneously live values in the schedule θ.

Define $DS = \{\vec{i} - \vec{j} \mid \vec{i} \bowtie \vec{j}\}$. ☛ Can be over-approximated.

Modular mapping and admissible lattice

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Define $DS = \{\vec{i} - \vec{j} \mid \vec{i} \bowtie \vec{j}\}$. ☛ Can be over-approximated.

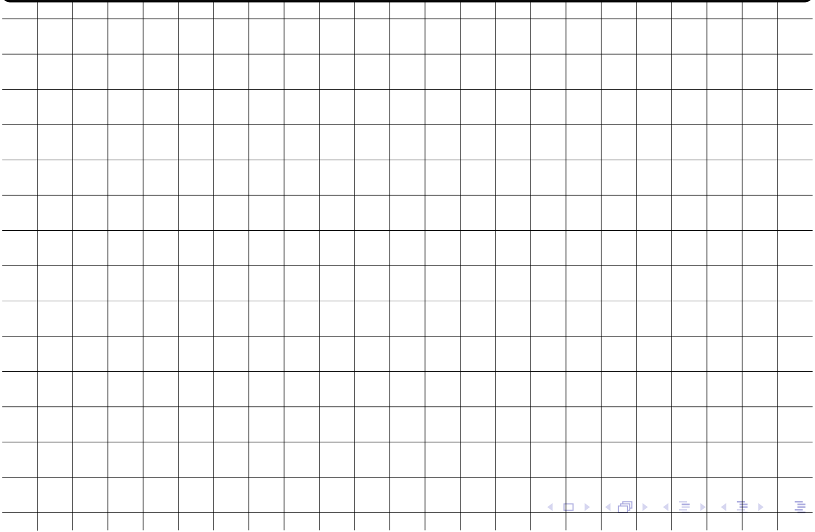
Lemma 6

The modular mapping $\sigma = (M, \vec{b})$ *is valid iff* $DS \cap \ker \sigma = \{\vec{0}\}$

☛ $\ker \sigma$ admissible lattice for DS.

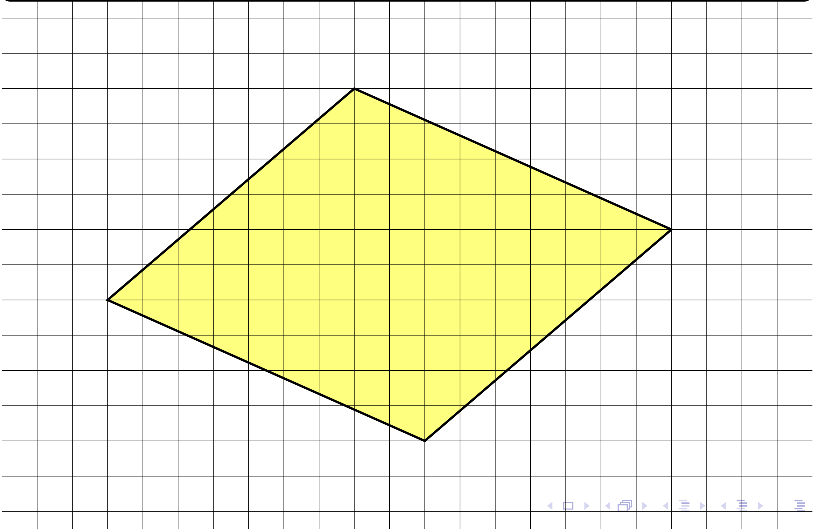
Critical and admissible lattices

Integer points

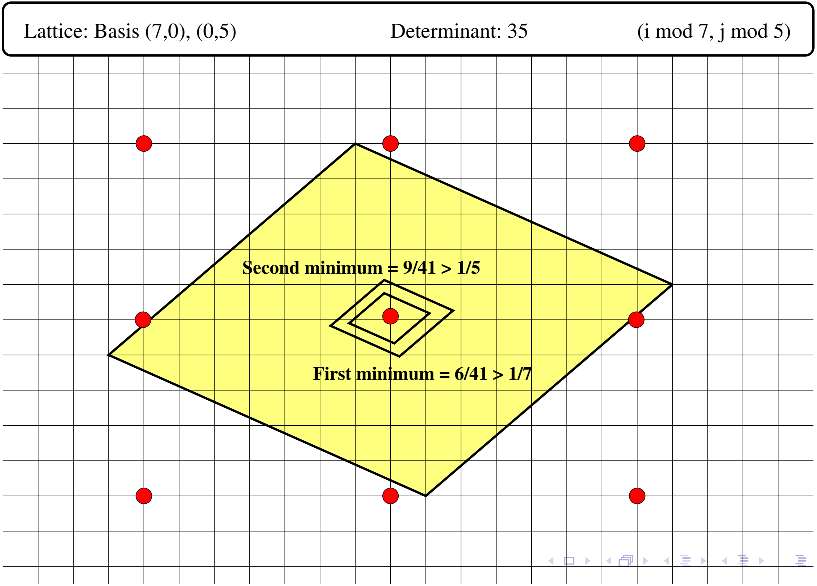


Critical and admissible lattices

0–Symmetric Polytope: vertices (8,1), (–8,–1), (–1,5), and (1,–5)

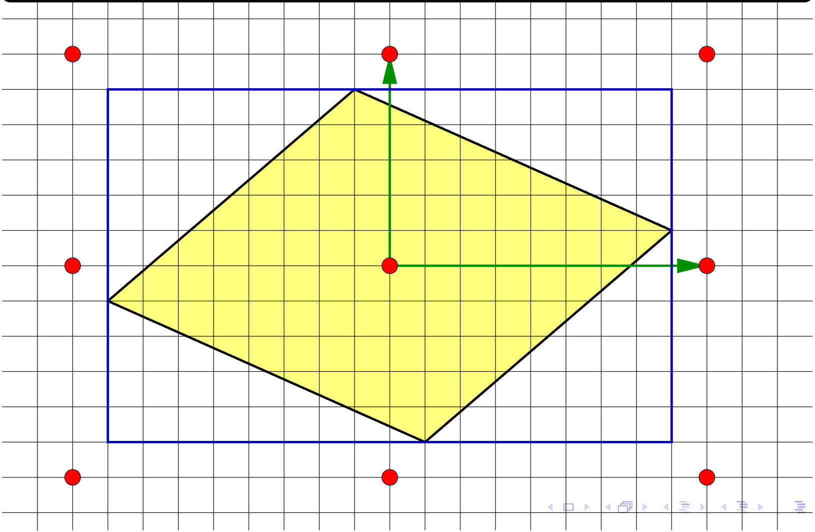


Critical and admissible lattices

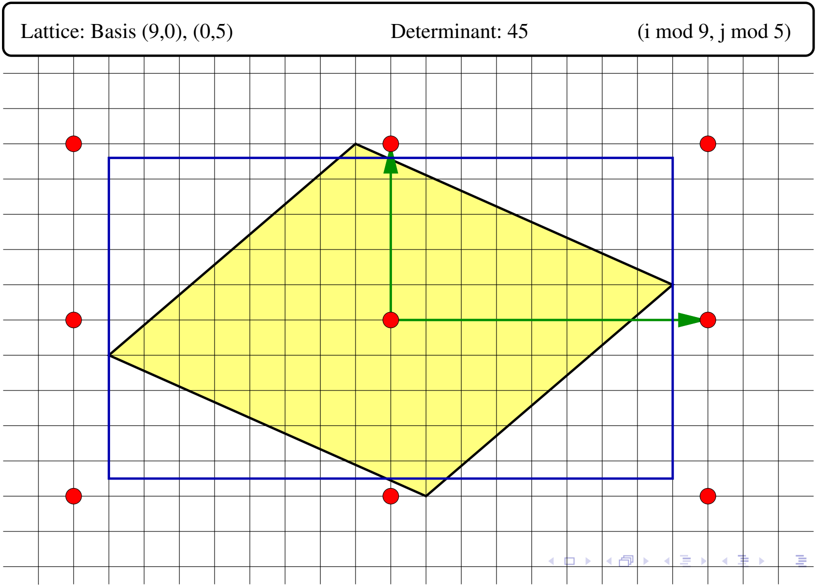


Critical and admissible lattices

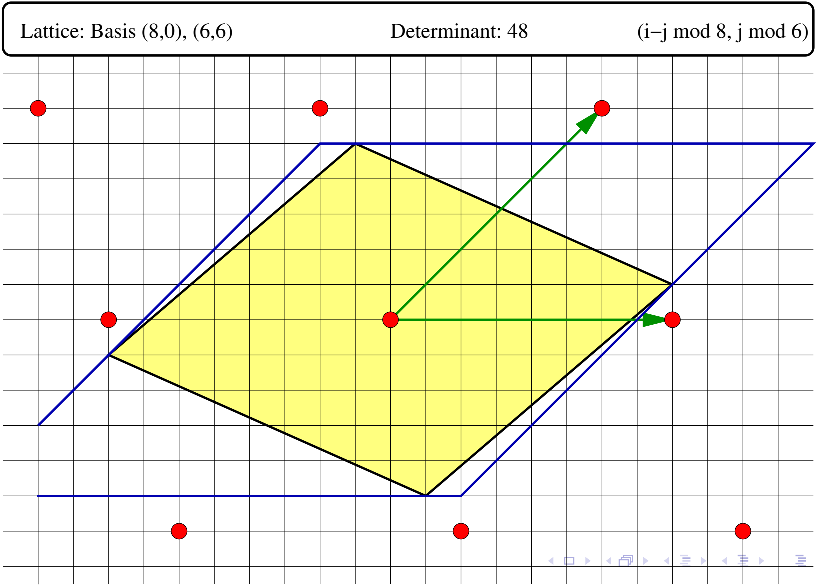
Lattice: Basis (9,0), (0,6) Determinant: 54 (i mod 9, j mod 6)



Critical and admissible lattices

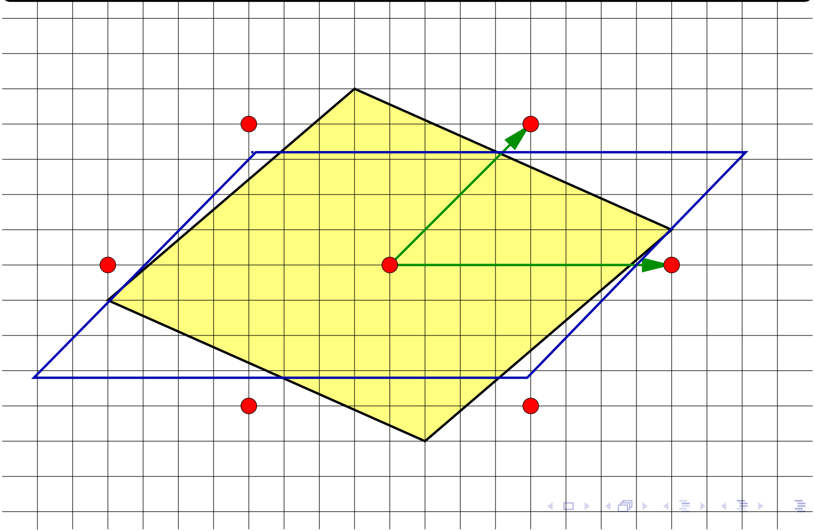


Critical and admissible lattices



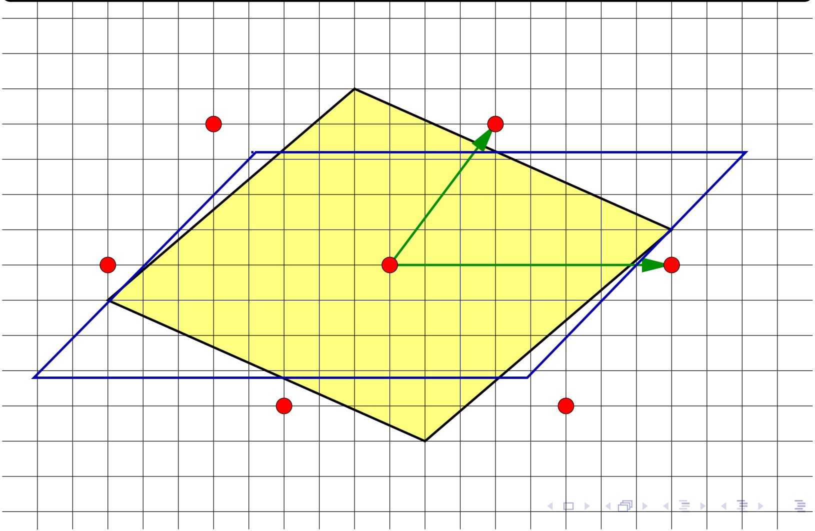
Critical and admissible lattices

Lattice: Basis (8,0), (4,4) Determinant: 32 (i−j mod 8, j mod 4)

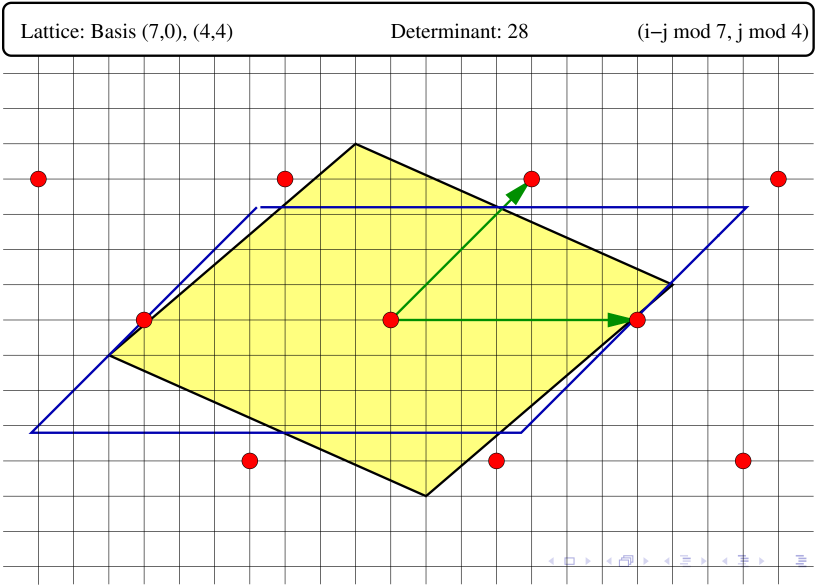


Critical and admissible lattices

Lattice: Basis (8,0), (3,4)	Determinant: 32	4i–3j mod 32

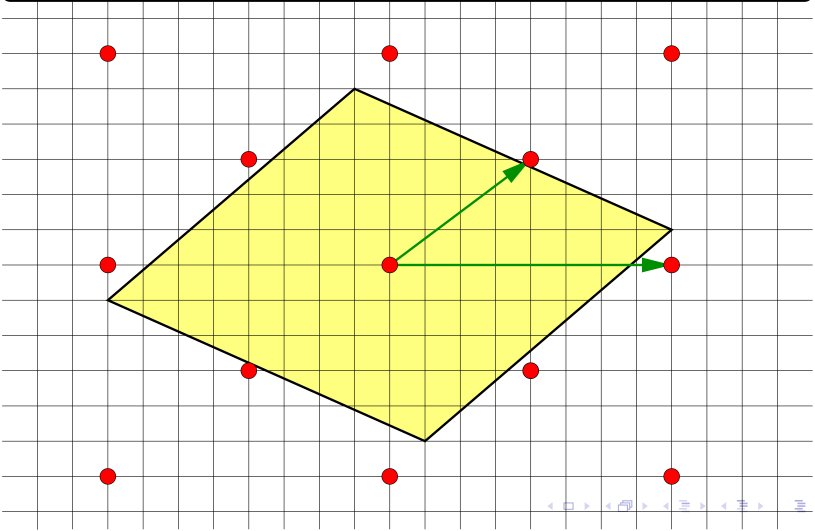


Critical and admissible lattices



Critical and admissible lattices

Critical Lattice: Basis (4,3), (8,0) Determinant: 24 3i–4j mod 24



Lattice-based memory allocation: process

1. Lifetime analysis of the array elements of A, w.r.t. θ.
2. Relation $\bowtie$: Build the polytope of conflicting differences.
3. Admissible lattice: Build an admissible Λ of small determinant.
4. Modulo function: Compute $\sigma = (M, \vec{b})$ such that $\ker \sigma = \Lambda$.
5. Code generation: Define new array A' and replace each occurrence of $A(\vec{i})$ with $A'(M\vec{i} \bmod \vec{b})$.

☛ Not a perfect scheme, does not reach minimal size, but: robust, expressed in terms of θ, usable with approximations.

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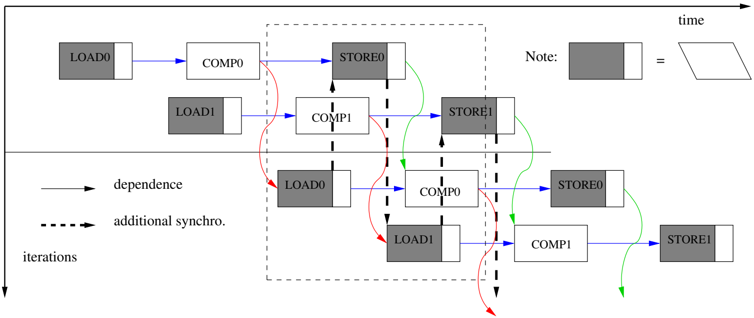
Source-to-source communication optimizations for HLS

Optimize DDR accesses for bandwidth-bound accelerators.

- Use tiling for data reuse and to enable burst communication.
- Use fine-grain software pipelining to pipeline DDR requests.
- Use double buffering to hide DDR latencies.
- Use coarse-grain software pipelining to hide computations.



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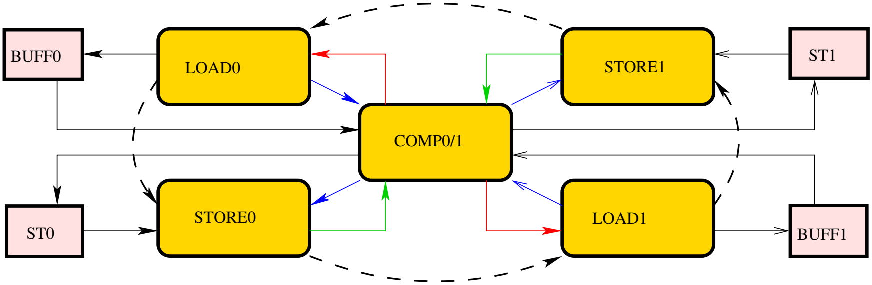
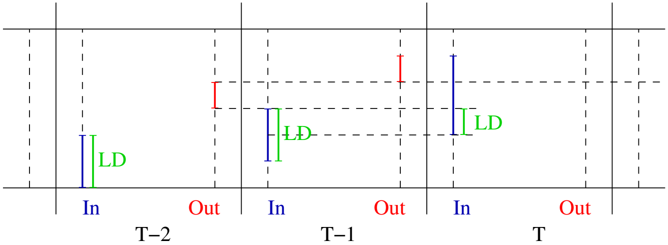


Overview of the method (for C2H Altera HLS tool)

Derive automatically C2H-compliant C functions for the pipelined accelerators: load, store, and compute. Blocks are obtained by loop tiling, pipelined in a "double-buffering" scheme.

1. Communication coalescing: prefetches data out of tile, following rows, and exploits data reuse.
   - Array access analysis.
   - Tiling + software pipelining = schedule θ.
2. Local memory management: defines memory elements, reduces size, and computes access functions.
   - Lifetime analysis w.r.t. θ.
   - Lattice-based memory reduction of intermediate buffers.
3. Code generation: generates final C code in a linearized form while optimizing accesses to the DDR.
   - Placement of FIFO synchronizations.
   - Boulet-Feautrier's method for polytope scanning.

Formalization of valid, exact, and approximated load



Valid load

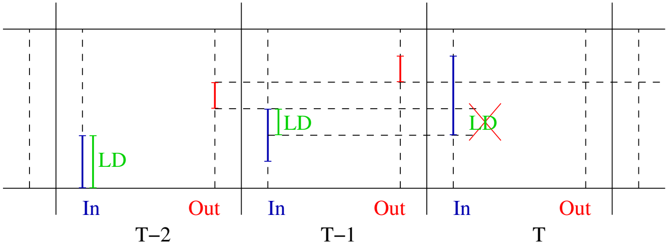
(i) Load at least what is needed but not previously produced:

$$\cup_{t\leq T}\left\{\mathrm{In}(t)\setminus \mathrm{Out}(t' < t)\right\} \subseteq \mathrm{Load}(t \leq T)$$

(ii) Do not overwrite locally-defined data:

$$\mathrm{Out}(t < T) \cap \mathrm{Load}(T) = \emptyset$$

Formalization of valid, exact, and approximated load



Exact load

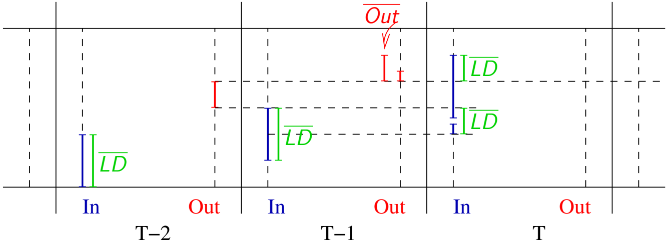
(i) Load exactly what is needed but not previously produced:

$$\forall T, \cup_{t \leq T} \{\mathrm{In}(t) \setminus \mathrm{Out}(t' < t)\} = \mathrm{Load}(t \leq T)$$

(ii) All loads should be disjoint (no redundant transfers):

$$\mathrm{Load}(T) \cap \mathrm{Load}(T') = \emptyset, \forall T \neq T'$$

Formalization of valid, exact, and approximated load



Valid approximated load

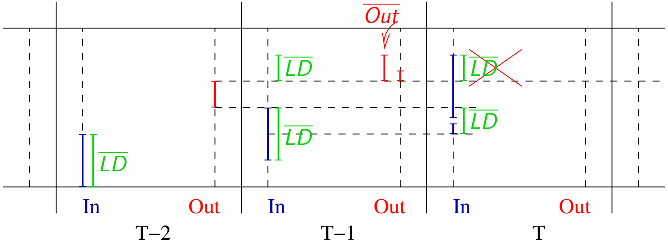
(i) Load at least the exact amount of data:

$$\cup_{t \leq T} \{\overline{\mathrm{In}}(t) \setminus \underline{\mathrm{Out}}(t' < t)\} \subseteq \mathrm{Load}(t \leq T)$$

(ii) Do not overwrite possibly locally-defined data:

$$\overline{\mathrm{Out}}(t < T) \cap \mathrm{Load}(T) = \emptyset$$

Formalization of valid, exact, and approximated load



Valid approximated load

(i) Load at least the exact amount of data:

$$\cup_{t \leq T} \left\{ \overline{\mathrm{In}}(t) \setminus \underline{\mathrm{Out}}(t' < t) \right\} \subseteq \mathrm{Load}(t \leq T)$$

(ii) Do not overwrite possibly locally-defined data:

$$\overline{\mathrm{Out}}(t < T) \cap \mathrm{Load}(T) = \emptyset$$

Handling approximations of data accesses

Exact situation:

$$\begin{aligned}
\mathrm{Load}(T) &= \mathrm{In}(T) \setminus \{\mathrm{In}(t < T) \cup \mathrm{Out}(t < T)\} \\
&= \mathrm{FirstReadBeforeWrite} \cap T \\
\mathrm{Store}(T) &= \mathrm{Out}(T) \setminus \mathrm{Out}(t > T) = \mathrm{LastWrite} \cap T
\end{aligned}$$

Approximated situation:

$$\begin{aligned}
\mathrm{Load}(T) &= \overline{\mathrm{In}}(T) \setminus \{\overline{\mathrm{In}}(t < T) \cup \overline{\mathrm{Out}}(t < T)\} \\
\mathrm{Store}(T) &= \overline{\mathrm{Out}}(T) \setminus \overline{\mathrm{Out}}(t > T)
\end{aligned}$$

Theorem 8 (Valid approximated load and store operators)

The previous load and store operators are valid, for any tile T:

(i) $\overline{\mathrm{Out}}(T) \subseteq \overline{\mathrm{In}}(t \leq T) \cup \overline{\mathrm{Out}}(t > T) \cup \underline{\mathrm{Out}}(t \leq T)$.

(ii) $\overline{\mathrm{In}}(T) \cap \{\overline{\mathrm{Out}}(t < T) \setminus \underline{\mathrm{Out}}(t < T)\} \subseteq \overline{\mathrm{In}}(t < T)$.

Possible solution: $\begin{cases} \overline{\mathrm{Out}}(T) \setminus \underline{\mathrm{Out}}(T) \subseteq \overline{\mathrm{In}}(T) \\ \overline{\overline{\mathrm{In}}}(T) = \cup_{t > T} \{\overline{\mathrm{In}}(t) \cap (\overline{\mathrm{Out}}(t' \leq T) \setminus \underline{\mathrm{Out}}(t' < t))\} \cup \overline{\mathrm{In}}(T) \end{cases}$

Outline

1. The polyhedral model
2. Scheduling, SURES, and approximated loops
3. Data mapping & communication optimizations
   - Lattice-based memory reduction
   - Communication optimizations for remote data
   - Conclusion

The polytope model: more than an exact representation

Discuss correctness and optimality with respect to a description.

- Parallelism detection with respect to dependence abstraction.
- More accurate for uniform dependences and Allen & Kennedy.
- Optimality in a class of functions.

Try to not assume that some information is exactly described, i.e., take into account approximations. Think conservative!

- Dependence and lifetime analysis.
- Array references and sets of data.
- Memory mapping and conflicts.
- Iteration domains? If conversion? Non-determinism?

☛ Approximating the control remains a major difficulty.

☛ Incorporate more techniques such as abstract interpretation.