Sparse graphs with bounded induced cycle packing number have logarithmic treewidth

Marthe Bonamy$^1$ Édouard Bonnet$^2$ Hugues Déprés$^2$ Louis Esperet$^3$
Colin Geniet$^2$ Claire Hilaire$^1$ Stéphan Thomassé$^2$ Alexandra Wesolek$^4$

$^1$Univ. Bordeaux, France
$^2$ENS de Lyon, France
$^3$Univ. Grenoble, France
$^4$Simon Fraser Univ., Canada

Symposium on Discrete Algorithms 2023
Erdős Pósa property

Cycle packing $cp(G)$: maximum size of a collection of vertex-disjoint cycles in $G$.

Feedback-vertex-set $fvs(G)$: minimum number of vertices required to intersect all cycles of $G$.

Theorem (Erdős–Pósa) $fvs(G) \leq f(cp(G))$ with $f(k) = O(k \log k)$.

A graph with $cp(G)$ bounded is a tree plus a bounded number of vertices.
Erdős Pósa property

Cycle packing $cp(G)$: maximum size of a collection of vertex-disjoint cycles in $G$.

Feedback-vertex-set $fvs(G)$: minimum number of vertices required to intersect all cycles of $G$.

\[ fvs(G) \leq f(cp(G)) \text{ with } f(k) = O(k \log k) \]

A graph with $cp(G)$ bounded is a tree plus a bounded number of vertices.

$\Rightarrow$ algorithmically simple graphs
Odd cycles packing

Odd cycle packing $ocp(G)$: same with only odd cycles.

**Theorem (Fiorini, Joret, Weltge, Yuditsky, '21)**

In graphs with $ocp(G) \leq k$, maximum independent set can be solved in polynomial time.
Odd cycles packing

Odd cycle packing $ocp(G)$: same with only odd cycles.

**Theorem (Fiorini, Joret, Weltge, Yuditsky, '21)**

*In graphs with $ocp(G) \leq k$, maximum independent set can be solved in polynomial time.*

Induced odd cycle packing $iopc(G)$: only consider packings of non-adjacent odd cycles.

**Theorem (Bonamy et al., '18)**

*For graphs with $iopc(G)$ bounded, VC-dimension bounded, and linear size independent sets, there is an EPTAS for maximum independent set.*

Applications to disk and unit ball graphs.
Induced cycle packing

*Induced cycle packing* $icp(G)$: maximum number of vertex-disjoint *and non-adjacent* cycles in $G$. 

We study the class of graphs with $icp(G) \leq k$ ($k$ constant). 

Problems:
- Testing $icp(G) \leq k$
- Algorithms in this class for independent set, …

Question: Does $icp(G) \leq k$ imply $fvs(G) \leq f(k)$?

No: cliques have $icp(K_t) = 1$ but $fvs(K_t) = t - 2$. 

Colin Geniet (ENS de Lyon)
**Induced cycle packing**

*Induced cycle packing* $icp(G)$: maximum number of vertex-disjoint *and non-adjacent* cycles in $G$.

We study the class of graphs with $icp(G) \leq k$ ($k$ constant).

Problems:

- Testing $icp(G) \leq k$
- Algorithms in this class for independent set, …
Induced cycle packing

*Induced cycle packing* $icp(G)$: maximum number of vertex-disjoint *and non-adjacent* cycles in $G$.

We study the class of graphs with $icp(G) \leq k$ ($k$ constant).

Problems:
- Testing $icp(G) \leq k$
- Algorithms in this class for independent set, ...

**Question**

Does $icp(G) \leq k$ imply $fvs(G) \leq f(k)$?
Induced cycle packing

*Induced* cycle packing $icp(G)$: maximum number of vertex-disjoint *and* non-adjacent cycles in $G$.

We study the class of graphs with $icp(G) \leq k$ ($k$ constant).

Problems:
- Testing $icp(G) \leq k$
- Algorithms in this class for independent set, ... 

**Question**

Does $icp(G) \leq k$ imply $fvs(G) \leq f(k)$?

No: cliques have $icp(K_t) = 1$ but $fvs(K_t) = t - 2$. 
Induced cycle packing

*Induced* cycle packing $icp(G)$: maximum number of vertex-disjoint *and non-adjacent* cycles in $G$.

We study the class of graphs with $icp(G) \leq k$ ($k$ constant).

Problems:
- Testing $icp(G) \leq k$
- Algorithms in this class for independent set, ...

**Question**

Does $icp(G) \leq k$ and no $K_{t,t}$ subgraph imply $fvs(G) \leq f(k, t)$?

Still no!
$icp(G) = 1$ and FVS unbounded.
Feedback vertex set is logarithmic

**Theorem**

If $G$ is a graph with $icp(G) \leq k$ and without $K_{t,t}$ subgraph, then

$$fvs(G) \leq f(k, t) \cdot \log n$$
Feedback vertex set is logarithmic

**Theorem**

If $G$ is a graph with $icp(G) \leq k$ and without $K_{t,t}$ subgraph, then

$$fvs(G) \leq f(k, t) \cdot \log n$$

Some problems with algorithms in time $2^{O(tw(G))} \cdot poly(n)$:

- Maximum independent set
- 3-coloring
- Hamiltonian cycle
- ...
- Testing $icp(G) \leq k$ [Mi. Pilipczuk, '22]

When $fvs(G)$ is logarithmic in $n$, these algorithms are polynomial.
Solving Maximum Independent Set

$F$ feedback vertex set of size $O(\log n)$.
Solving Maximum Independent Set

For each \( v \in F \), branch on \( v \):
- either pick \( v \in I \), and delete \( N[v] \),
- or \( v \notin I \), and delete \( v \).

After this, only a forest is left \( \Rightarrow \) pick leafs greedily.

Branching is polynomial because \( F \) is logarithmic.

\( F \) feedback vertex set of size \( O(\log n) \).

We construct \( I \) independent.

Colin Geniet (ENS de Lyon)
Graphs with bounded induced cycle packing
SODA23  6 / 10
Solving Maximum Independent Set

$F$ feedback vertex set of size $O(\log n)$.

We construct $I$ independent.
For each $v \in F$, branch on $v$:
- either pick $v \in I$, and delete $N[v]$,  
- or $v \notin I$, and delete $v$.

After this, only a forest is left $\Rightarrow$ pick leafs greedily.
Solving Maximum Independent Set

$F$ feedback vertex set of size $O(\log n)$.

We construct $I$ independent.
For each $v \in F$, branch on $v$:

- either pick $v \in I$, and delete $N[v]$,
- or $v \notin I$, and delete $v$.

After this, only a forest is left $\Rightarrow$ pick leafs greedily.

Branching is polynomial because $F$ is logarithmic.
Solving MIS in the dense case

Theorem

For any fixed $k$, Maximum Independent Set can be solved in quasipolynomial time $n^{O \left( \log n \right)}$ on graphs with $icp(G) \leq k$. 
Solving MIS in the dense case

Theorem

For any fixed $k$, Maximum Independent Set can be solved in quasipolynomial time $n^{O(\log n)}$ on graphs with $icp(G) \leq k$.

Let $S$ the set of cycles with length 4.
Solving MIS in the dense case

**Theorem**

For any fixed $k$, Maximum Independent Set can be solved in quasipolynomial time $n^{O(\log n)}$ on graphs with $\text{icp}(G) \leq k$.

Let $S$ the set of cycles with length 4.

Fix $C \in S$. All cycles are adjacent to $C$. 
Theorem

For any fixed $k$, Maximum Independent Set can be solved in quasipolynomial time $n^{O(\log n)}$ on graphs with $icp(G) \leq k$.

Let $S$ the set of cycles with length 4.

Fix $C \in S$. All cycles are adjacent to $C$.
Thus some $v \in C$ is adjacent to $1/4$ of the cycles of $S$. 

C

v

Graphs with bounded induced cycle packing

SODA23 7/10
Solving MIS in the dense case

Theorem

For any fixed $k$, **Maximum Independent Set** can be solved in quasipolynomial time $n^{O(\log n)}$ on graphs with $\text{icp}(G) \leq k$.

Let $S$ the set of cycles with length 4.

Fix $C \in S$. All cycles are adjacent to $C$. Thus some $v \in C$ is adjacent to $1/4$ of the cycles of $S$.

Branch on $v$:

- Take $v$ and delete $N(v) \Rightarrow$ destroys $1/4$ cycles in $S$,
- Delete $v \Rightarrow$ destroys $C$

This kind of branching is quasipolynomial.
Solving MIS in the dense case

**Theorem**

For any fixed $k$, **Maximum Independent Set** can be solved in quasipolynomial time $n^{O(\log n)}$ on graphs with $icp(G) \leq k$.

Let $S$ the set of cycles with length 4.

Fix $C \in S$. All cycles are adjacent to $C$.

Thus some $v \in C$ is adjacent to $1/4$ of the cycles of $S$.

Branch on $v$:
- Take $v$ and delete $N(v)$ ⇒ destroys $1/4$ cycles in $S$,
- Delete $v$ ⇒ destroys $C$

This kind of branching is quasipolynomial.

When $S = \emptyset$, we are in the $K_{2,2}$-free case.
Theorem

If $G$ is a graph with $icp(G) \leq k$ and without $K_{t,t}$ subgraph, then $fvs(G) \leq f(k, t) \cdot \log n.$
Theorem

If $G$ is a graph with $\text{icp}(G) \leq k$ and with girth $> 10$, then $G$ has average degree $\leq 2k + 2$. 
Back to the main theorem

**Theorem**

If $G$ is a graph with $\text{icp}(G) \leq k$ and with girth $> 10$, then $G$ has average degree $\leq 2k + 2$.

Pick $C$ cycle with minimal length.
Theorem

If $G$ is a graph with $\text{icp}(G) \leq k$ and with girth $> 10$, then $G$ has average degree $\leq 2k + 2$.

Pick $C$ cycle with minimal length.

Let $N$ its neighbourhood, $R = G \setminus (C \cup N)$,
If $G$ is a graph with $icp(G) \leq k$ and with girth $> 10$, then $G$ has average degree $\leq 2k + 2$.

Pick $C$ cycle with minimal length. Let $N$ its neighbourhood, $R = G \setminus (C \cup N)$, and $S$ the second neighbourhood of $C$. 
Back to the main theorem

**Theorem**

If $G$ is a graph with $icp(G) \leq k$ and with girth $> 10$, then $G$ has average degree $\leq 2k + 2$.

Pick $C$ cycle with minimal length. Let $N$ its neighbourhood, $R = G \setminus (C \cup N)$, and $S$ the second neighbourhood of $C$.

$C$ is the only cycle in $G[C \cup N \cup S]$, otherwise $C$ would not be minimal $\Rightarrow$ average degree $\leq 2$. 

Colin Geniet (ENS de Lyon)  
Graphs with bounded induced cycle packing  
SODA23 8/10
Theorem

If $G$ is a graph with $icp(G) \leq k$ and with girth $> 10$, then $G$ has average degree $\leq 2k + 2$.

Pick $C$ cycle with minimal length. Let $N$ its neighbourhood, $R = G \setminus (C \cup N)$, and $S$ the second neighbourhood of $C$.

$C$ is the only cycle in $G[C \cup N \cup S]$, otherwise $C$ would not be minimal $\Rightarrow$ average degree $\leq 2$.

$G[R]$ is disjoint from $C$, so $icp(G[R]) \leq k - 1$. $\Rightarrow$ average degree $\leq 2k$ by induction.
Summary

For graphs with $icp(G) \leq k$ and no $K_{t,t}$ subgraph: (sparse setting)

- Feedback vertex set is logarithmic + tight up to the constant.
Summary

For graphs with \( icp(G) \leq k \) and no \( K_{t,t} \) subgraph: (sparse setting)

- Feedback vertex set is logarithmic + tight up to the constant.
- Polynomial algorithm for independent set, and many other problems.
- Polynomial algorithm to compute \( icp(G) \).

For graphs with \( icp(G) \leq k \): (dense setting)

- Quasi-polynomial algorithms for independent set and 3-coloring.

Related result:
Theorem (Nguyen, Scott, Seymour + Le, '22)

In graphs with \( icp(G) \leq k \), there are at most \( |V(G)| f(k) \) induced paths.

Implies a polynomial algorithm to test \( icp(G) \leq k \).
Summary

For graphs with $icp(G) \leq k$ and no $K_{t,t}$ subgraph: (sparse setting)

- Feedback vertex set is logarithmic + tight up to the constant.
- Polynomial algorithm for independent set, and many other problems.
- Polynomial algorithm to compute $icp(G)$.

For graphs with $icp(G) \leq k$: (dense setting)

- Quasi-polynomial algorithms for independent set and 3-coloring.

Related result:

Theorem (Nguyen, Scott, Seymour + Le, ’22)

In graphs with $icp(G) \leq k$, there are at most $|V(G)|^{f(k)}$ induced paths.

Implies a polynomial algorithm to test $icp(G) \leq k$. 
Summary

For graphs with $icp(G) \leq k$ and no $K_{t,t}$ subgraph: (sparse setting)
- Feedback vertex set is logarithmic + tight up to the constant.
- Polynomial algorithm for independent set, and many other problems.
- Polynomial algorithm to compute $icp(G)$.

For graphs with $icp(G) \leq k$: (dense setting)
- Quasi-polynomial algorithms for independent set and 3-coloring.

Related result:

**Theorem (Nguyen, Scott, Seymour + Le, ’22)**

In graphs with $icp(G) \leq k$, there are at most $|V(G)|^{f(k)}$ induced paths.

Implies a polynomial algorithm to test $icp(G) \leq k$. 
Open Questions

- In the dense settings, can quasi-polynomial algorithms be improved to be polynomial?
- Any FPT algorithms with $icp(G)$ as parameter?
- What about restricting packing of specific types of cycles? (E.g., packing nonadjacent induced cycles of length $\geq 4$.)
Open Questions

- In the dense settings, can quasi-polynomial algorithms be improved to be polynomial?
- Any FPT algorithms with $icp(G)$ as parameter?
- What about restricting packing of specific types of cycles? (E.g., packing nonadjacent induced cycles of length $\geq 4$.)

Thank you!