Completion to chordal distance-hereditary graphs: a quartic vertex-kernel

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Abstract. Given a class of graphs $\mathcal{G}$ and a graph $G = (V, E)$, the aim of the $\mathcal{G}$-COMPLETION problem is to find a set of at most k non-edges whose addition in G results in a graph that belongs to $\mathcal{G}$. Completion to chordal or to natural subclasses of chordal graphs cover a broad range of classical NP-Complete problems, that have been extensively studied. When $\mathcal{G}$ coincides with the class of chordal graphs, the problem is the well-known MINIMUM FILL-IN problem. Other notable examples include completion to proper interval, threshold or trivially perfect graphs. Aforementioned problems are known to admit polynomial kernels, and it has been conjectured that completion to subclasses of chordal graphs further characterized by a finite number of forbidden induced subgraphs admit polynomial kernels. We investigate this line of research by considering completion to an important subclass of chordal graphs, namely chordal distance-hereditary graphs. Chordal distance-hereditary graphs are a natural generalization of trivially perfect graphs and have been extensively studied from the structural viewpoint. However, to the best of our knowledge, completion to chordal distance-hereditary graphs has not received attention so far. We thus initiate the first algorithmic study of this problem, and prove its NP-Completeness and that it admits a kernel with $O(k^4)$ vertices. To that aim, we rely on several known characterizations of chordal distance-hereditary graphs. In particular, such graphs admit a tree-like decomposition, so-called *clique laminar tree*. Unlike all aforementioned subclasses of chordal graphs, this decomposition does not correspond to a partition of the vertex set at hand. To circumvent this, we propose an approach based on the notion of clique (minimal) separator decomposition and a new characterization of chordal distance-hereditary graphs that might be of independent interest.

Keywords: Parameterized complexity · Kernelization algorithms · Chordal graphs · Distance-hereditary graphs

1 Introduction

Given a class of graphs $\mathcal{G}$ and a graph $G = (V, E)$, the aim of the parameterized $\mathcal{G}$-COMPLETION problem is to find a set of at most k non-edges whose addition in G results in a graph that belongs to $\mathcal{G}$. Completion problems cover a broad range of NP-Complete problems [15, 18, 19, 28, 31, 37] and have been extensively studied in the last decades. When $\mathcal{G}$ coincides with the class of chordal graphs, this is the well-known Minimum Fill-In problem [18, 37]. Minimum Fill-In has been tackled from the parameterized complexity viewpoint by Kaplan et al. [24] who gave both parameterized and kernelization algorithms. Following this line of research, many completion problems towards subclasses of chordal graphs have been studied [4, 5, 6, 13, 14, 21]. The motivation for the study of such completion problems mainly comes from practical applications, covering a wide range of fields such as bioinformatics, database management or artificial intelligence (see for instance [6]). Notable examples include completion to 3-leaf power, threshold or trivially perfect graphs, which are known to admit polynomial kernels [4, 14, 21]. In this work we consider an important subclass of chordal graphs, namely chordal distance-hereditary graphs [8, 25]. These graphs do not contain any induced cycle of length at least 4 (or *hole*) and are moreover distance-hereditary: the distances in every induced subgraph are the same than in the original graph. Such graphs are a natural generalization of chordal cographs (*i.e.* trivially perfect graphs) and contain all aforementioned classes. Moreover, they are known to admit a laminar structure [36], *i.e.* a tree-like decomposition that captures the subset relation on nonempty intersections of maximal cliques. Chordal distance-hereditary graphs have been extensively investigated from the structural viewpoint [2, 7, 9, 22, 23, 25, 32, 34]. However, to the best of our knowledge, completion to chordal distance-hereditary graphs has not received any attention so far. We thus initiate the algorithmic study of this problem, mainly from the parameterized complexity viewpoint.

Parameterized complexity. A parameterized problem Π is a problem whose input is a pair (I, k), where $k \in \mathbb{N}$ is called *parameter*. A parameterized problem Π is *fixed-parameter tractable* whenever any instance (I, k) of Π can be decided in time $f(k) \times p(|I|)$, where f is a computable function and p is a polynomial in the input size. A kernelization algorithm (*kernel* for short) for a parameterized problem Π is an algorithm that given any instance (I, k) of Π outputs in polynomial time an equivalent instance $(I', g(k))$ such that $|I'| \leqslant h(k)$ for some function h and $g(k) \leqslant k$. A kernel is said to be *polynomial* whenever h is a polynomial. It is well known that a parameterized problem Π is FPT if and only if it admits a kernel (see e.g. [17]). Formally, we consider the following problem:

Chordal Distance-Hereditary Completion
- **Input**: A graph $G = (V, E)$, $k \in \mathbb{N}$
- **Question**: Does there exist a set $F \subseteq (V \times V)$ of size at most k such that $H = (V, E \cup F)$ is chordal distance-hereditary?

Chordal distance-hereditary graphs are also known as *ptolemaic* graphs in the literature [25]. For the sake of readability, we will henceforth mainly refer to such graphs as *ptolemaic graphs* and consider the Ptolemaic Completion problem.

Related work. Cai [10] provided a dichotomy result for parameterized complexity of more general graph modification problems. Whenever the target graph class $\mathcal{G}$ can be characterized by finitely many *obstructions* (*i.e.* forbidden induced subgraphs), the corresponding modification problem (including $\mathcal{G}$-COMPLETION) is fixed-parameter tractable. While several polynomial kernels are known for graph modification problems [4, 5, 12, 14, 20, 21], there exist such problems that do not admit polynomial kernels [11, 20, 26]. When $\mathcal{G}$ is characterized by a single obstruction, several recent results towards a dichotomy have been obtained [1, 11, 30]. We refer the reader to [12, 17, 27] for recent surveys on the subject. Regarding (sub)classes of chordal graphs, Kaplan et al. [24] considered completion problems to chordal and proper interval graphs, providing a polynomial kernel for the former problem. Guo [21] provided several kernelization algorithms for completion problems towards split, threshold and trivially perfect graphs. A recent result of Drange and Pilipczuk extended the latter to the TRIVIALLY PERFECT EDITING problem [14]. Other examples of such kernels are the ones for 3-LEAF POWER COMPLETION [4] and PROPER INTERVAL COMPLETION [5]. Bessy and Perez [5] conjectured that completion problems to subclasses of chordal graphs further characterized by a finite number of obstructions admit polynomial kernels. In all aforementioned problems, the kernelization algorithms rely on the finite set of obstructions and on tree-like decompositions of the graph classes at hand that provide a *partition* of the vertex set which is exploited by reduction rules. However, the laminar structure of ptolemaic graphs is defined on intersections of maximal cliques and hence does not provide such a partition. This implies that standard techniques (such as the notion of *branches* [4, 5]) cannot be applied directly.

Our results. We prove that PTOLEMAIC COMPLETION is NP-Complete and admits a kernel with $O(k^4)$ vertices. Our method is inspired by the notion of clique (minimal) separator decomposition introduced by Tarjan [35]. This allows us to detect and reduce parts of the instance that are properly connected to the rest of the graph. This process can actually be reproduced on most previously mentioned kernelization algorithms for completion problems to subclasses of chordal graphs. This might bring new insights towards the design of kernelization algorithms for completion problems to other subclasses of chordal graphs.

Outline. We begin with preliminary definitions and results about ptolemaic graphs (Section 2). We then provide structural properties and a new decomposition theorem for such graphs (Section 3). We next describe the main structures that will be used (Section 4). Finally, we give our set of reduction rules (Section 5) and we conclude by bounding the size of a reduced instance (Section 6). Due to lack of space, results marked with ($\star$) are deferred to the appendix.

2 Preliminaries

We consider simple undirected graphs $G = (V, E)$ where V denotes the *vertex set* and E the *edge set* of G. We will sometimes use $V(G)$ and $E(G)$ to clarify the context. Given a vertex $u \in V$, $N_G(u)$ denotes the (open) *neighborhood* of u in G, that is $N_G(u) = \{v \in V : uv \in E\}$. We similarly define $N^i_G(u)$, $i \geqslant 2$, as the set of vertices at distance at most i from u in G. The closed neighborhood of u is defined as $N_G[u] = N_G(u) \cup \{u\}$. Two vertices u and v are *true twins* whenever $N_G[u] = N_G[v]$ and *false twins* whenever $uv \notin E$ and $N_G(u) = N_G(v)$. Given a subset $S \subseteq V$ of vertices, $N_G(S)$ is the set $\cup_{v\in S}(N_G(v) \setminus S)$. Similarly, $N^i_G(S)$, $i \geqslant 2$, is the set $\cup_{v\in S}(N^i_G(v) \setminus S)$. We moreover consider the closed neighborhoods $N_G[S]$ and $N^i_G[S]$ as natural extensions of the previous definitions. The *frontier* of S is defined as $\delta_G(S) = \{v \in S : N(v) \cap (V \setminus S) \neq \emptyset\}$. We omit the mention to graph G whenever the context is clear. The subgraph $G[S] = (S, E_S)$ *induced* by S is defined as $E_S = \{uv \in E : u \in S, v \in S\}$. For the sake of readability, given a subset $S \subseteq V$ we define $G \setminus S$ as $G[V \setminus S]$. A subset of vertices $C \subseteq V$ is a *connected component* of G if $G[C]$ is a maximal connected subgraph of G. We will sometimes refer to $G[C]$ as C. A *semi-split* of G is a subset $C \subseteq V$ such that the edges between $\delta_G(C)$ and $V \setminus C$ induce a complete bipartite graph. Let $G = (V, E)$ be a connected graph. A set $S \subseteq V$ is a *separator* of G if $G \setminus S$ is not connected. Given two vertices u and v of G, the separator S is a uv-separator if u and v lie in distinct connected components of $G \setminus S$. Moreover, S is a *minimal* uv-separator if no proper subset of S is a uv-separator. Finally, a separator S is *minimal* if there exists a pair $\{u, v\}$ such that S is a minimal uv-separator.

Ptolemaic graphs. We use a forbidden induced subgraph characterization of ptolemaic graphs [23] as well as a tree decomposition defined on intersections of maximal cliques [29, 36]. Hereafter, the gem is the graph on 5 vertices with an induced $P_4 = \{p_1, p_2, p_3, p_4\}$ and a universal vertex t.

Theorem 1 ([2, 23]). *The following conditions are equivalent:*

(i) *G is chordal distance-hereditary (or* ptolemaic*)*
(ii) *G does not contain any hole nor gem as an induced subgraph*
(iii) *Given two maximal cliques P, Q of G such that $P \cap Q \neq \emptyset$, $P \cap Q$ separates $P \setminus Q$ and $Q \setminus P$ in G.*
(iv) *G can be obtained from a single vertex by repeating the following operations: adding a degree-one vertex, a true twin to some vertex u or a false twin to some vertex v, in which case $N_G(v)$ must be a clique.*

An instance $(G = (V, E), k)$ of Ptolemaic Completion is a YES-instance whenever there exists a set $F \subseteq (V \times V)$ of size at most k such that $H = (V, E \cup F)$ is ptolemaic. The set F is called a k-completion of G (into a ptolemaic graph), and we will denote the resulting graph $H = G+F$. A *completion* refers to any set $F \subseteq (V \times V)$ such that $H = G+F$ is ptolemaic. Moreover, an *optimal completion* is a minimum-sized completion of G. A vertex is *affected* by a completion F whenever it is contained in some pair of F.

Lemma 1 ($\star$). Ptolemaic Completion *is NP-Complete.*

Clique laminar tree of ptolemaic graphs. We now describe a canonical tree representation for ptolemaic graphs due to Uehara and Uno [36].

Definition 1 ((Strong) Laminar family). *Let U be a universe and $\mathcal{F} \subseteq 2^{|U|}$ a family of subsets of U. The family $\mathcal{F}$ is* laminar *iff for all $A, B \in \mathcal{F}$, A and B are either disjoint or comparable by inclusion,* i.e. *either $A \cap B = \varnothing$, $A \subseteq B$ or $B \subseteq A$ holds. We say that $\mathcal{F}$ is a* strong *laminar family whenever there exists a set $A \in \mathcal{F}$ that contains all other sets of $\mathcal{F}$.*

Let U be a universe of n elements, and $\mathcal{F} \subseteq 2^{|U|}$ a family of subsets of U. Let $\vec{D}_\mathcal{F} = (V, A)$ be a directed graph where V contains a vertex x for every set $X \in \mathcal{F}$ and there is an arc from x to y in A if and only if $Y \subsetneq X$ (sets corresponding to y and x, respectively) and there does not exist $Z \in \mathcal{F}$ such that $Y \subsetneq Z \subsetneq X$. The digraph $\vec{D}_\mathcal{F}$ is called the *transitive reduction digraph* of $\mathcal{F}$. We use $D_\mathcal{F}$ to denote the underlying undirected graph. Moreover, if $\mathcal{F}$ is defined as a collection of subsets of vertices of a graph $G = (V, E)$, we refer to the vertices of $\vec{D}_\mathcal{F}$ as *bags* in order to avoid confusion with vertices of G. The notation t will henceforth denote a bag, while V_t will stand for the vertices of G contained in the set of $\mathcal{F}$ corresponding to t. Given a ptolemaic graph $G = (V, E)$, let $\mathcal{M}(G)$ be the set of maximal cliques of G and $\mathcal{C}(G)$ be the set of nonempty intersections of some maximal cliques of G. Notice in particular that $\mathcal{C}(G)$ contains the set $\mathcal{M}(G)$. Let $\mathcal{L}(G) = \mathcal{C}(G) \setminus \mathcal{M}(G)$ be the set of all nonempty intersections of at least two distinct maximal cliques of G. Any family $\mathcal{F}$ of sets of $\mathcal{L}(G)$ that are contained in a same maximal clique of $\mathcal{M}(G)$ is a laminar family [29, 36]. This leads to the following characterization.

Theorem 2 ([29, 36]). *A graph $G = (V, E)$ is ptolemaic iff $D_{\mathcal{C}(G)}$ is a tree.*

We say that $D_{\mathcal{C}(G)}$ is the *clique laminar tree* of G (see Figure 1). For the sake of readability, we will use $\vec{T}_G$ (resp. T_G) to denote $\vec{D}_{\mathcal{C}(G)}$ (resp. $D_{\mathcal{C}(G)}$).

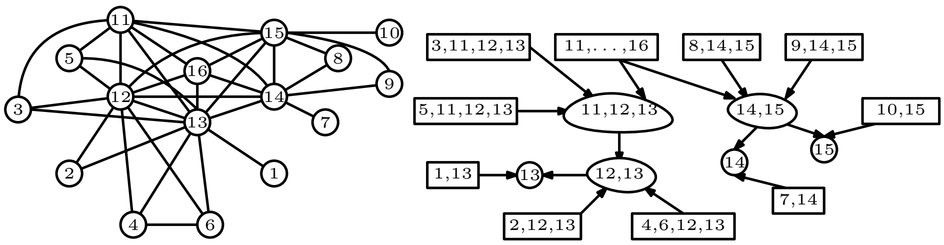

Fig. 1: A ptolemaic graph together with its laminar tree [36].

3 Structural properties of ptolemaic graphs

In this section, we provide a complete characterization of ptolemaic graphs in terms of laminar family and semi-splits (Theorem 3). This result was partially known to exist [16, 29, 36], but we present a stronger statement. The following observation will be useful when considering decomposition of ptolemaic graphs.

Observation 1 ($\star$) *Let $G = (V, E)$ be a chordal graph, $S \subseteq V$ a clique of G and $\{C_1, \cdots, C_\ell\}$, with $\ell \geq 1$, the set of connected components of $G \setminus S$. Then for every $1 \leqslant i \leqslant \ell$, $G[\delta(C_i)]$ is connected.*

Definition 2 (Footprint and trace). *Let $G = (V, E)$ be a graph, $S \subseteq V$ a clique of G and $\{C_1, \dots, C_\ell\}$ the set of connected components of $G \setminus S$. The* footprint *of C_i, $1 \leqslant i \leqslant \ell$, is defined as the family of sets:*

$$\Phi_S^G(C_i) = \{N(v) \cap S : \ v \in \delta(C_i)\} \cup \{N(\delta(C_i)) \cap S\}$$

The set $N(\delta(C_i)) \cap S$ is called the trace *of component C_i, and denoted $\tau_S^G(C_i)$.*

In both notations we omit the reference to the clique S or the graph G whenever the context is clear. Notice that $\{\tau(C_i)\} = \Phi(C_i)$ whenever C_i is a semi-split.

Definition 3 (Overlap and $\oslash$ notation). *Let U be a universe and A, B two subsets of U. The sets A and B overlap, denoted $A \oslash B$, whenever A and B are neither disjoint nor comparable by inclusion. Two families $\mathcal{A}, \mathcal{B} \subseteq 2^{|U|}$ of subsets of U overlap, denoted $\mathcal{A} \oslash \mathcal{B}$, if there exist $A \in \mathcal{A}$ and $B \in \mathcal{B}$ such that $A \oslash B$.*

For the sake of readability, we will abuse this notation for a set A and a family $\mathcal{B}$, that is $A \oslash \mathcal{B}$ rather than $\{A\} \oslash \mathcal{B}$.

Theorem 3 ($\star$). *Let $G = (V, E)$ be a graph, $S \subseteq V$ a clique of G and $\mathcal{C}$ the set of connected components of $G \setminus S$. The graph G is ptolemaic if and only if there is an order $\{C_1, \dots, C_\ell\}$ on the components of $\mathcal{C}$ such that:*

(i) $G[S \cup C_i]$ is ptolemaic for every $1 \leqslant i \leqslant \ell$
(ii) $\bigcup_{1 \leqslant i \leqslant \ell} \Phi_S(C_i)$ is a laminar family
(iii) C_i is a semi-split of G for every $1 \leqslant i \leqslant \ell - 1$
(iv) if C_ℓ is not a semi-split then:
- *there exists $v \in C_\ell$ such that $N(v) \cap S = \tau(C_\ell) = S$ and*
- *no set $\Gamma \in \Phi(C_\ell)$ satisfies $\Gamma \subsetneq \tau(C_i)$, $1 \leqslant i \leqslant \ell - 1$.*

Corollary 1 ($\star$). *Let $G = (V, E)$ be a graph, and $S \subseteq V$ a clique minimal separator or a maximal clique of G. Let $\mathcal{C} = \{C_1, \dots, C_\ell\}$ be the set of connected components of $G \setminus S$. The graph G is ptolemaic iff the conditions (i) and (ii) of Theorem 3 hold, and if condition (iii) of Theorem 3 holds for every $1 \leqslant i \leqslant l$ (*i.e. *condition (iv) of Theorem 3 does not occur).*

4 Decomposing the instance

We first give some known and new results about completions into chordal distance-hereditary graphs. The soundness of the following result comes from the fact that such graphs are hereditary and closed under true twin addition (Theorem 1 (iv)).

Lemma 2 ([4]). *Let $\mathcal{G}$ be an hereditary class of graphs closed under true twin addition. For every graph $G = (V, E)$, there exists an optimal completion F into a graph of $\mathcal{G}$ such that for any two maximal sets of true twins M and M' either $(M \times M') \subseteq F$ or $(M \times M') \cap F = \emptyset$.*

Lemma 3 ($\star$). *Let G be a graph and S a clique of $G = (V, E)$ such that any connected component C of $G \setminus S$ is a semi-split. Then, there exists an optimal completion F of G, with $H = G + F$, such that any connected component C' of $H \setminus S$ is a semi-split.*

Corollary 2. *Let $G = (V, E)$ be a graph and $S \subseteq V$ a clique of G. Let $S' \subseteq S$ be such that for every connected component C of $G \setminus S$ the following holds:*

(i) S' and $\Phi_S(C)$ do not overlap and
(ii) if C is not a semi-split, then no set $\Gamma \in \Phi_S(C)$ satisfies $\Gamma \subsetneq S'$.

Then there exists an optimal completion F of G such that for any connected component C' of $H \setminus S$, with $H = G + F$, the following conditions hold:

(i) S' and $\Phi^H_S(C')$ do not overlap and
(ii) if C' is not a semi-split, then no set $\Gamma \in \Phi^H_S(C')$ satisfies $\Gamma \subsetneq S'$.

Proof. Consider G, S and S' satisfying the conditions of Corollary 2. Then, G and S' satisfy the conditions of Lemma 3. Consider the optimal completion F given by Lemma 3 and let $H = G + F$. We show that H satisfies the conclusions of Corollary 2. To prove Conclusion (i), assume for a contradiction that there exists a component C' of $H \setminus S$ and a vertex $x \in C'$ such that $N_H(x) \cap S$ overlaps S'. Then, there exists $y \in (N_H(x) \cap S) \setminus S'$. As x and y are adjacent, they belong to the same component C'' of $H \setminus S'$. But y is adjacent to every vertex of S' while x is adjacent to some but not all of them, contradicting that C'' is a semi-split of $H \setminus S'$ (which holds from the conclusion of Lemma 3). Thus, Conclusion (i) of Corollary 2 holds. To show Conclusion (ii) of Corollary 2, observe that if C' is not a semi-split, then, from Theorem 3 applied with clique S, there exists a vertex $x \in C'$ that is adjacent to all the vertices of S. Therefore, there cannot exist some vertex $y \in C'$ such that $N_H(y) \cap S \subsetneq S'$, as this would imply that the component C'' of $H \setminus S'$ that contains C' is not a semi-split, contradicting the conclusion of Lemma 3. Thus, Conclusion (ii) of Corollary 2 holds. □

4.1 Clams and tentacles

We now introduce the main structures that will be considered by our kernelization algorithm. We consider parts of the graph that are *properly* connected to the rest of the graph (in terms of overlap, Definition 3).

Definition 4 (Clam). *Let $G = (V, E)$ be a graph, $S \subseteq V$ a clique separator of G and $\mathcal{C} = \{C_1, \ldots, C_p\}$, with $p \geq 1$, a maximal collection of connected components of $G \setminus S$ such that:*

(i) $G[S \cup C_i]$ *is ptolemaic,* $1 \leqslant i \leqslant p$
(ii) C_i *is a semi-split of* G*,* $1 \leqslant i \leqslant p$
(iii) $\tau(C_i) = \tau(C_j)$ *for every* $1 \leqslant i < j \leqslant p$

*The collection $\mathcal{C}$ is called an S-*clam *of G.*

Let $G = (V, E)$ be a graph, $S \subseteq V$ a clique separator of G and $\mathcal{C} = \{C_1, \ldots, C_p\}$ an S-clam of G. We let $V(\mathcal{C}) = \cup_{i=1}^{p} C_i$, and $\delta(\mathcal{C}) = \cup_{i=1}^{p} \delta(C_i)$. The set $\delta(\mathcal{C})$ is called the *frontier* of the S-clam $\mathcal{C}$. We first prove that there always exists an optimal completion that affects only (and uniformly) the frontier of a given clam.

Lemma 4 ($\star$). *Let $G = (V, E)$ be a graph, $S \subseteq V$ a clique separator of G and $\mathcal{C} = \{C_1, \ldots, C_p\}$ an S-clam of G. Any inclusion-minimal completion F of G into a ptolemaic graph satisfies the two following properties:*

(i) if F contains a pair $\{u, v\}$ where $u \in V(\mathcal{C})$ then $u \in \delta(\mathcal{C})$ and $v \in V \setminus V(\mathcal{C})$
(ii) if F contains a pair $\{u, v\}$, where $u \in \delta(\mathcal{C})$ and $v \in V \setminus V(\mathcal{C})$, then F contains all pairs $\{u', v\}$ with $u' \in \delta(\mathcal{C})$.

We now give a reduction rule to deal with *clams*. This is needed to define properly the other structures considered by our kernelization algorithm. As we shall see, using Lemma 4 on clique *minimal* separators will ease the polynomial-time application of our reduction rules.

Rule 1 (Clams) *Let $S \subseteq V$ be a clique minimal separator of G and $\mathcal{C} = \{C_1, \ldots, C_p\}$ an S-clam of G. Replace $V(\mathcal{C})$ by a clique C_S of size* $\min(k + 1, |\delta(\mathcal{C})|)$ *having the same neighborhood as $\delta(\mathcal{C})$.*

Lemma 5 ($\star$). *Rule 1 is sound.*

In the remaining of this section, we assume that the instance at hand is reduced under Rule 1. In particular, this means that any clam of the given instance contains exactly one connected component. In order to avoid confusion, we henceforth refer to such clams as *tentacles*.

Definition 5 (Tentacle). *Let $(G = (V, E), k)$ be an instance of* Ptolemaic Completion *reduced under Rule 1 and $S \subseteq V$ be a clique separator of G. Let $\mathcal{C}$ be an S-clam. Since G is reduced under Rule 1, $\mathcal{C}$ contains exactly one connected component C. This component is called an S-*tentacle *of G.*

In a slight abuse of notation we will henceforth identify a given S-tentacle $\mathcal{C}$ by its only connected component C. We refine the notion of tentacles by considering different *types* of such structures according to hypotheses of Corollary 2.

Definition 6. *Let $(G = (V, E), k)$ be an instance of* Ptolemaic Completion *reduced under Rule 1, $S \subseteq V$ a clique separator of G and C an S-tentacle of G. If there does not exist a component C' of $G \setminus (S \cup C)$ such that $\Phi(C) \oslash \Phi(C')$ then C is a:*

(i) ***type-α S-tentacle*** *if for every component C'' of $G \setminus (S \cup C)$ that is not a semi-split, no set $\Gamma \in \Phi(C'')$ satisfies $\Gamma \subsetneq \tau(C)$.*
(ii) ***type-β S-tentacle*** *if there exists a component C'' of $G \setminus (S \cup C)$ that is not a semi-split and a set $\Gamma \in \Phi(C'')$ such that $\Gamma \subsetneq \tau(C)$.*

Otherwise C is a ***type-γ S-tentacle****.*

5 Reducing the instance

We now provide the set of reduction rules that will constitute our kernelization algorithm. For the sake of readability, we assume in the remaining of this section that we are given an instance $(G = (V, E), k)$ of Ptolemaic Completion.

Rule 2 *Let $C \subseteq V$ be a subset of vertices such that $G[C]$ is a ptolemaic connected component of G. Remove C from G.*

Rule 3 *Let $\{X_1, \ldots, X_p\}$, $p > k$, be a set of distinct induced cycles of length 4 having a non-edge $\{u, v\}$ in common. Add the pair $\{u, v\}$ to the solution and decrease k by 1.*

The soundness of the following rule comes from Lemma 2 since ptolemaic graphs are hereditary and closed under true twin addition (Theorem 1 (iv)).

Rule 4 *Let $C \subseteq V$ be a set of true twins of G such that $|C| > k + 1$. Remove $|C| - (k + 1)$ arbitrary vertices in C from G.*

Definition 7 (Gem-breaker). *Let $G = (V, E)$ be a graph and $X = \{t, p_1, p_2, p_3, p_4\}$ an induced gem of G. Each of the two pairs $\{p_1, p_3\}$ and $\{p_2, p_4\}$ is called a* gem-breaker *of X.*

Definition 8 (Gem-sunflower). *A collection $\mathcal{X} = \{X_1, \ldots, X_p\}$ of induced gems with $X_i \subseteq V$, $1 \leqslant i \leqslant p$, is a* gem-sunflower *if $p > k$ and:*

(i) there exist $u, v \in \cap_{i=1}^{p} X_i$ such that $\{u, v\}$ is a gem-breaker of each X_i, and
(ii) X_i and X_j do not share a gem-breaker $\{u', v'\} \neq \{u, v\}$, $1 \leqslant i < j \leqslant p$.

The gem-breaker $\{u, v\}$ is called the center *of $\mathcal{X}$.*

Rule 5 *Let $\mathcal{X} = \{X_1, \ldots, X_p\}$ be a gem-sunflower. Add the center of $\mathcal{X}$ to the solution and decrease k by 1.*

Lemma 6 ($\star$). *Rules 2 to 5 are sound and can be applied in polynomial time.*

5.1 Fishing and eating the seafood

We now turn our attention to reduction rules that consider structures defined Section 4.1. In the remaining of this section we assume that we are given an instance $(G = (V, E), k)$ of Ptolemaic Completion reduced under Rule 1.

Rule 6 (Type-α tentacles) *Let $S \subsetneq V$ be a clique minimal separator of G, and $C \in G \setminus S$ be a type-α S-tentacle of G. Remove C from G.*

Rule 7 (Type-β tentacles) *Let $S \subseteq V$ be a clique minimal separator of G and $\mathcal{C} = \{C_1, \dots, C_p\}$, $p \geqslant 2k+1$, be a set of type-β S-tentacles. If there exist a non semi-split component C' of $G \setminus S$ and a set $\Gamma \in \Phi(C')$ with $\Gamma \subsetneq \tau(C_i)$ for every $1 \leqslant i \leqslant p$, then add all pairs of $(\delta(C') \times \tau(C')) \setminus E$ to the solution and decrease k accordingly.*

Rule 8 *Let $S \subseteq V$ be a clique minimal separator of G such that:*

1. *S separates G into exactly two connected components C_1 and C_2 and*
2. *$G[S \cup N^2_G(S)]$ is ptolemaic and*
3. *$\forall i \in \{1, 2\}$, there exist a clique minimal separator $S_i \subseteq C_i \setminus N^2_G(S)$ of $G[C_i]$ and a connected component C'_i of $G[C_i] \setminus S_i$ such that $N^2_G(S) \cap C_i \subseteq C'_i$ and C'_i is an S_i-clam.*

Then, remove S from G.

Lemma 7 ($\star$). *Rules 6 to 8 are sound.*

We now state that above reductions rules can be applied in polynomial time. We rely on the following result.

Theorem 4 ([3, 35]). *Given a graph $G = (V, E)$, a clique minimal separator decomposition can be obtained in $O(nm)$ time. Moreover, all clique minimal separators are used in the decomposition.*

We would like to mention that our objective here is to prove that all reduction rules can be applied in polynomial time. In particular, we do not give the explicit running time of our algorithms nor try to optimize them.

Lemma 8 ($\star$). *Rules 1, 6, 7 and 8 can be applied in polynomial time.*

6 Bounding the size of reduced instances

We are now ready to bound the size of reduced instances (G = (V,E), k) of Ptolemaic Completion. We need the following results.

Lemma 9 (⋆). *Let $(G = (V, E), k)$ be a YES-instance of* Ptolemaic Completion *reduced under Rules 1 and Rules 5 to 7. Let $S \subsetneq V$ be a clique separator of G and $\mathcal{C} = \{C_1, \ldots, C_\ell\}$ the connected components of $G \backslash S$. Then $|\mathcal{C}| = O(k^2)$.*

Lemma 10 (⋆). *Let $G = (V, E)$ be a connected ptolemaic graph without true twins. Let $\vec{T}_G$ be its clique laminar tree with $\left|V(\vec{T}_G)\right| = p$. Then $|V(G)| \leqslant p$.*

Theorem 5 (⋆). Ptolemaic Completion *admits a kernel with $O(k^4)$ vertices.*

Proof (sketch). We say that an instance $(G = (V, E), k)$ of Ptolemaic Completion is *reduced* whenever none of the described reduction rules can be applied to G. Let $(G = (V, E), k)$ be a reduced YES-instance of Ptolemaic Completion, and F a k-completion of G. Let $\{C_1, \ldots, C_c\}$ denote the connected components of G. We work on the ptolemaic graph $H = G + F$ with connected components $\{H_1, \ldots, H_c\}$. Since G is reduced under Rule 2, we know that $c \leqslant k$. We assume that $F = \cup_{i=1}^{c} F_i$ with $H_i = G[C_i] + F_i$ and $|F_i| = k_i$ for $1 \leqslant i \leqslant c$. By Theorem 2, let $\vec{T}_H$ be the clique laminar forest of H, and $\vec{T}_{H_i}$ be the clique laminar tree of H_i, $1 \leqslant i \leqslant c$. Let $\mathcal{F}_i$ be the set of all *filled bags* of $\vec{T}_{H_i}$, that is $\mathcal{F}_i = \{t \in V(\vec{T}_{H_i}) : \exists \{u, v\} \in F, \{u, v\} \subseteq V_t\}$ and $\mathcal{F} = \cup_{i=1}^{c} \mathcal{F}_i$.

Claim 1. For every $1 \leqslant i \leqslant c$, $|\mathcal{F}_i| \leqslant k_i \cdot (6k - 1)$. Hence $|\mathcal{F}| \leqslant 6k^2 - k$.

Tentacles. Let T_i denote a minimal tree spanning vertices of $\mathcal{F}_i$ in T_{H_i}, $1 \leqslant i \leqslant c$. Let $\mathcal{S}_i$ be the set of maximal subtrees of $T_{H_i} \setminus T_i$. We will count the bags of $\mathcal{S}_i$ together with $\mathcal{A}_i$, the set of bags of $T_i \setminus \mathcal{F}_i$ containing at least one affected vertex.

Claim 2. $|V(\mathcal{S}_i) \cup \mathcal{A}_i| = O(k_i \cdot k^2)$ with $V(\mathcal{S}_i)$ the set of bags of subtrees in $\mathcal{S}_i$.

Let $\mathcal{R}_i^{\geqslant 3}$ be the set of bags of $T_i \setminus \mathcal{F}_i$ having degree at least 3 in T_i. One can see that $\left|\mathcal{R}_i^{\geqslant 3}\right| \leqslant k_i$. Finally, let P be a path of $T_i \setminus (\mathcal{F}_i \cup \mathcal{R}_i^{\geqslant 3} \cup \mathcal{A}_i)$. Since G is reduced under Rule 8, one can see that $|V(P)| \leqslant 15$ where $V(P)$ denotes the bags of P. To conclude the proof, notice that by construction, any bag of $\vec{T}_H$ is either in $\mathcal{F}$, or in the set $V_i = V(T_i) \setminus \mathcal{F}_i$ of bags of the minimal tree T_i spanning $\mathcal{F}_i$ or in a maximal subtree of $T_{H_i} \setminus T_i$ for some $1 \leqslant i \leqslant c$. We thus obtain:

$$\left|V(\vec{T}_H)\right| = \left|\mathcal{F} \bigcup (\cup_{i=1}^{c} V(\mathcal{S}_i)) \bigcup (\cup_{i=1}^{c} V_i)\right| \leqslant O(k^2) + \sum_{i=1}^{c} O(k_i \cdot k^2) + \sum_{i=1}^{c} (k_i + O(k_i \cdot k^2) + 15k_i)$$

In turn, since $\sum_{i=1}^{c} k_i = k$, we have $\left|V(\vec{T}_H)\right| \in O(k^3)$. By Lemma 10 and Rule 4, we thus conclude that G contains $O(k^4)$ vertices. Since all reduction rules can be applied in polynomial time (Lemmata 6 and 8), the result follows. □

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Bibliography

[1] Aravind, N., Sandeep, R., Sivadasan, N.: Dichotomy results on the hardness of H-free edge modification problems. SIAM Journal on Discrete Mathematics **31**(1), 542–561 (2017)

[2] Bandelt, H.J., Mulder, H.M.: Distance-hereditary graphs. Journal of Combinatorial Theory, Series B **41**(2), 182–208 (1986)

[3] Berry, A., Pogorelcnik, R., Simonet, G.: An introduction to clique minimal separator decomposition. Algorithms **3**(2), 197–215 (2010)

[4] Bessy, S., Paul, C., Perez, A.: Polynomial kernels for 3-leaf power graph modification problems. Discrete Applied Mathematics **158**(16), 1732–1744 (2010). https://doi.org/10.1016/j.dam.2010.07.002, `https://doi.org/10.1016/j.dam.2010.07.002`

[5] Bessy, S., Perez, A.: Polynomial kernels for proper interval completion and related problems. Information and Computation **231**, 89–108 (2013)

[6] Bliznets, I., Cygan, M., Komosa, P., Pilipczuk, M., Mach, L.: Lower bounds for the parameterized complexity of minimum fill-in and other completion problems. ACM Transactions on Algorithms (TALG) **16**(2), 1–31 (2020)

[7] Brandstädt, A., Hundt, C.: Ptolemaic graphs and interval graphs are leaf powers. In: Latin American Symposium on Theoretical Informatics. pp. 479–491. Springer (2008)

[8] Brandstädt, A., Le, V.B., Spinrad, J.P.: Graph Classes: A Survey. SIAM (1999)

[9] Brešar, B., Changat, M., Klavžar, S., Kovše, M., Mathews, J., Mathews, A.: Cover-incomparability graphs of posets. Order **25**(4), 335–347 (2008)

[10] Cai, L.: Fixed-parameter tractability of graph modification problems for hereditary properties. Information Processing Letters **58**(4), 171–176 (1996)

[11] Cai, L., Cai, Y.: Incompressibility of h-free edge modification problems. Algorithmica **71**(3), 731–757 (2015)

[12] Crespelle, C., Drange, P.G., Fomin, F.V., Golovach, P.A.: A survey of parameterized algorithms and the complexity of edge modification. CoRR **abs/2001.06867** (2020), `https://arxiv.org/abs/2001.06867`

[13] Drange, P.G., Dregi, M.S., Lokshtanov, D., Sullivan, B.D.: On the threshold of intractability. In: Algorithms-ESA 2015. pp. 411–423. Springer (2015)

[14] Drange, P.G., Pilipczuk, M.: A polynomial kernel for trivially perfect editing. Algorithmica **80**(12), 3481–3524 (2018)

[15] El-Mallah, E.S., Colbourn, C.J.: The complexity of some edge deletion problems. IEEE transactions on circuits and systems **35**(3), 354–362 (1988)

[16] Fagin, R.: Degrees of acyclicity for hypergraphs and relational database schemes. Journal of the ACM (JACM) **30**(3), 514–550 (1983)

[17] Fomin, F.V., Lokshtanov, D., Saurabh, S., Zehavi, M.: Kernelization: theory of parameterized preprocessing. Cambridge University Press (2019)

[18] Garey, M.R., Johnson, D.S.: Computers and intractability, vol. 29. wh freeman New York (2002)

[19] Golumbic, M.C., Kaplan, H., Shamir, R.: On the complexity of DNA physical mapping. Advances in Applied Mathematics **15**(3), 251–261 (1994)

[20] Guillemot, S., Havet, F., Paul, C., Perez, A.: On the (non-)existence of polynomial kernels for P_l-free edge modification problems. Algorithmica **65**(4), 900–926 (2013). https://doi.org/10.1007/s00453-012-9619-5, `https://doi.org/10.1007/s00453-012-9619-5`

[21] Guo, J.: Problem kernels for NP-complete edge deletion problems: Split and related graphs. In: Tokuyama, T. (ed.) ISAAC. Lecture Notes in Computer Science, vol. 4835, pp. 915–926. Springer (2007)

[22] Howorka, E.: On metric properties of certain clique graphs. Journal of Combinatorial Theory, Series B **27**(1), 67–74 (1979)

[23] Howorka, E.: A characterization of ptolemaic graphs. Journal of Graph Theory **5**(3), 323–331 (1981)

[24] Kaplan, H., Shamir, R., Tarjan, R.E.: Tractability of parameterized completion problems on chordal, strongly chordal, and proper interval graphs. SIAM Journal on Computing **28**(5), 1906–1922 (1999)

[25] Kay, D.C., Chartrand, G.: A characterization of certain ptolemaic graphs. Canadian Journal of Mathematics **17**, 342–346 (1965)

[26] Kratsch, S., Wahlström, M.: Two edge modification problems without polynomial kernels. In: International Workshop on Parameterized and Exact Computation. pp. 264–275. Springer (2009)

[27] Liu, Y., Wang, J., Guo, J.: An overview of kernelization algorithms for graph modification problems. Tsinghua Science and Technology **19**(4), 346–357 (2014)

[28] Mancini, F.: Graph modification problems related to graph classes. PhD degree dissertation, University of Bergen Norway **2** (2008)

[29] Markenzon, L., Waga, C.F.E.M.: New results on ptolemaic graphs. Discrete Applied Mathematics **196**, 135–140 (2015)

[30] Marx, D., Sandeep, R.B.: Incompressibility of h-free edge modification problems: Towards a dichotomy. In: Grandoni, F., Herman, G., Sanders, P. (eds.) 28th Annual European Symposium on Algorithms, ESA 2020, September 7-9, 2020, Pisa, Italy (Virtual Conference). LIPIcs, vol. 173, pp. 72:1–72:25. Schloss Dagstuhl - Leibniz-Zentrum für Informatik (2020). https://doi.org/10.4230/LIPIcs.ESA.2020.72

[31] Natanzon, A., Shamir, R., Sharan, R.: Complexity classification of some edge modification problems. Discrete Applied Mathematics **113**(1), 109–128 (2001)

[32] Nieminen, J.: The center and the distance center of a ptolemaic graph. Operations research letters **7**(2), 91–94 (1988)

[33] Sharan, R.: Graph modification problems and their applications to genomic research. Ph.D. thesis, Tel-Aviv University (2002)

[34] Takahara, Y., Teramoto, S., Uehara, R.: Longest path problems on ptolemaic graphs. IEICE transactions on information and systems **91**(2), 170–177 (2008)

[35] Tarjan, R.E.: Decomposition by clique separators. Discrete Mathematics **55**, 221–232 (1985)

[36] Uehara, R., Uno, Y.: Laminar structure of ptolemaic graphs with applications. Discrete Applied Mathematics **157**(7), 1533–1543 (2009)
[37] Yannakakis, M.: Computing the minimum fill-in is NP-complete. SIAM Journal on Algebraic Discrete Methods **2**(1), 77–79 (1981)

A Proofs of Section 2

Proof (of Lemma 1). Notice first that the problem is in NP since it can be verified in polynomial time whether a given set of non-edges of any instance (G = (V,E), k) is a k-completion using any polynomial-time recognition algorithm for such graphs [36]. We now give a reduction from the known NP-Complete Trivially Perfect Completion problem [33, 37]. A graph is *trivially perfect* iff it does not contain any C_4 nor P_4 as an induced subgraph. Given an instance $(G = (V, E), k)$ of Trivially Perfect Completion we compute an instance $(G' = (V', E'), k)$ of Ptolemaic Completion by adding a universal vertex to G.

We claim that G admits a k-completion into a trivially perfect graph if and only if G' admits a k-completion into a ptolemaic graph. Assume first that there exists a k-completion F of G into a trivially perfect graph. By definition, $G + F$ is chordal and does not contain any P_4 as an induced subgraph. Now consider the graph $G' + F$: since $G'[V]$ is trivially perfect, and since $V' \setminus V = \{u\}$ is a universal vertex, $G' + F$ is chordal and cannot contain any induced gem, and is hence ptolemaic (Theorem 1 (ii)). On the other hand, let F' be a k-completion of G' into a ptolemaic graph. Since $G' + F$ is chordal and does not contain any induced gem, it follows that $G'[V] + F = G + F$ is chordal and does not contain any induced P_4, and is hence trivially perfect. □

Proof (of Observation 1). Assume for a contradiction that there exists $1 \leqslant i \leqslant l$ such that $G[\delta(C_i)]$ is not connected. Then, there exist two vertices u, v of $\delta(C_i)$ such that $uv \notin E$, and $P_{uv} = \{u, \pi_{uv}, v\}$ an induced path from u to v with π_{uv} in $V(G[C_i \setminus \delta(C_i)])$. If $N(u) \cap N(v) \cap S = \varnothing$, choose $s_u \in N(u) \cap S$ and $s_v \in N(v) \cap S$, otherwise choose $s_u \in N(u) \cap N(v) \cap S$ and $s_v = s_u$. In both cases, the set $\{s_u, u, \pi_{uv}, v, s_v\}$ induces a cycle of length at least 4 in G, leading to a contradiction. □

Proof (of Theorem 3). We will need the following results.

Lemma 11. *Let $G = (V, E)$ be a ptolemaic graph and $S \subsetneq V$ a clique of G. For any connected component C of $G \setminus S$, the family $\mathcal{F}_C = \{N(x) \cap S \ : \ x \in \delta(C)\}$ is a strong laminar family.*

Proof. We first show that $\mathcal{F}_C$ is laminar. For any $x \in \delta(C)$, we denote $S_x = N(x) \cap S$. Consider two vertices $x, y \in \delta(C)$. If $xy \in E$, since $G[S \cup C]$ is chordal, we must have $S_x \subseteq S_y$ or $S_y \subseteq S_x$. If $xy \notin E$, S_x and S_y cannot overlap since otherwise the set $\{t, x, y, s_x, s_y\}$, with $s_x \in S_x \setminus S_y$, $s_y \in S_y \setminus S_x$ and $t \in S_x \cap S_y$ induces a gem. Therefore, $\mathcal{F}_C$ is a laminar family. To show that $\mathcal{F}_C$ is a strong laminar family, consider a vertex $x \in \delta(C)$ whose S_x is inclusion-wise maximal and consider another arbitrary vertex $y \in \delta(C)$. From Observation 1, $G[\delta(C)]$ is connected. Hence there exists a path π in $\delta(C)$ from x to y. From what precedes, for any two consecutive vertices on π their neighborhoods in S are ordered by inclusion. Since $\mathcal{F}_C$ is laminar and since S_x is inclusion-wise maximal, it follows that the neighborhoods in S of all vertices of π are included in S_x. In particular, $S_y \subseteq S_x$ and consequently $\mathcal{F}_C$ is a strong laminar family. □

Proposition 1. *Let $G = (V, E)$ be a ptolemaic graph, $S \subseteq V$ a clique of G and C a connected component of $G \setminus S$. Let $x, y \in \delta(C)$ be such that $xy \in E$, $S \not\subseteq N(x)$ and $S \not\subseteq N(y)$. Then, we have $N(x) \cap S = N(y) \cap S$.*

Proof. Notice that if C is a semi-split then the statement follows directly. Otherwise, Lemma 11 gives that $\Phi_S(C)$ is laminar. This implies that $N(x) \cap S$ and $N(y) \cap S$ are either disjoint or included in one another. If they are disjoint, take $x' \in N(x) \cap S$ and $y' \in N(y) \cap S$: $\{x, x', y', y\}$ forms an induced C_4, which cannot be since G is chordal. Therefore, $N(x) \cap S$ and $N(y) \cap S$ are included one in another. Assume w.l.o.g. that $N(x) \cap S \subseteq N(y) \cap S$ and assume for a contradiction that $N(x) \cap S \neq N(y) \cap S$. Consider the following vertices: $x' \in N(x) \cap S$, $y' \in (N(y) \setminus N(x)) \cap S$ and $z \in S \setminus N(y)$. Notice that such a vertex z exists since we have $S \not\subseteq N(y)$ by hypothesis. Since $xy', xz, yz \notin E(G)$, the set $\{x', x, y, y', z\}$ induces a gem with universal vertex x', contradicting the fact that G is ptolemaic. Thus, $N(x) \cap S = N(y) \cap S$. □

We are now ready to prove Theorem 3.

$\Leftarrow$ Assume first that the four conditions hold. Observe that condition (i) implies that the graph G is chordal. We now assume for a contradiction that G contains a gem $X = \{t, p_1, p_2, p_3, p_4\}$. By condition (i) we have that X contains at least two vertices from different components C_i and C_j, $1 \leqslant i < j \leqslant \ell$. It follows that $S_X = S \cap X$ is a separator for X. This means that either $S_X = \{t, p_2\}$, $S_X = \{t, p_3\}$ or $S_X = \{t, p_2, p_3\}$ holds. Since the first two cases are symmetric, we have:

1. $S_X = \{t, p_2\}$: p_3 and p_4 belongs to the same connected component. W.l.o.g. assume this component is C_j. Since $N(p_3) \cap S \neq N(p_4) \cap S$, we have $j = \ell$. Moreover, $p_2 \in N(p_1) \setminus N(p_4)$ and $t \in N(p_1) \cap N(p_4) \cap S$, and thus condition (ii) implies $N(p_4) \cap S \subsetneq N(p_1) \cap S$. Now, since C_i is a semi-split, we have $N(p_1) \cap S = \tau(C_i)$ which contradicts condition (iv).
2. $S_X = \{t, p_2, p_3\}$: in that case $N(p_1) \cap S$ overlaps $N(p_4) \cap S$ contradicting condition (ii).

It follows that G is a chordal gem-free graph, and hence G is ptolemaic.

$\Rightarrow$ We now assume that $S \subseteq V$ is a clique and G is ptolemaic, and prove that the four conditions hold. Notice that we may assume $S \subsetneq V$ since otherwise conditions trivially hold. We let $\mathcal{C} = \{C_1, \ldots, C_\ell\}$ be the set of connected components of $G \setminus S$. Notice that condition (i) trivially holds since ptolemaic graphs are hereditary. We now prove (ii). Lemma 11 implies in particular that for any $1 \leqslant i \leqslant \ell$, there exists $v \in C_i$ such that $N(v) \cap S = \tau(C_i)$. Hence the footprint of C_i is $\Phi(C_i) = \{N(v) \cap S : v \in \delta(C_i)\}$, $1 \leqslant i \leqslant \ell$. By Lemma 11, we know that $\Phi(C_i)$ is a laminar family for $1 \leqslant i \leqslant \ell$. Assume for a contradiction that condition (ii) does not hold. In such a case, there exist $x, y \in V$ in different connected components of $G \setminus S$ such that $xy \notin E$ and $N(x) \cap S$ and $N(y) \cap S$ overlap. This means that $(N(x) \cap S \cap N(y)) \neq \emptyset$, $((N(x) \cap S) \setminus N(y)) \neq \emptyset$ and $((N(y) \cap S) \setminus N(x)) \neq \emptyset$. We thus let $t \in N(x) \cap S \cap N(y)$, $p_2 \in (N(x) \cap S) \setminus N(y)$ and $p_3 \in (N(y) \cap S) \setminus N(x)$. The graph $G[\{t, x, p_2, p_3, y\}]$ induces a gem, which contradicts the fact that G is ptolemaic. Hence condition (ii) holds.

We now prove that conditions (iii) and (iv) hold. Notice that if every component of $\mathcal{C}$ is a semi-split, conditions (iii) and (iv) hold immediately and we are done. We hence assume this is not the case. We first prove condition (iv) in order to have properties on non semi-split components. We choose a component C' of $\mathcal{C}$ that is not a semi-split and assume w.l.o.g. that $C' = C_\ell$. We first show that there exists $v \in C_\ell$ such that $N(v) \cap S = S$. From Observation 1, and since C_ℓ is not a semi-split, we can choose $x, y \in \delta(C_\ell)$ such that $xy \in E$ and $N(x) \cap S \neq N(y) \cap S$. The contraposition of Proposition 1 directly gives us that $N(x) \cap S = S$ or $N(y) \cap S = S$. We now prove that for every $1 \leqslant i \leqslant \ell - 1$, no set $\Gamma \in \Phi(C_\ell)$ satisfies $\Gamma \subsetneq \tau(C_i)$. Assume for a contradiction that it is not the case, *i.e.* there exist $1 \leqslant i \leqslant \ell - 1$ and $\Gamma \in \Phi(C_\ell)$ such that $\Gamma \subsetneq \tau(C_i)$. By definition there exists $w \in C_\ell$ such that $N(w) \cap S = \Gamma$. Moreover, thanks to Lemma 11, we know that there exist $p_1 \in C_i$ such that $N(p_1) \cap S = \tau(C_i)$. Since C_ℓ is not a semi-split there exists $z \in \delta(C_\ell)$ such that $N(w) \cap S \neq N(z) \cap S$. From Observation 1, there exists a path $\{w = q_1, \ldots, q_p = z\}$ in $\delta(C_\ell)$. On this path we select the first index i such that $N(q_i) \cap S \neq N(q_{i+1}) \cap S$ and set $x = q_i$, $y = q_{i+1}$. We hence have $x, y \in \delta(C_\ell)$ with $xy \in E$, $N(x) \cap S \neq N(y) \cap S$ and $N(x) \cap S = N(w) \cap S = \Gamma$ by choice of x. Since $\Gamma \subsetneq \tau(C_i)$ implies $\Gamma \subsetneq S$, we have $S \not\subseteq N(x)$ and hence the contraposition of Proposition 1 gives $N(y) \cap S = S$. So we know that there exist $p_2 \in (N(p_1) \cap S \cap N(y)) \setminus N(x)$ and $t \in N(x) \cap N(p_1) \cap N(y) \cap S = N(x) \cap S$. Hence the set $\{t, p_1, p_2, y, x\}$ induces a gem, contradicting the fact that G is ptolemaic. Hence condition (iv) holds.

It remains to prove condition (iii). Assume that there exist two different components C_i and C_j, $1 \leqslant i < j \leqslant \ell$ which are not semi-splits. Since condition (iv) holds, we now know that $\tau(C_i) = \tau(C_j) = S$. We also know that since C_i is not a semi-split there exists $\Gamma \in \Phi(C_i)$ such that $\Gamma \neq \tau(C_i)$, and thus $\Gamma \subsetneq S = \tau(C_j)$, which contradicts condition (iv). This concludes the proof of this Theorem. □

Proof (of Corollary 1). We first prove the reverse direction for both cases (that is S being a clique minimal separator or a maximal clique). Notice that satisfying conditions (i)—(iii) implies satisfying the conditions of Theorem 3 and hence we are done. For the forward direction, assume first that S is a clique minimal separator and that G is ptolemaic. We use the fact that a set S is a clique minimal separator if and only if $G \setminus S$ contains at least two connected components C_1 and C_2 such that $N(C_1) = N(C_2) = S$. Such components are called *full components* of S in G. By Theorem 3, we know that G fulfills conditions (i) to (iv). We claim that the last condition cannot happen. Indeed, since S is a clique minimal separator there exist two full components C_i and C_j of $G \setminus S$. If the component C_ℓ is not a semi-split, $\tau(C_i)$ strictly contains Γ for some set $\Gamma \in \Phi(C_\ell)$, a contradiction. We now deal with the case where S is a maximal clique and G is ptolemaic. By Theorem 3, we know that G fulfills conditions (i) to (iv). Once again, we claim that the last condition cannot happen. Indeed, if there exists a non semi-split component $C_\ell \in \mathcal{C}$, and a vertex $v \in C_\ell$ such that $N(v) \cap S = S$, then the set $S \cup \{v\}$ is a clique containing S, contradicting its maximality. □

B Proofs of Section 3

Proof (of Lemma 3). Let F be an optimal completion of G and let $H = G + F$. We denote $\mathcal{C}$ the set of connected components of $G \setminus S$. From Theorem 3, $\{N_H(v) \cap S \mid v \in V \setminus S\}$ is laminar. Moreover, since any $C \in \mathcal{C}$ is a semi-split in G, then for any vertex $v \in \delta_G(C)$, $N_H(v) \cap S \supseteq \tau_S^G(C)$. It follows that $\Phi_S^H(C)$, which is laminar, is even nested. Therefore, for each $C \in \mathcal{C}$, we choose a vertex v of $\delta_G(C)$ such that $N_H(v) \cap S$ is minimum for inclusion and we denote $N_C = N_H(v) \cap S$. Note that because $\{N_H(v) \cap S \mid v \in V \setminus S\}$ is laminar, so is $\{N_C \mid C \in \mathcal{C}\}$. For any $C \in \mathcal{C}$, we denote $\tilde{F}_C = F \cap (\delta_G(C) \times N_C)$ and $\tilde{H}_C = G[C \cup N_C] + \tilde{F}_C$. Since the vertices of N_C are true twins in $\tilde{H}_C$, Lemma 2 implies that there exists an optimal completion $\hat{F}_C$ of $\tilde{H}_C$ such that the vertices of N_C are true twins in $\hat{H}_C = \tilde{H}_C + \hat{F}_C$. We denote $F'_C = \tilde{F}_C \cup \hat{F}_C$ (note that $\tilde{F}_C$ and $\hat{F}_C$ are disjoint) and $F' = \bigcup_{C \in \mathcal{C}} F'_C$ and we denote $H' = G + F'$.

Notice first that, by construction, the connected components of $H' \setminus S$ are exactly the components $C \in \mathcal{C}$ of $G \setminus S$ and that for any $C \in \mathcal{C}$, we have $\tau_S^{H'}(C) = N_C$. Observe also that, by definition, $\hat{H}_C = H'[C \cup N_C]$. Since the vertices of N_C are true twins in $\hat{H}_C$, it follows that C is a semi-split in H'. Thus, H' satisfies the conclusion of the lemma. We now show that H' is ptolemaic. Since any $C \in \mathcal{C}$ is a semi-split in H', then Condition (iii) of Theorem 3 is satisfied (and Condition (iv) does not occur). Moreover, as $\tau_S^{H'}(C) = N_C$ and $H'[C \cup N_C] = \hat{H}_C$ is ptolemaic, Condition (i) of Theorem 3 is also satisfied. Finally, as noted above, $\bigcup_{C \in \mathcal{C}} \Phi_S^{H'}(C) = \{N_C \mid C \in \mathcal{C}\}$ is a laminar family and so Condition i of Theorem 1 is satisfied. This implies that H' is ptolemaic.

In order to achieve the proof of the lemma, we just need to show that F' is an optimal completion, which we do by showing that $|F'| \leq |F|$. Let $C \in \mathcal{C}$ and let $F_C = F \cap ((C \cup N_C) \times (C \cup N_C))$. We have $G[C \cup N_C] + F_C = H[C \cup N_C]$ is ptolemaic. Moreover, observe that $\tilde{F}_C \subseteq F_C$ and remember that $\hat{F}_C$ has been defined as an optimal completion of $G[C \cup N_C] + \tilde{F}_C$ with $\tilde{F}_C \cap \hat{F}_C = \emptyset$.

Consequently, we have $|\tilde{F}_C| + |\hat{F}_C| \leq |F_C|$ and so $|F'_C| \leq |F_C|$ (as $|F'_C| = |\tilde{F}_C| + |\hat{F}_C|$), for any $C \in \mathcal{C}$. This gives $|F'| \leq \sum_{C \in \mathcal{C}} |F'_C| \leq \sum_{C \in \mathcal{C}} |F_C|$. Note that any pair in F_C is incident to some vertex of C, which implies that such completions are pairwise disjoint. Since they are all included in F, we have $\sum_{C \in \mathcal{C}} |F_C| \leq |F|$. We then obtain $|F'| \leq |F|$, which implies that F' is an optimal completion and concludes the proof. □

C Proofs of Section 4

Proof (of Lemma 4). We will need the following auxiliary result.

Observation 2 *Let $G = (V, E)$ be a ptolemaic graph and $S \subseteq V$ a clique of G that is also a semi-split in G. Then, the graph G' obtained by adding a vertex x in G adjacent to all (and only to) the vertices of S is a ptolemaic graph.*

Proof. If there exists a vertex $y \in S \setminus \delta(S)$, this follows from Theorem 1 (iv) as x is a true twin of y in G'. Otherwise, let $z \in S$. By heredity of ptolemaic graphs, $G[\{z\} \cup (V \setminus S)]$ is ptolemaic and, from Theorem 1 (iv), so is $G'[\{x, z\} \cup (V \setminus S)]$, as x has degree one in $G'[\{x, z\} \cup (V \setminus S)]$. Finally, observe that the vertices in $S \setminus \{z\}$ are true twins of z in G'. Theorem 1 (iv) concludes that G' is ptolemaic. □

Let F be an inclusion-minimal completion of G into a ptolemaic graph and $H = G + F$ the resulting ptolemaic graph. We denote $\tau(\mathcal{C}) = \tau(C_1) \subseteq S$ and we let $S' \supseteq \tau(\mathcal{C})$ be a maximal clique of $H[V \setminus V(\mathcal{C})]$ that contains $\tau(\mathcal{C})$. Since H is ptolemaic, from Theorem 3, we have that the family $\{N_H(x) \cap S' \mid x \in \delta(\mathcal{C})\}$ is laminar, and even nested since for every $x \in \delta(\mathcal{C})$, $N_H(x) \cap S' \supseteq \tau(\mathcal{C})$. Let N be the minimum element of this nested family, we then have for every $x \in \delta(\mathcal{C})$, $N_H(x) \cap S' \supseteq N$. We define a new completion H' of G by $H'[V \setminus V(\mathcal{C})] = H[V \setminus V(\mathcal{C})]$ and $H'[V(\mathcal{C})] = G[V(\mathcal{C})]$ and for every $x \in \delta(\mathcal{C})$, $N_{H'}(x) \cap S' = N$. See Figure 2 for an illustration. Observe that H' is indeed a completion of G (i.e. all edges of G are in H') and that the set F' added to G to obtain H' is such that $F' \subseteq F$. Moreover, observe that the completion H' (which has yet to be proven ptolemaic) satisfies properties (i) and (ii) of the lemma. We now prove that H' is ptolemaic, which implies that $H' = H$ by minimality of H (and because $F' \subseteq F$).

First, observe that the connected components of $H' \setminus S'$ are exactly the connected components of $H[V \setminus V(\mathcal{C})] \setminus S'$ plus the components $\{C_1, \ldots, C_p\}$ of the S-clam. Moreover, since $H[V \setminus V(\mathcal{C})]$ is ptolemaic and since S' is a maximal clique of $H[V \setminus V(\mathcal{C})]$, Corollary 1 implies that any component C' of $H[V \setminus V(\mathcal{C})] \setminus S'$ is a semi-split in H and so in H'. Since $N_{H'}(x) \cap S' = N$ holds for every $x \in \delta(C_i)$, any component $C_i \in \mathcal{C}$ of the S-clam is a semi-split in H'. Thus, Condition (iii) of Theorem 3 is satisfied for H' and S'. Observe also that since H is ptolemaic, Theorem 3 implies that the family $\{N_H(x) \cap S' \mid x \in V \setminus S'\}$ is laminar and the set N belongs to this laminar family. Therefore the family

$\{N_{H'}(x) \cap S' \mid x \in V \setminus S'\}$ is laminar in H' and Condition (ii) of Theorem 3 is satisfied for H' and S'.

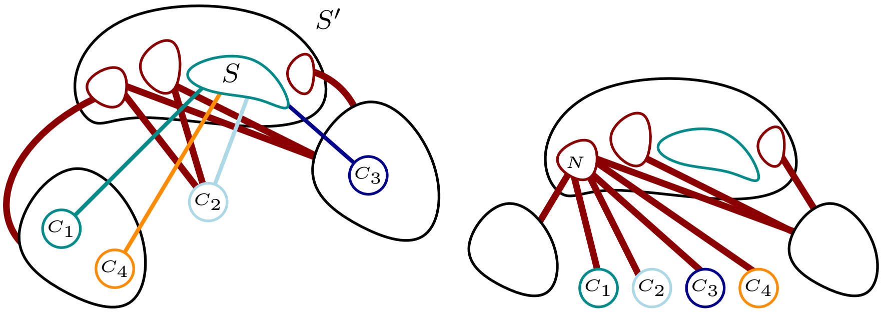

Fig. 2: Illustration of the two steps of the proof. The graphs represented are H and H', respectively. The graph H' is obtained from H by connecting every component of the S-clam to the set N. Notice that the edges towards S' are still present in H' but not represented for the sake of clarity.

Finally, since H is ptolemaic, we have by heredity that any component C' of $H[V \setminus V(\mathcal{C})] \setminus S'$ is ptolemaic as $H'[C' \cup S'] = H[C' \cup S']$. Let us now consider a component $C_i \in \mathcal{C}$ of the S-clam. We know by hypothesis of the lemma that $G[C_i \cup \tau(\mathcal{C})] = H'[C_i \cup \tau(\mathcal{C})]$ is ptolemaic. Because, in $H'[C_i \cup N]$, the vertices of N are true twins of the vertices of $\tau(\mathcal{C})$ then Theorem 1 (iv) gives that $H'[C_i \cup N]$ is ptolemaic. From Observation 2, it follows that $H'[C_i \cup S']$ is ptolemaic. Thus, Condition (i) of Theorem 3 is satisfied for H' and S'.

As a conclusion, we have that all first three conditions of Theorem 3 are satisfied by H' and S'. Now, since every component of $H' \setminus S'$ is a semi-split we conclude that H' is ptolemaic. By minimality of H, it follows that $H = H'$ and that H satisfies properties (i) and (ii) of the lemma. □

Proof (of Lemma 5). Let $G' = (V', E')$ be the reduced graph obtained from G, and consider first a k-completion F' of G'. Notice that C_S is a set of true twins in G'. Hence by Lemma 2 we may assume that F' affects uniformly all vertices of C_S. Let $H' = G' + F'$ and S' be a maximal clique of H' containing the set $N = N_{H'}(C_s)$. We know that conditions of Corollary 1 are fulfilled around S' in H'. In particular, notice that every component of $H' \setminus S'$ is a semi-split. We claim that the completion F obtained from F' by preserving all pairs in $((V \setminus C_s) \times (V \setminus C_s))$ and replacing the pairs in $(C_s \times (V \setminus C_s))$ by corresponding pairs in $(\delta(C) \times (V \setminus V(\mathcal{C})))$ is a k-completion for G. To see this, notice first that since $|\delta(C)| = |C_s|$ we have $|F| = |F'|$. Moreover, since neighborhoods towards S' are the same in $H = G + F$ and in H' we have that the last three conditions of Theorem 3 are fulfilled around S' in H. As in the proof of Lemma 4, we know by definition of the S-clam that $G[C_i \cup \tau(\mathcal{C})] = H[C_i \cup \tau(\mathcal{C})]$ is ptolemaic. Because, in $H[C_i \cup N]$, the vertices of N are true twins of the vertices of $\tau(\mathcal{C})$ then Theorem 1 (iv) gives that $H[C_i \cup N]$ is ptolemaic. From Observation 2 (see the proof of Lemma 4), it follows that $H[C_i \cup S']$ is ptolemaic. We thus conclude that all first three conditions of Theorem 3 are satisfied by H and S'. Now, since every component of $H \setminus S'$ is a semi-split we conclude that H is ptolemaic.

We now consider a k-completion F of G and let $H = G + F$. Recall that by Lemma 4 the only affected vertices of $\mathcal{C}$ are contained in $\delta(\mathcal{C})$ and are affected uniformly by F. Let $v \in \delta(\mathcal{C})$. Since ptolemaic graphs are hereditary, we know that $H_v = H[(V \setminus V(\mathcal{C})) \cup \{v\}]$ is ptolemaic. Moreover, since ptolemaic graphs are closed under true twin addition, adding $\min(k, |\delta(\mathcal{C})| - 1)$ vertices as true twins of v yields a ptolemaic graph. Notice that the resulting graph is a completion for G'. Let F' be the completion needed to obtain such a graph and $H' = G' + F'$. By construction, the completion F' is obtained from F by preserving pairs of $((V \setminus C_s) \times (V \setminus C_s))$ and replacing all pairs $(\delta(\mathcal{C}) \times (V \setminus V(\mathcal{C}))$ by corresponding pairs $(C_s \times (V \setminus V(\mathcal{C}))$. Since $|C_s| = |\delta(\mathcal{C})|$ we have $|F'| = |F|$. This concludes the proof. □

D Proofs of Section 5

Proof (of Lemma 6). The soundness of Rule 2 is classical and comes from the fact that ptolemaic graphs are closed under disjoint union. Rule 3 is a well-known *sunflower-based* reduction rule and has already been used in several works (see *e.g.* [5]). Finally, the soundness of Rule 4 comes from [4, Lemma 1.4] since ptolemaic graphs are hereditary and closed under true twin addition (Theorem 1 (iv)). To prove the soundness of Rule 5, notice that for any gem, at least one of its gem-breakers belongs to any completion of G. Since $\mathcal{X}$ is a gem-sunflower, any completion that does not take its center must take the remaining $p > k$ gem-breakers, and hence have size greater than k. Notice moreover that all gems can be bruteforcely found in polynomial time. □

Proof (of Lemma 7). We first prove the soundness of Rule 6. Let G' be the graph obtained from G after the removal of C. It must be clear that if G admits a k-completion, then so does G', by heredity of ptolemaic graphs.

We now prove the converse implication. Let $\tilde{S} = \tau_S^G(C) \subseteq S$. Notice that since C is a type-α S-tentacle, then $\tilde{S}$ fulfills conditions of Lemma 3 in graph G'. As a consequence, if there exists a k-completion of G', then there exists a k-completion F' of G' such that any connected component C' of $H' \setminus \tilde{S}$ is a semi-split, where $H' = G + F'$. We claim that the graph $H = G + F'$ is ptolemaic. Indeed, the components C' of $H \setminus \tilde{S}$ are exactly the components of $H' \setminus \tilde{S}$ plus C. Thus, they are all semi-splits in H and they all satisfy $H[C' \cup \tilde{S}]$ is ptolemaic. Moreover, since H' is ptolemaic, Theorem 3 gives that $\{\tau(C'') \mid C''$ component of $H' \setminus \tilde{S}\}$ is laminar. So we also have that $\{\tau(C') \mid C'$ component of $H \setminus \tilde{S}\} = \{\tau(C'') \mid C''$ component of $H' \setminus \tilde{S}\} \cup \{\tilde{S}\}$ is laminar and Corollary 1 concludes that $H = G + F'$ is ptolemaic, implying that G admits a k-completion.

We now turn our attention to Rule 7. Notice first that since G is reduced under Rule 6, $\tau(C_i) \neq \tau(C_j)$ holds for any $1 \leqslant i < j \leqslant p$. Moreover, if there does not exist $v \in C'$ such that $N(v) \cap S = \Gamma$, this means that $\Gamma = \tau(C')$. Hence we can replace Γ by $N(v') \cap S$ for any $v' \in C'$ and still satisfy hypotheses of the rule. Hence we assume that such a v exists. In order to prove the soundness of this rule, we only have to prove that given a k-completion F of G such that $H = G + F$ is ptolemaic, $(\delta(C') \times \tau(C')) \subseteq E \cup F$. We denote C_H the component of $H \setminus S$ containing C'. We know that S is a separating clique of H, since at least $k+1$ components of $\mathcal{C}$ are not in C_H. We assume w.l.o.g. that the components $\{C_1, \ldots, C_{k+1}\}$ of $\mathcal{C}$ are not in C_H. Notice that the family $\mathcal{F} = \{\tau^G(C_1), \ldots, \tau^G(C_{k+1})\}$ is laminar, since otherwise such sets would overlap one another and would thus be type-γ tentacles. Since all sets of $\mathcal{F}$ share Γ, they form a nested family, *i.e.* for every $1 \leqslant i \neq j \leqslant k$, $\tau(C_i) \subseteq \tau(C_j)$ or $\tau(C_j) \subseteq \tau(C_i)$ holds. Up to a renaming of the components, we may hence assume that for every $1 \leqslant i \leqslant k$, $\tau^G(C_i) \subsetneq \tau^G(C_{i+1})$ (recall that no two type-β S-tentacles have the same trace). In particular, we hence have $\left|\tau^G(C_{k+1})) \setminus \Gamma\right| > k$. This means that $N_H(v) \cap S \subsetneq \tau^H(C_{i+1})$ since $N_H(v)$ has at most k more vertices than $N_G(v)$. Thus, the component C_H is a semi-split, since otherwise it would violate condition (iv) of Theorem 3. We also have that $\tau^G(C') \subseteq \tau^H(C_H)$, which in turn implies that $(\delta(C') \times \tau^G(C')) \subseteq E \cup F$.

We finally turn our attention to the soundness of Rule 8 (see Figure 3). Let $G' = G \setminus S$. By heredity of ptolemaic graphs, if G admits a k-completion

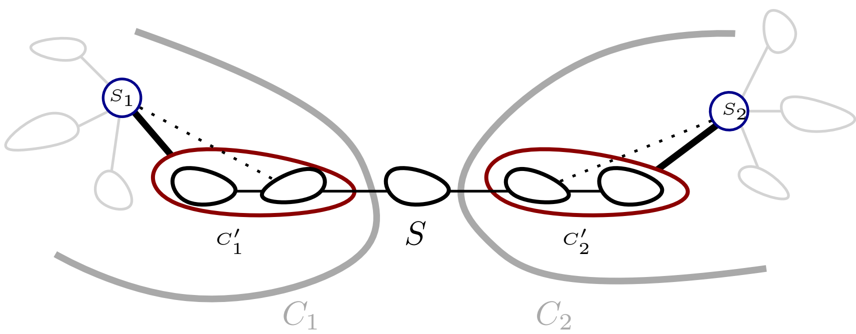

Fig. 3: Illustration of Rule 8. The sets C_i' containing $N_G^2(S) \cap C_i$ in $G[C_i]$ are S_i-clams. The bold edges represent semi-split.

then so does G'. Let us now show that the converse also holds. If G' admits a k-completion, then $G[C_1]$ admits a k_1-completion and $G[C_2]$ admits a k_2-completion with $k_1 + k_2 \leq k$. For $i \in \{1, 2\}$, since C_i' is an S_i-clam and $N_G(S) \cap$

$\delta(C'_i) = \varnothing$, then from Lemma 4, there exists an optimal completion F_i of $G[C_i]$ such that the vertices of $N_G(S)$ are unaffected and there are no pairs of F_i contained in $(C'_i \times C'_i)$. In particular, since F_i is minimum-sized, we have $|F_i| \leq k_i$ and consequently $|F_1| + |F_2| \leq k$. We claim that $H = G + (F_1 \cup F_2)$ is ptolemaic.

By construction, $H \setminus S$ has exactly two connected components $H[C_1]$ and $H[C_2]$. To show that H is ptolemaic, we will show that the conditions of Corollary 1 are satisfied for S in H. Let $i \in \{1, 2\}$. We first show that C_i is a semi-plit in H. Observe that $\delta(C_i) = N_G(S)$. Moreover, since $G[S \cup N^2_G(S)]$ is ptolemaic and since S is a minimal separator of $G[S \cup N^2_G(S)]$ (as S is a minimal separator of G), then Corollary 1 gives that $N^2_G(S) \cap C_i$ is a semi-split in $G[S \cup N^2_G(S)]$ and so is C_i in H: Condition (iii) of Corollary 1 is satisfied for S in H. In addition, as for $i \in [\![1, 2]\!]$, $\tau^H_S(C_i) = S$, then Condition (ii) of Corollary 1 is also satisfied for S in H.

Finally, we show that for $i \in [\![1, 2]\!]$, $H[C_i \cup S]$ is ptolemaic. First observe that if $H[C_i \cup S]$ contains an induced gem I, then I necessarily contains at least one vertex $v \in S$, as $H[C_i]$ is ptolemaic. It follows that $I \subseteq N^2_H(v)$, as a gem is included in the second neighborhood of any of its vertices. By definition of F_i, the vertices of $N_G(S) \cap C_i$ are unaffected and there are no edges of F_i in $N^2_G(S)$. This gives that for any vertex $v \in S$, $N^2_H(v) \cap C_i = N^2_G(v) \cap C_i$ and $H[(N^2_G(S) \cap C_i) \cup S] = G[(N^2_G(S) \cap C_i) \cup S]$. Therefore, the induced gem I is also induced in $G[(N^2_G(S) \cap C_i) \cup S]$, which cannot be since $G[N^2_G(S) \cup S]$ is ptolemaic by hypothesis. Thus, $H[C_i \cup S]$ does not contain any induced gem.

In order to prove that $H[C_i \cup S]$ is chordal, we first show that in $H[N_G(S) \cap C_i]$ all vertices are at distance at most 2 from each other. We already noticed in what precedes that $H[N_G(S) \cap C_i] = G[N_G(S) \cap C_i]$. Since $G[N^2_G(S) \cup S]$ is ptolemaic, it is in particular chordal and Observation 1 applied on S gives that $G[N_G(S) \cap C_i]$ is connected. Assume for a contradiction that there exist two vertices $a, b \in N_G(S) \cap C_i$ that are at distance 3. Then, there exists an induced path a, c, d, b in $G[N_G(S) \cap C_i]$ and because $N_G(S)$ is complete to S in G (remember that $N^2_G(S) \cap C_i$ is a semi-split in $G[S \cup N^2_G(S)]$), then for any vertex $s \in S$, the subset of vertices $\{a, b, c, d, s\}$ is an induced gem in $G[N^2_G(S) \cup S]$, which is a contradiction with the fact that $G[N^2_G(S) \cup S]$ is ptolemaic. It follows that any two vertices in $H[N_G(S) \cap C_i]$ are at distance at most 2.

We now turn to proving that $H[C_i \cup S]$ is chordal and we assume for a contradiction that there exists an induced cycle X of length at least 4 in $H[C_i \cup S]$. Since $H[C_i]$ is chordal, X must contain at least one vertex of S. On the other hand, since $N_G(S)$ is complete to S in G (and so in H), there cannot be more than one vertex of S in X, since otherwise X would contain at least two chords. Therefore, X contains exactly one vertex $s \in S$. In addition, since $G[(N^2_G(S) \cap C_i) \cup S]$ is ptolemaic (and hence chordal) and since $G[(N^2_G(S) \cap C_i) \cup S] = H[(N^2_G(S) \cap C_i) \cup S]$, X must contain at least one vertex $a \in C_i \setminus N^2_G(S)$. Let b and c be the two neighbors of s in X. Since X is an induced cycle, bc is not an edge of H and since the maximum distance between any two vertices in $H[N_G(S) \cap C_i]$ is at most 2 by the previous arguments, there exists a vertex $d \in N_G(S)$ that is adjacent to both b and c. Note that da is not an edge of H, as the vertices in $N_G(S)$ are unaffected in H. Let us denote P_b (resp. P_c) the path in X from a to b (resp. from a to c). Consider the vertex $b' \in P_b$ (resp. $c' \in P_c$) that is adjacent to d and that is the closest from a on P_b (resp. on P_c). Finally, denote P'_b (resp. P'_c) the subpath of P_b (resp. P_c) from a to b' (resp. c'). Then, the subset of vertices $X' = P'_b \cup P'_c \cup \{d\}$ is an induced cycle of length at least 4 in $H[C_i]$, which is ptolemaic: a contradiction. As a conclusion, there is no induced cycle X of length at least 4 in $H[C_i \cup S]$, which is then ptolemaic (recall that we previously showed that it contains no induced gem). Thus, Condition (i) of Corollary 1 is satisfied for S in H and H is ptolemaic. As H is a k-completion of G, this achieves the proof of the lemma. □

Proof (of Lemma 8). Thanks to Theorem 4 we can compute in polynomial time the set $\mathcal{S}$ of all clique minimal separators of G. For every such separator $S \in \mathcal{S}$, one can check in polynomial time if some components separated by S belong to clams (Definition 4). More precisely, we can consider equivalence classes of semi-split components of $G \setminus S$ inducing ptolemaic graphs based on their neighborhoods towards S. If there exists a class containing more that one element, this corresponds to a clam of G. This allows to reduce exhaustively the instance under Rule 1.

We next turn our attention to Rules 6 and 7. Note that the type of a given tentacle may depend on the clique minimal separator around which it is defined. The following result shows that it is sufficient to determine the type of a tentacle C on the clique minimal separator $\tau(C)$.

Lemma 12. *Let $G = (V, E)$ be a graph, $S \subseteq V$ a clique minimal separator and C an S-tentacle of G such that $S = \tau(C)$. If C is a type-α S-tentacle, then C is a type-α S'-tentacle for any clique minimal separator $S' \supseteq S$.*

Proof. Let S' be a clique minimal separator containing $S = \tau(C)$, and assume in a first place that C is a type-γ S'-tentacle, *i.e.* there exists a component C_o of $G \setminus (S' \cup C)$ such that $\Phi^G_{S'}(C_o) \oslash \Phi^G_{S'}(C)$. Let C' be the component of $G \setminus (S \cup C)$ that contains C_o. Notice that by construction, C' contains a vertex $x \in S' \setminus S$ which is thus such that $S \subseteq N(x)$. In particular, since $\Phi^G_{S'}(C_o) \oslash \Phi^G_{S'}(C)$ for any $C \in C$, this means that C' is not a semi-split since otherwise C_o would be adjacent to every vertex of S due to $S \subseteq N(x)$. We thus deduce that there exists a vertex $v_o \in C_o$ such that $N(v_o) \cap S \subsetneq S = \tau(C)$, implying that C is a type-β S-tentacle, a contradiction. The case where no component C_o of $G \setminus (S' \cup C)$ satisfies $\Phi^G_{S'}(C_o) \oslash \Phi^G_{S'}(C)$ is similar. Indeed, we may assume in this case that C is a type-β S'-tentacle, which in turn implies that there exists a component C' of $G \setminus (S' \cup C)$ with a set $\Gamma \in \Phi_{S'}(C') \subsetneq \tau(C)$. Notice that the component C' exists also in $G \setminus (S \cup C)$, implying once again that $\mathcal{C}$ is a type-β S-tentacle, a contradiction. □

We can check once again on every clique minimal separator S if a component separated by S is a tentacle, that is a semi-split that induces a complete graph.

Moreover, thanks to Lemma 12 we can directly decide its type during this process. Hence we can exhaustively reduce the instance under Rule 6. Regarding Rule 7, we compute for every $S \in \mathcal{S}$ the set $\mathcal{C}$ of non-semi-split components of $G \setminus S$. For every $C \in \mathcal{C}$ and for every set Γ in $\Phi_S(C)$, we can compute the number of type-β S-tentacles which strictly contain Γ. If there are at least $2k+1$ such type-β S-tentacles then Rule 7 can be applied in polynomial time. Applying Rule 8 can be done by enumerating all clique minimal separator, and checking, for every triplet of clique minimal separators, if they satisfy the conditions of the rule. There is a polynomial number of clique minimal separators, so a polynomial number of triplet of clique minimal separators. To conclude the proof, notice that at each step the graph is reduced (Rules 1, 6 and 8) or the parameter decreases (Rule 7). We thus need to repeat the above steps at most $O(n+k)$ times. □

E Proofs of Section 6

Proof (of Lemma 9). Let $I = (G = (V, E), k)$ be a YES-instance of Ptolemaic Completion reduced under Rules 1 and Rules 5 to 7. We partition the connected components $\{C_1, \ldots, C_\ell\}$ into four parts:

$\mathcal{P}_1$ contains components C such that $G[C \cup S]$ is not ptolemaic,
$\mathcal{P}_2$ contains components C such that $G[C \cup S]$ is ptolemaic but C is not a semi-split of G,
$\mathcal{P}_3$ contains components C such that C is a type-γ S-tentacle
$\mathcal{P}_4$ contains components C such that C is a type-β S-tentacle

Recall that G is type-α tentacle-free since it is reduced under Rule 6. We first show that $\mathcal{P}_1 \cup \mathcal{P}_2$ contains at most $2k+1$ components.

Claim 3. For any completion F of G into a ptolemaic graph, every connected component of $\mathcal{P}_1$ contains an affected vertex. Moreover, there is at most one connected component C_s of $\mathcal{P}_2$ that does not contain an affected vertex, in which case C_s satisfies $\tau(C_s) = S$.

Proof. Let $H = G + F$ and $C \in \mathcal{P}_1$. By definition $G[C \cup S]$ is not ptolemaic and since $H[C \cup S]$ is ptolemaic C must contain some affected vertex. Moreover, using Theorem 3 on H and S, we know that at most one component C_H of $H \setminus S$ is not a semi-split in H, and this component satisfies $\tau^H_S(C_H) = S$. It follows that at most one component C_s of $\mathcal{P}_2$ has not been affected, and it satisfies $\tau^G_S(C_s) = S$. ◇

Recall that since any k-completion contains at most $2k$ affected vertices, Claim 3 directly implies that the number of components contained in $\mathcal{P}_1 \cup \mathcal{P}_2$ is at most $2k+1$. In order to show that the number of components in $\mathcal{P}_3$ is $O(k^2)$ we need the following property.

Claim 4. Let $C_i, C_j \in \mathcal{C}$ be such that $\tau_S^G(C_i) \oslash \tau_S^G(C_j)$ and let F be any completion of G into a ptolemaic graph. If no vertex of C_i is affected then C_j contains some affected vertex and every vertex of $\tau_S^G(C_i) \setminus \tau_S^G(C_j)$ is affected.

Proof. We forget the mention to both S and G in the proof for the sake of simplicity. If no vertex of C_i is affected, then $\tau(C_i)$ minimally separates C_i from $C_j \cup (\tau(C_j) \setminus \tau(C_i))$ in $H_1 = H[\tau(C_i) \cup C_i \cup \tau(C_j) \cup C_j]$. Hence $\tau(C_i)$ is a clique minimal separator in H_1. Since H_1 is ptolemaic Corollary 1 implies that $C_j \cup (\tau(C_j) \setminus \tau(C_i))$ is a semi-split in H_1. The frontier of this semi-split contains at least the vertices of $S_2 = C_j \cap N(\tau(C_i) \cap \tau(C_j)) \neq \varnothing$ and the vertices of $\tau(C_j) \setminus \tau(C_i) \neq \varnothing$. Since the vertices of $\tau(C_j) \setminus \tau(C_i)$ are adjacent to every vertex of $\tau(C_i)$ in H, then so are the vertices of S_2. It follows that vertices of $\tau(C_i) \setminus \tau(C_j) \neq \varnothing$ and of $S_2 \subseteq C_j$ are all affected. $\diamond$

We now consider the graph $G_{\mathcal{O}}$ whose edges are the pairs $\{C_i, C_j\} \in \mathcal{C}^2$, $1 \leqslant i < j \leqslant l$, such that $\tau(C_i)$ and $\tau(C_j)$ overlap and whose vertices are precisely the elements of $\mathcal{C}$ involved in these pairs. We first show the two following properties.

Claim 5. $G_{\mathcal{O}}$ admits a vertex cover of size at most $2k$.

Proof. Due to Claim 4, any edge $\{C_i, C_j\}$ of $G_{\mathcal{O}}$ is such that C_i or C_j contains an affected vertex. Since G admits a k-completion, $G_{\mathcal{O}}$ admits a vertex cover of size at most $2k$. $\diamond$

Claim 6. $G_{\mathcal{O}}$ does not contain any induced star (a tree on n vertices with exaclty one vertex of degree greater than 1 and $n-1$ so-called satellites) with more than $10k$ satellites.

Proof. Consider an induced star of $G_{\mathcal{O}}$. Let C_c be the *center* of the star, that is the unique vertex of degree greater than 1. For every satellite C_s, $\tau_S^G(C_s)$ overlaps with $\tau_S^G(C_c)$. Moreover, no two satellites have their traces that overlap each other, by definition of $G_{\mathcal{O}}$ and because the star is induced. Therefore, the traces of the satellites form a laminar family. Let F be a k-completion of G. Remove from $G_{\mathcal{O}}$ the satellites C_s such that C_s contains some affected vertex (recall there cannot be more than $2k$ such satellites). Let $G_{\mathcal{R}}$ be the resulting graph. Moreover, let $\mathcal{C}_{\mathcal{R}}$ be the set of satellites of $G_{\mathcal{R}}$. Notice that since for every satellite C_s of $G_{\mathcal{O}}$ the traces $\tau(C_s)$ and $\tau(C_c)$ overlap, none of them satisfy $\tau(C_s) = S$. Therefore, from Claim 3, any satellite C_s^r of $G_{\mathcal{R}}$ belongs to $\mathcal{P}_3$. For any such satellite, let $p_1 \in \delta(C_s^r)$, $p_2 \in \tau(C_s^r) \setminus \tau(C_c)$, $t \in \tau(C_c) \cap \tau(C_s^r)$, $p_3 \in \tau(C_c) \setminus \tau(C_s^r)$ and $p_4 \in \delta(C_c)$. One can see that the set $\{t, p_1, p_2, p_3, p_4\}$ induces a gem in G. If $|\tau(C_c) \setminus \tau(C_s^r)| > k$, then replacing p_3 by vertices of $\tau(C_c) \setminus \tau(C_s^r)$ yields a gem-sunflower in G. Since G has been reduced under Rule 5, we hence have $|\tau(C_c) \setminus \tau(C_s^r)| \leqslant k$. Let I_s be the intersection of the traces of all satellites of $G_{\mathcal{R}}$. Observe that since the family $\{\tau(C) \ : \ C \in \mathcal{C}_{\mathcal{R}}\}$ is laminar and since $|\tau(C_c) \setminus \tau(C_s^r)| \leqslant k$ for any satellite C_s^r of $G_{\mathcal{R}}$, we have

$|\tau(C_c) \setminus I_s| \leqslant 2k$. Moreover, $\mathcal{F_R} = \{\tau(C) \setminus I_s \ : \ C \in \mathcal{C_R}\}$ is laminar and all traces are different. Furthermore, Claim 4 implies that vertices of $\tau(C_s^r) \setminus \tau(C_c)$ are affected for any satellite C_s^r of $\mathcal{C_R}$. Hence the family $\mathcal{F_R}$ spans a universe composed of vertices that are either affected or in $\tau(C_c) \setminus I_s$. By the previous arguments, such a universe is of size at most $4k$. Since a laminar family $\mathcal{L}$ with different sets satisfies $2|\bigcup_{L\in\mathcal{L}} L| \geq |\mathcal{L}|$, we have at most $8k$ satellites in $\mathcal{C_R}$, and hence at most $10k$ in $G_\mathcal{O}$. ◇

Let X be a vertex cover of $G_\mathcal{O}$. By Claim 5, we know that X contains at most $2k$ vertices. By definition, the complement Y of X is an independent set. Since $G_\mathcal{O}$ does not have any induced star with more than $6k+1$ satellites (Claim 6), it follows that any vertex of X has at most $10k$ neighbors in Y. Consequently, $G_\mathcal{O}$ has $O(k^2)$ edges. Moreover, since $G_\mathcal{O}$ does not contain any isolated vertex by definition, it contains $O(k^2)$ vertices. Let $C \in \mathcal{C}$ be a component of $\mathcal{P}_3$. By definition of type-γ tentacles, $\tau(C)$ overlaps the trace of some other components in $\mathcal{C}$. Then C is a vertex of $G_\mathcal{O}$ and it follows from what precedes that the number of components in $\mathcal{P}_3$ is $O(k^2)$.

To conclude the proof we bound the number of components in $\mathcal{P}_4$. Notice that for any type-β S-tentacle C, there exist a component C' of $G \setminus S$ that is not a semi-split and hence a vertex $v \in C'$ such that $N(v) \cap S \subsetneq \tau(C)$ and $N(v) \cap S \subsetneq \tau(C')$.

Claim 7. Let C be a type-β S-tentacle. Then for any completion F of G, the vertex $v \in C'$ such that $N(v) \cap S \subsetneq \tau(C)$ and $N(v) \cap S \subsetneq \tau(C')$ is affected or there is a vertex in C that is affected.

Proof. Assume for a contradiction that neither C nor v is affected in $H = G+F$. Since v is unaffected and $N(v) \cap S \neq \tau(C')$, v is in a non semi-split component of $H \setminus S$. Now, $N_H(v) \cap S \subsetneq \tau(C)$ and C is a component of $H \setminus S$ since it was not affected. This contradicts condition (iv) of Theorem 3. ◇

Let F be a k-completion of G. We classify components of $\mathcal{P}_4$ into two categories, namely the ones that are affected ($\mathcal{B}_a$) and the ones that are not ($\mathcal{B}_u$). We know that $|\mathcal{B}_a| \leqslant 2k$. Let C be a component of $\mathcal{B}_u$, and C' a component of $G \setminus S$ that is not a semi-split. Moreover, let $v \in C'$ be such that $N(v) \cap S \subsetneq \tau(C)$ and $N(v) \cap S \neq \tau(C')$. Thanks to Claim 7, we know that v has been affected. Hence there exist at most $2k$ vertices $\{v_1, \dots, v_p\}$ such that v_i belongs to a non semi-split component of $G \setminus S$, v_i has been affected and there exists a type-β S-tentacle B_i with $N(v_i) \cap S \subsetneq \tau(B_i)$. Since G has been reduced under Rule 7, we know that for each $1 \leqslant i \leqslant p$, at most $2k$ different type-β S-tentacles C satisfy $N(v_i) \cap S \subsetneq C$. This means that $|\mathcal{B}_u| \leqslant 4k^2$. So we can conclude that the number of components in $\mathcal{P}_4$ is $O(k^2)$. □

Proof (of Lemma 10). We consider the partially ordered set $P_u = (\{U \in V(\vec{T}_G) \mid u \in U\}, \subseteq)$ for any vertex $u \in V(G)$. We claim that P_u has a minimum. Indeed, let

A and B be two incomparable elements of P_u. By definition of T_G, there exists $C = (A \cap B)$ in P_u, such that $C \subseteq A$ and $C \subseteq B$. Since P_u is finite, we can support our claim. Let us call this minimum S_u. Let $v \in V(G)$, and define similarly S_v. Assume for a contradiction that $S_v = S_u$, and let $\mathcal{M}_u$ and $\mathcal{M}_v$ be the sets of all maximal cliques containing respectively u and v. Moreover, let $M_u \in \mathcal{M}_u$. By definition of S_u, we have that $S_v = S_u \subseteq M_u$, and since $v \in S_v$, we have $v \in M_u$, which in turn implies $M_u \in \mathcal{M}_v$. Similarly one can see that every $M_v \in \mathcal{M}_v$ is also in $\mathcal{M}_u$. Thus $\mathcal{M}_u = \mathcal{M}_v$, and hence u and v are true twins, leading to a contradiction. So for every $u, v \in V(G)$, $S_u \neq S_v$ holds, and hence $|V(G)| \leqslant \left|V(\vec{T}_G)\right| = p$. □

Proof (of Theorem 5). We say that an instance $(G = (V, E), k)$ of Ptolemaic Completion is *reduced* whenever none of the described reduction rules can be applied to G. Let $(G = (V, E), k)$ be a reduced YES-instance of Ptolemaic Completion, and F a k-completion of G. Let $\{C_1, \ldots, C_c\}$ denote the connected components of G. We work on the ptolemaic graph $H = G + F$ with connected components $\{H_1, \ldots, H_c\}$. Since G is reduced under Rule 2, we know that $c \leqslant k$. We assume that $F = \cup_{i=1}^{c} F_i$ with $H_i = G[C_i] + F_i$ and $|F_i| = k_i$ for $1 \leqslant i \leqslant c$. By Theorem 2, let $\vec{T}_H$ be the clique laminar forest of H, and $\vec{T}_{H_i}$ be the clique laminar tree of H_i, $1 \leqslant i \leqslant c$. In order to bound the size of the instance, we will count the number of bags of the clique laminar tree $\vec{T}_H$. This approach is justified by Lemma 10. For any $t \in V(\vec{T}_H)$, we let $anc(t)$ and $desc(t)$ denote the set of ancestors and descendants of t in $\vec{T}_H$, respectively. For the sake of readability, we assume $t \in anc(t)$ and $t \notin desc(t)$. In a slight abuse of notation we will use $anc(V_t)$ (resp. $desc(V_t)$) to denote the set of vertices $\cup_{t' \in anc(t)} V_{t'}$ (resp. $\cup_{t' \in desc(t)} V_{t'}$). The following proposition will be useful to bound the size of the instance.

Lemma 13. *Let $G = (V, E)$ be a connected ptolemaic graph and $\vec{T}_G$ its clique laminar tree. Let t be a bag of $\vec{T}_G$. Let $\{C_1, \cdots, C_\ell\}$ be the collection of connected components of $G \setminus V_t$. Then, $|desc(t)| \leqslant \ell$.*

Proof. Let $d \in desc(t)$, d is a descendant of t so it has at least one parent, so by definition of $\vec{T}_G$ it has at least 2 parents. Moreover, d can not have 2 parents in $desc(t) \cup \{t\}$, otherwise there would be a cycle in T_G. So d has a parent not in $desc(t) \cup \{t\}$, we denote this parent p. Now, p is neither t nor in $desc(t)$ so $V_p \nsubseteq V_t$, by definition of $\vec{T}_G$. So $V_p \setminus V_t \neq \emptyset$. So we proved that we can associate a connected component of $G \setminus V_t$ to each bag $d \in desc(t)$, we now only have to prove that for two different bags $d, d' \in desc(t)$, $V_p \setminus V_t$ and $V_{p'} \setminus V_t$ are in different connected components of $G \setminus V_t$.

We can prove this easily using the fact that T_G is a clique tree. Indeed, since T_G is a tree and $\{t\} \cup desc(t)$ is connected, the path from d to d' in T_G has to go through d (and d'). But, by definition of $\vec{T}_G$, $V_d \subset V_t$, so $V_d \setminus V_t = \emptyset$. By removing V_t from G we removed all the vertices in a bag on the path from d to d' in a clique tree of G, so the remaining vertices of d and d' are disconnected in $G \setminus V_t$, thus in different connected components. □

Let $\mathcal{F}_i$ be the set of all *filled bags* of $\vec{T}_{H_i}$, that is $\mathcal{F}_i = \{t \in V(\vec{T}_{H_i}) : \exists \{u,v\} \in F, \{u,v\} \subseteq V_t\}$ and $\mathcal{F} = \cup_{i=1}^{c} \mathcal{F}_i$. Moreover, let $\mathcal{M}_i$ be the set of *minimal filled bags* of $\vec{T}_{H_i}$, *i.e.* bags that do not have any filled bag in their descendants. Finally, we set $\mathcal{M} = \cup_{i=1}^{c} \mathcal{M}_i$. We will use the following results, which are straightforward implications of Theorem 2 on the ptolemaic graph H.

Observation 3 *Let $t \in V(\vec{T}_H)$ and $\{u,v\} \in V_t$. For every ancestor $t' \in anc(t)$, $\{u,v\} \in V_{t'}$.*

Claim 8. Let $t \in V(\vec{T}_H)$, and $\{t_1, \dots, t_p\}$ denote its children. Any vertex $u \in V(G)$ that belongs to V_t is contained in at most one set $desc(V_{t_i})$, $1 \leqslant i \leqslant p$.

Proof. Assume for a contradiction that there exist $1 \leqslant i < j \leqslant p$ such that $u \in desc(V_{t_i})$ and $u \in desc(V_{t_j})$. In this case, $u \in V_{t_i}$ and $u \in V_{t_j}$, which means that $V_{t_i} \cap V_{t_j} \neq \emptyset$. Hence there exists a bag $t' \in desc(t_i) \cap desc(t_j)$ such that $\{t, t_i, t_j, t'\}$ induces a cycle in T_{H_i}, a contradiction. ◇

Notice that Observation 3 and Claim 8 imply that a pair $\{u,v\}$ of F cannot occur in two different minimal filled bags t and t' of $\mathcal{M}$, since otherwise $V_{t''}$ would contain $\{u,v\}$ and t'' would have two descendants containing u, where t'' is the least common ancestor of t and t'. Since $|F_i| = k_i$, we have $|\mathcal{M}_i| \leqslant k_i$ and hence $|\mathcal{M}| = \sum_{i=1}^{c} |\mathcal{M}_i| \leqslant k$. For every $1 \leqslant i \leqslant c$, let $\mathcal{M}_i = \{t_i^1, \dots, t_i^q\}$, $1 \leqslant q \leqslant k_i$, be an arbitrary order over the minimal filled bags of $\vec{T}_{H_i}$. We denote the ancestors of $\mathcal{M}_i$ in the following way: for every $1 \leqslant p \leqslant q$ we let $\mathcal{B}_i^p = anc(t_i^p)$. By Observation 3 we have that $\mathcal{F}_i = \cup_{j=1}^{q} \mathcal{B}_i^j$.

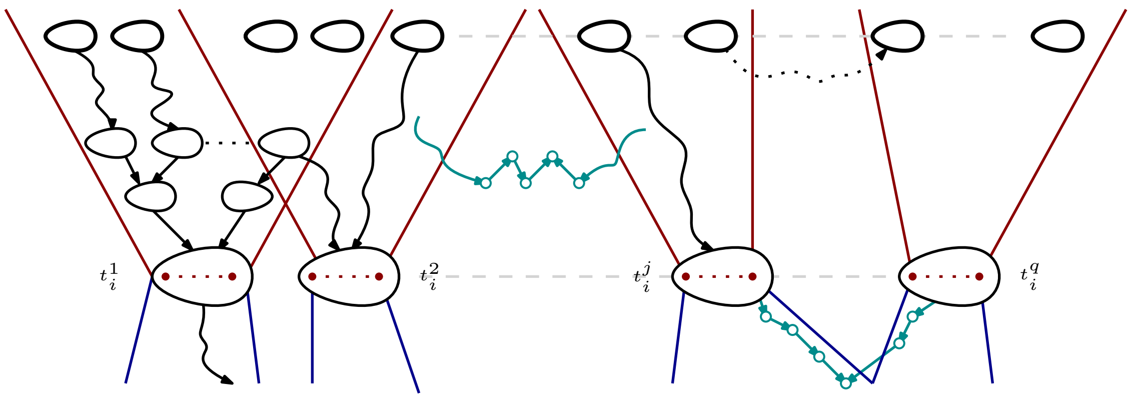

Fig. 4: Illustration of the configuration of $\vec{T}_{H_i}$, $1 \leqslant i \leqslant c$. Top bags represent maximal cliques of H. Notice that whenever two minimal filled bags have an ancestor in common, their corresponding sets of descendants cannot intersect nor be connected by a path, and vice-versa. Whenever such sets are disjoint, either ancestors or descendants are connected by a path (recall $G[C_i]$ is connected).

Claim 9. Let $t \in V(\vec{T}_{H_i})$ be a bag that contains no affected vertex, that is $V_t \cap \{u \in V(G) : \exists v \in V(G), \{u,v\} \in F\} = \emptyset$. Let $(C_i^j)_{1 \leqslant j \leqslant \ell}$ be the collection of connected components of $G[C_i] \setminus V_t$. Then, for every $1 \leqslant j \leqslant \ell$, C_i^j contains an affected vertex.

Proof. We will assume that there exists a $1 \leqslant j \leqslant l$, such that C_i^j does not contain any affected vertex. As neither C_i^j nor V_t contain an affected vertex, $G[C_i^j \cup V_t] = H[C_i^j \cup V_t]$ is ptolemaic and C_i^j is a connected component of $H \setminus V_t$. Moreover, V_t is either a minimal separating clique or a maximal clique of H, as t is a bag of $\vec{T}_{H_i}$. By Corollary 1 (iii), C_i^j is a semi-split of H. We know that $N_G(C_i^j) \subseteq V_t$ and $G[C_i^j \cup V_t] = H[C_i^j \cup V_t]$, so C_i^j is a semi-split of G. Thanks to Claims 4 and 7, we know that C_i^j is neither a type-β V_t-tentacle nor a type-γ V_t-tentacle, so C_i^j is a type-α V_t-tentacle. But since G is reduced under Rule 6, this leads to a contradiction. ◇

Claim 10. For every $1 \leqslant i \leqslant c$, $1 \leqslant j \leqslant q$, $\left|\mathcal{B}_i^j\right| \leqslant 6k - 1$.

Proof. Let $\{u,v\}$ be a pair of F that belongs to $V_{t_i^j}$. Moreover, let $\mathcal{C}_i^j$ be the collection of maximal cliques of H_i contained in $\mathcal{B}_i^j$. By Observation 3, we know that $\{u,v\} \in V_t$ for every $t \in \mathcal{B}_i^j$. Let $\{s_1, \dots s_p\}$ be the set of children in $\mathcal{B}_i^j$ of bags corresponding to maximal cliques in $\mathcal{C}_i^j$. By Theorem 2, the bag of each clique in $\mathcal{C}_i^j$ has exactly one s_i as a child, otherwise there would be a cycle in T_{H_i}. Thus, the removal of $\cup_{i=1}^{p} V_{s_i}$ leaves none of the cliques in $\mathcal{C}_i^j$ empty, each of them in a different connected component, and all adjacent to both u and v. Since $|F| \leqslant k$, at most $2k$ of these connected components contain vertices from F. Hence, assuming $\left|\mathcal{C}_i^j\right| > 3k$, there exists a subset of $k+1$ vertices $\{c_1, \dots, c_{k+1}\}$ that are adjacent to both u and v and induce an independent set in G. Hence, the collection $\{c_1, c_2, u, v\}, \{c_2, c_3, u, v\} \dots \{c_{k+1}, c_1, u, v\}$ is a C_4-sunflower of G, contradicting the fact that G is reduced.

We may hence assume that $\left|\mathcal{C}_i^j\right| \leqslant 3k$. Since any set in $\mathcal{B}_i^j$ is obtained as the intersection of two or more sets of $\mathcal{C}_i^j$ and since T_{H_i} is a tree, we have that $\left|\mathcal{B}_i^j\right| \leqslant (2 \times \left|\mathcal{C}_i^j\right|) - 1 \leqslant 6k - 1$. ◇

Claim 11. For every $1 \leqslant i \leqslant c$, $|\mathcal{F}_i| \leqslant k_i \cdot (6k-1)$. Hence $|\mathcal{F}| \leqslant 6k^2 - k$.

Proof. Since $\mathcal{F}_i = \cup_{j=1}^{q} \mathcal{B}_i^j$, Claim 10 implies that $|\mathcal{F}_i| = \sum_{j=1}^{q} \left|\mathcal{B}_i^j\right| \leqslant k_i \cdot (6k-1)$. Hence we have that $|\mathcal{F}| = \sum_{i=1}^{c} |\mathcal{F}_i| \leqslant \sum_{i=1}^{c} k_i \cdot (6k-1) \leqslant k \cdot (6k-1)$ since $\sum_{i=1}^{c} k_i = k$. ◇

Tentacles. Let T_i denote a minimal tree spanning vertices of $\mathcal{F}_i$ in T_{H_i}, $1 \leqslant i \leqslant c$. Moreover, let $\mathcal{S}_i$ be the set of maximal subtrees of $T_{H_i} \setminus T_i$. In order to count the bags of those subtrees properly, we are partitioning $\mathcal{S}_i$ into two parts, namely $\mathcal{S}_i^{\mathcal{F}}$ which contains the subtrees connected to T_i by a bag of $\mathcal{F}_i$ and $\mathcal{S}_i^{\mathcal{I}} = \mathcal{S}_i \setminus \mathcal{S}_i^{\mathcal{F}}$.

Claim 12. Let $T_{\mathcal{S}}$ be any subtree of $\mathcal{S}_i^{\mathcal{F}}$. The vertices of G contained in the bags of $T_{\mathcal{S}}$ can be partitioned into a clique separator S and a collection $\mathcal{C}$ of S-tentacles such that $\cup_{C \in \mathcal{C}} \Phi_S^G(C)$ is a laminar family. Moreover, $T_{\mathcal{S}}$ contains an affected vertex.

Proof. By definition of T_{H_i} and $\mathcal{F}_i$, exactly one bag t of $T_{\mathcal{S}}$ has a parent p in T_i, and p is the only bag of T_i that has a child in $T_{\mathcal{S}}$. (otherwise T_{H_i} would contain a cycle). We will denote by $V_{\mathcal{S}}$ the set $\cup_{t \in V(T_{\mathcal{S}})} V(t)$. Since any affected vertex in $V_{\mathcal{S}}$ must be contained in V_t, the components of $H \setminus V_t$ containing vertices of $V_{\mathcal{S}}$ are unaffected. Thus each of them is a component of $G \setminus V_t$. Since H is ptolemaic, we directly have that those components form a collection $\mathcal{C}$ of V_t-tentacles such that $\cup_{C \in \mathcal{C}} \Phi_S^G(C)$ is a laminar family in G. Notice that at least one such component exists since t is not the only bag of $T_{\mathcal{S}}$, as it has at least two parents by definition of $\vec{T}_{H_i}$. It follows that V_t is a separating clique of H and thus of G. Now, if $V_{\mathcal{S}}$ does not contain any affected vertex then so does V_t, which contradicts Claim 9. ◇

Claim 13. Let $\mathcal{S}^{\mathcal{F}} = \cup_{i=1}^{c} \mathcal{S}_i^{\mathcal{F}}$. Then $\left|V(\mathcal{S}^{\mathcal{F}})\right| = O(k^3)$, where $V(\mathcal{S}^{\mathcal{F}})$ denotes the bags of $\mathcal{S}^{\mathcal{F}}$.

Proof. Using both Claim 8 and the last part of Claim 12 one can observe that there are at most $2k_i$ maximal subtrees in $\mathcal{S}_i^{\mathcal{F}}$. Moreover, Claim 12 implies that each subtree $T_{\mathcal{S}}$ of $\mathcal{S}_i^{\mathcal{F}}$ is a bag t and a collection of V_t-tentacles, so each tentacle is at most in one bag of $T_{\mathcal{S}}$ since G has been reduced under Rule 1. The other bags of $T_{\mathcal{S}}$ are in $desc(t)$ because they are subsets of V_t. Using Lemma 9, we know that there are $O(k^2)$ components in $G \setminus V_t$, thus $O(k^2)$ components in $H \setminus V_t$. Lemma 13 allows us to conclude that $|desc(t)| = O(k^2)$, so $T_{\mathcal{S}}$ contains $O(k^2)$ bags. Hence $\left|V(\mathcal{S}^{\mathcal{F}})\right| = \sum_{i=1}^{c} \left|V(\mathcal{S}_i^{\mathcal{F}})\right| = \sum_{i=1}^{c} 2k_i \cdot O(k^2)$, which implies $\left|V(\mathcal{S}^{\mathcal{F}})\right| = O(k^3)$ since $\sum_{i=1}^{c} k_i = k$. ◇

We now consider the set $\mathcal{R}_i^{\geqslant 3}$ of bags of $T_i \setminus \mathcal{F}_i$ having degree at least 3 in T_i.

Claim 14. $\left|\mathcal{R}_i^{\geqslant 3}\right| \leqslant k_i$.

Proof. Let $\tilde{T}_i$ be the tree obtained from T_i by contracting each maximum subtree of T_i containing only bags in $\mathcal{F}_i$ into one single bag, called a special bag. In other words, for each such maximum subtree T' of T_i replace T' by a single special bag t' and make t' adjacent to the subset N' of bags of T_i defined as $N' = \{u \in T_i \setminus T' \mid \exists v \in T', uv \in E(T_i)\}$. Note that since by definition $\mathcal{R}_i^{\geqslant 3} \cap \mathcal{F}_i = \varnothing$, all the bags in $\mathcal{R}_i^{\geqslant 3}$ are in $\tilde{T}_i$. Moreover, since T_i is the minimum subtree of T_{H_i} spanning bags in $\mathcal{F}_i$, then all the leaves of T_i belong to $\mathcal{F}_i$ and it follows that the leaves of $\tilde{T}_i$ are all special bags. Finally, observe that each edge $uv \in F_i$ belong to exactly one special bag, as the bags of T_{H_i} containing edge uv form a subtree of T_{H_i}. As a consequence, the number of leaves in $\tilde{T}_i$ is at most k_i. As the number of vertices having degree at least 3 in a tree having p leaves is at most $p-2$ (folklore), we obtain that in particular $|\mathcal{R}_i^{\geqslant 3}| \leq k_i$. ⋄

Claim 15. Any subtree in $\mathcal{S}_i^{\mathcal{I}}$ is adjacent to a bag in $\mathcal{A}_i$.

Proof. Let T' be a subtree in $\mathcal{S}_i^{\mathcal{I}}$ and assume for contradiction that T' is adjacent to a bag $t \in T_i \setminus \mathcal{F}_i$ that is not in $\mathcal{A}_i$. Then, Claim 9 gives that $V(T')$ must contain some affected vertex v. As any affected vertex also belongs to some bag in $\mathcal{F}_i \subseteq T_i$, then $v \in V_t$, which contradicts the fact that $t \notin \mathcal{A}_i$. ⋄

We can now count the number of bags of $\mathcal{S}_i^{\mathcal{I}}$. To that aim, we will count them together with $\mathcal{A}_i$, the set of bags of $T_i \setminus \mathcal{F}_i$ containing at least one affected vertex.

Claim 16. The number of bags of $\mathcal{S}_i^{\mathcal{I}} \cup \mathcal{A}_i$ is $O(k_i \cdot k^2)$.

Proof. First, notice that from Claim 15, each subtree in $\mathcal{S}_i^{\mathcal{I}}$ has to be adjacent to a bag in $\mathcal{A}_i$. We let $\mathcal{A}_i^{\mathcal{M}}$ be the set of maximal bags of $\mathcal{A}_i$: a bag is in $\mathcal{A}_i^{\mathcal{M}}$ if and only if it has no parent in $\mathcal{A}_i$. So we can associate to each subtree $T_{\mathcal{S}} \in \mathcal{S}_i^{\mathcal{I}}$, a bag $t \in \mathcal{A}_i^{\mathcal{M}}$ such that it is an ancestor of the bag in $\mathcal{A}_i$ adjacent to $T_{\mathcal{S}}$ (there might be several possibilities, then we pick any of them). We denote by $V_{\mathcal{S}}$ the set of all vertices in the bags of $T_{\mathcal{S}}$ and $V_{\mathcal{I}}$ the vertices in the bags of $\mathcal{S}_i^{\mathcal{I}} \cup \mathcal{A}_i$. Notice that by construction, $V_{\mathcal{I}}$ does not contain any pair of F and hence $G[V_{\mathcal{I}}] = H[V_{\mathcal{I}}]$ is ptolemaic. In particular, this means that $G[V_t \cup V_{\mathcal{S}}] = H[V_t \cup V_{\mathcal{S}}]$. Hence the vertices of $V_{\mathcal{S}} \setminus V_t$ form a collection $\mathcal{C}$ of V_t-tentacles in G and $\Phi^G_{V_t}(\mathcal{C})$ is laminar. For any $T'_{\mathcal{S}} \in \mathcal{S}_i^{\mathcal{I}}$ associated to the same bag t, we define, similarly as $\mathcal{C}$, the collection $\mathcal{C}'$ of V_t-tentacles formed by $V'_{\mathcal{S}}$, the vertices of $T'_{\mathcal{S}}$. Thus, $\Phi^G_{V_t}(\mathcal{C}) = \Phi^H_{V_t}(\mathcal{C})$ and $\Phi^G_{V_t}(\mathcal{C}') = \Phi^H_{V_t}(\mathcal{C})$ do not overlap, as $G[V_{\mathcal{I}}]$ is ptolemaic. Thanks to Lemma 9, for each bag $t \in \mathcal{A}_i^{\mathcal{M}}$ there are $O(k^2)$ components in $G \setminus V_t$, so also in $H \setminus V_t$, and we also know that the components of $G[V_{\mathcal{I}}] \setminus V_t = H[V_{\mathcal{I}}] \setminus V_t$ are V_t-tentacles. Since G is reduced under Rule 1, any tentacle is composed of only a clique. Then, the bags of $\mathcal{S}_i^{\mathcal{I}} \cup \mathcal{A}_i$ associated to t are either in $desc(t)$ or single bags connected to $desc(t) \cup \{t\}$. Using Lemma 13, we can conclude that $|desc(t)| = O(k^2)$. So there are $O(k^2)$ bags of $\mathcal{S}_i^{\mathcal{I}} \cup \mathcal{A}_i$ associated to t. Finally, the number of bags of $\mathcal{S}_i^{\mathcal{I}} \cup \mathcal{A}_i$ is $O(|\mathcal{A}_i^{\mathcal{M}}| \cdot k^2)$.

Notice now, that for any undirected path P in $\vec{T}_{H_i}$ and for any vertex $v \in V(G)$, the number of bags of P without any parent in P containing v is at most 2 (see Claim 8). Moreover if there are 2 such bags, they are *consecutive* (meaning that no other bag without parents in P is on the path between them. Using this observation, one can see that on every path in $T_i \setminus (\mathcal{F}_i \cup \mathcal{R}_i^{\geqslant 3})$, the bags of $\mathcal{A}_i^{\mathcal{M}}$ that are not on an endpoint of this path have to be consecutive to an endpoint of this path, as each affected vertex in the path has to reach $\mathcal{F}_i$, and thus is also in at least one of its endpoint. Hence, on each such path, there are at most 4 bags of $\mathcal{A}_i^{\mathcal{M}}$. Since there are at most $2k_i - 1$ paths in $T_i \setminus (\mathcal{F}_i \cup \mathcal{R}_i^{\geqslant 3})$, we have that $|\mathcal{A}_i^{\mathcal{M}}| \leqslant 4 \cdot (2k_i - 1) \leqslant 8k_i$. So we can conclude that $V(\mathcal{S}_i^{\mathcal{I}} \cup \mathcal{A}_i) = O(k_i \cdot k^2)$. ⋄

Claim 17. Let P be a path of $T_i \setminus (\mathcal{F}_i \cup \mathcal{R}_i^{\geqslant 3} \cup \mathcal{A}_i)$, then $|V(P)| \leqslant 15$ where $V(P)$ denotes the bags of P.

Proof. Assume for contradiction that the number of bags in P is at least 16. We will show that Rule 8 can be applied to G, contradicting the fact that G is reduced under Rule 8.

First, observe that since the bags in P, which have degree 2 in T_i, are not in $\mathcal{A}_i$, Claim 15 implies that they have degree 2 in T_{H_i} as well. Note that a bag of $\vec{T}_{H_i}$ that is not a source has at least two parents and that a bag of $\vec{T}_{H_i}$ that is not a sink has at least one child. Consequently, since all bags of P have degree 2 in $\vec{T}_{H_i}$, then P contains only sources and sinks of $\vec{T}_{H_i}$. Since P has at least 16 bags, there exists a subpath $P' = u_1, b_6, b_5, b_4, b_3, b_2, b_1, t, b'_1, b'_2, b'_3, b'_4, b'_5, b'_6, u_2$ of P such that $b_6, b_4, b_2, t, b'_2, b'_4, b'_6$ are sinks in $\vec{T}_{H_i}$ and $u_1, b_5, b_3, b_1, b'_1, b'_3, b'_5, u_2$ are sources in $\vec{T}_{H_i}$. For any $i \in [\![1, 6]\!]$, we denote $B_i = V_{b_i}$ and $B'_i = V_{b'_i}$. Let $S = V_t$, $S_1 = B_6$ and $S_2 = B'_6$. We show that S, S_1 and S_2 satisfy all conditions of Rule 8 in G.

First, note that S separates H in exactly 2 connected components: C_1 that contains S_1 and C_2 that contains S_2. We show that $G[C_1]$ is connected, the same holding for $G[C_2]$ by symmetry. Assume for a contradiction that $G[C_1]$ is not connected. Since vertices of S are in bag b_1 but not in bag b_2 (by Claim 8), then $N_H(S) \cap C_1 = B_1 \setminus S$. As $b_1 \notin \mathcal{F}_i \cup \mathcal{A}_i$, b_1 is not affected and it follows that B_1 is a clique in G as well, which implies that $N_G(S) \cap C_1 = B_1 \setminus S$ is a clique. Since $G[C_1]$ is not connected, then there exists a connected component C of $G[C_1]$ that does not contain any vertex of $B_1 \setminus S$ (which is connected in G). Therefore, in G, the vertices of C are not adjacent to the vertices of S, they are not adjacent to the vertices of $C_1 \setminus C$ and they are not adjacent to the vertices of C_2 (since in H they are not): C is a connected component of G. By the choice of F, F does not contain any pair between connected components of G, so $C \subseteq C_1$ must be a connected component of H as well, which is not: a contradiction. Thus, $G[C_1]$ is connected and so is $G[C_2]$ by symmetry. Condition 1 of Rule 8 is satisfied by S in G.

As explained above, $N_H(S) \cap C_1 = B_1 \setminus S$. Observe that none of the vertices of $B_1 \setminus S$ are in bag b_4, because there is no common descendant to b_1 and b_4. On the other hand, some of the vertices in $B_1 \setminus S$, namely the vertices in B_2, are in bag b_3. It follows that $N_H^2(S) \cap C_1 = (B_3 \cup B_1) \setminus S$. Moreover, note that because none of the bags of P' are in $\mathcal{F}_i \cup \mathcal{A}_i$, i.e. they are neither filled nor affected bags, then we have that $H[\bigcup_{1 \leq i \leq 6} B_i \cup S \cup \bigcup_{1 \leq i \leq 6} B'_i] = G[\bigcup_{1 \leq i \leq 6} B_i \cup S \cup \bigcup_{1 \leq i \leq 6} B'_i]$. In particular, we obtain that $N_H^2(S) = N_G^2(S)$ and $G[S \cup N_G^2(S)] = H[S \cup N_G^2(S)]$ is ptolemaic, by heredity: Condition 2 of Rule 8 is satisfied by S in G.

As mentioned previously, none of the vertices of B_1 are in B_4 and for the same reason (b_3 and b_6 have no common descendant in $\vec{T}_{H_i}$), none of the vertices in B_3 are in B_6. It follows that $B_6 = S_1$ is disjoint from $B_3 \cup B_1$ and so it is from $N^2_G(S) \cap C_1 = N^2_H(S) \cap C_1 = (B_3 \cup B_1) \setminus S$. One can see that in $G[C_1]$, S_1 separates the vertices of $B_5 \setminus S_1$ from the vertices of $V_{u_1} \setminus S_1$. Moreover, $V_{u_1} \setminus S_1$ is complete to S_1 in H as $S_1 \subseteq V_{u_1}$ and V_{u_1} is a clique in H. As bag u_1 contains no added edges in H, then $V_{u_1} \setminus S_1$ is also complete to S_1 in G. For the same reasons, $B_5 \setminus S_1$ is complete to S_1 in G and S_1 is thus a clique minimal separator of $G[C_1]$. We let $C^S_1 = (\bigcup_{1 \leq i \leq 5} B_i) \setminus (S \cup S_1)$. Observe that $C^S_1 \supseteq N^2_G(S) \cap C_1 = (B_3 \cup B_1) \setminus S$ and that C^S_1 is connected in H, and so also in G, since $H[\bigcup_{1 \leq i \leq 6} B_i] = G[\bigcup_{1 \leq i \leq 6} B_i]$. Again because H and G coincides on $C^S_1 \cup S_1$, we obtain that $G[C^S_1 \cup S_1]$ is ptolemaic. Furthermore, we have $\delta_{G[C_1]}(C^S_1) = B_5 \setminus S_1$ and $\delta_{G[C_1]}(C^S_1)$ is complete to S_1 as $S_1 \subseteq B_5$ and B_5 is a clique in G. This gives that C^S_1 is an S_1-clam in G: Condition 3 of Rule 8 is satisfied by S and S_1 in G. By symmetry, we obtain that S and S_2 satisfy Condition 3 of Rule 8 in G.

Therefore, all conditions of Rule 8 are satisfied with S, S_1 and S_2 in G and since G is reduced under Rule 8, S should have been removed from G: a contradiction. This proves that the number of bags in P is at most 15. $\diamond$

To conclude the proof, notice that by construction, any bag of $\vec{T}_H$ is either in $\mathcal{F}$, or in the set $V_i = V(T_i) \setminus \mathcal{F}_i$ of bags of the minimal tree T_i spanning $\mathcal{F}_i$ or in a maximal subtree of $T_{H_i} \setminus T_i$ for some $1 \leqslant i \leqslant c$. Recall that any minimal tree T_i is partitioned into three sets, namely $\mathcal{R}^{\geqslant 3}_i$, $\mathcal{A}_i$ and a set of at most $2k_i$ paths of $T_i \setminus (\mathcal{F}_i \cup \mathcal{R}^{\geqslant 3}_i \cup \mathcal{A}_i)$. Using respectively Claims 11, 13, 16, 14 and 17, we obtain that:

$$\left|V(\vec{T}_H)\right| = \left|\mathcal{F} \bigcup (\cup^c_{i=1} V(\mathcal{S}_i)) \bigcup (\cup^c_{i=1} V_i)\right| \leqslant O(k^2) + \sum_{i=1}^{c} O(k_i \cdot k^2) + \sum_{i=1}^{c} (k_i + O(k_i \cdot k^2) + 15k_i)$$

In turn, since $\sum^c_{i=1} k_i = k$, this implies that $\left|V(\vec{T}_H)\right| \in O(k^3)$. By Lemma 10 and Rule 4, we thus conclude that G contains $O(k^4)$ vertices. Since all reduction rules can be applied in polynomial time (Lemmata 6 and 8), the result follows. $\square$