Twin-Width of Groups and Graphs of Bounded Degree

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**Twin-Width**

Contractions:
- Any pair of vertices can be contracted (not just edges)
- Loops and double edges are removed

Contraction sequence: $G_n, \ldots, G_1$, where
- $G_i$ is result of a contraction in $G_{i+1}$
- $G_1$ has just one vertex

Twin-width:

$$
tw(G) = \min_{G=G_n, \ldots, G_1 \text{ contr. seq.}} \max_{i \in [n]} \max_{v \in V(G_i)} d_{red}(v)
$$

Simplified definition for graphs of bounded degree.
Examples

- Paths, cycles have $\text{tww} = 2$
- Trees have $\text{tww} = \Delta$
- Grids have $\text{tww} = 4$
- $d$-dimensional grids have $\text{tww} = O(d)$
Example: Bilu–Linial Expanders

2-lift of $G$:

- Duplicate each vertex $v \in V(G)$ into $v_0, v_1$.
- For $uv \in E(G)$ add either
  - the edges $u_0 v_0$ and $u_1 v_1$ (straight),
  - or the edges $u_0 v_1$ and $u_1 v_0$ (crossing).

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Theorem (Bilu and Linial, '06)
Iterated 2-lifts starting from $K_4$, with random choices of straight/crossing, yield cubic expanders almost surely.

All iterated 2-lifts of $K_4$ have $\delta(G) \leq 6$: reverse the lift sequence.
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Why Twin-Width

For classes of graphs with bounded twin-width:

- FPT first-order model checking (given a contraction sequence) [É.B., E.J. Kim, S.T., R.Watrigant].
- Quasi-polynomially $\chi$-bounded [Mi.Piliczuk, M.Sokołowski]
- Some FPT and approximation algorithms for independent set, dominating set [É.B., C.G., E.J. Kim, S.T., R.Watrigant].
Small Classes

When counting graphs on $n$ vertices in a class $C$, we count graphs in $C$ with vertices labeled from 1 to $n$.

A class is small if the number of graphs on $n$ vertices is

$$O(n! \cdot c^n) = 2^{n \log n + O(n)}$$

Examples:

- Trees
- Proper minor-closed classes [Norine, Seymour, Thomas, Wollan]

**Theorem (É.B., C.G., E.J. Kim, S.T., R.Watrigant)**

Any class with bounded twin-width is small.
Not Small Classes

Number of cubic graphs on $n$ vertices:

$$2^{3/2 \cdot n \log n + \Omega(n)}$$

Number of graphs of twin-width $k$ on $n$ vertices:

$$2^{n \log n + O_k(n)}$$

Corollary

*Expected twin-width of random cubic graphs is unbounded.*
Questions

1. Can we find explicit constructions of graphs with bounded degree and unbounded twin-width?
2. Do all small (hereditary) classes have bounded twin-width?
Power of Graphs

The $k$th power of $G$ is the graph $G^{(k)}$ with

- vertices $V(G)$
- an edge $xy$ whenever $d_G(x, y) \leq k$

**Lemma**

$$\text{tww} \left( G^{(k)} \right) \leq \text{tww}(G)^k$$

**Generalisation** (for the general definition of twin-width):

**Theorem**

*For any first-order transduction $\Phi$ and graph $G$,*

$$\text{tww}(\Phi(G)) \leq f(\text{tww}(G), \Phi)$$
Power of Graphs (Proof)

Contraction sequence of width $t$:

$$G = G_n, \ldots, G_1 = K_1$$

same sequence on $G^{(k)}$:

$$G^{(k)} = G'_n, \ldots, G'_1 = K_1$$

$G'_i$ is a subgraph of $G^{(k)}_i$:

$$\Delta(G'_i) \leq \Delta\left(G^{(k)}_i\right) \leq \Delta(G_i)^k \leq t^k$$
Coarse Geometry

$f : X \rightarrow Y$ is a $\lambda$-quasi-isometric embedding if

$$\lambda^{-1}d_X(x, y) - \lambda \leq d_Y(f(x), f(y)) \leq \lambda d_X(x, y) + \lambda$$
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Lemma

If $f : H \to G$ is a $\lambda$-quasi-isometric embedding of graphs of bounded degree,

\[ \text{tww}(H) \leq f(\lambda, \text{tww}(G)) \]
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If \( f : H \to G \) is a \( \lambda \)-quasi-isometric embedding of graphs of bounded degree,

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tww(H) \leq f(\lambda, tww(G))
\]

For \( G \) infinite, define

\[
tww(G) = \sup_{H \subset_{\text{fin}} G} tww(H)
\]

For infinite graphs with bounded degree, finite twin-width is preserved by quasi-isometries.
Cayley Graphs

Let $\Gamma$ group generated by $S$ finite.
The Cayley graph $\text{Cay}(\Gamma, S)$ has
- vertices $\Gamma$
- an edge from $x$ to $xs$ for every $x \in \Gamma$, $s \in S$. 

Examples:
- $\text{Cay}(\mathbb{Z}, \{1\})$ is the infinite path
- $\text{Cay}(\mathbb{Z}/n\mathbb{Z}, \{1\}) = \mathbb{C}_n$
- $\text{Cay}(\mathbb{Z}_2, \{(0,1), (1,0)\})$ is the infinite grid ($d$-dimensional grid for $\mathbb{Z}^d$)

If $F(S)$ is the group freely generated by $S$, $\text{Cay}(F(S), S)$ is the $2|S|$-regular tree.
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- If $\mathbb{F}(S)$ is the group freely generated by $S$, $\text{Cay}(\mathbb{F}(S), S)$ is the $2|S|$-regular tree.
Twin-Width of Groups

Lemma

All Cayley graphs of $\Gamma$ are quasi-isometric.

Finite twin-width is well-defined on groups.

Examples:

- $\mathbb{Z}$, $\mathbb{Z}/n\mathbb{Z}$
- Free groups
- Products of groups with finite twin-width
- (Finitely generated) commutative groups
Cayley Graphs

Let \( \Gamma \) group generated by \( S \).
Let \( C \) be the class of finite induced subgraphs of \( \text{Cay}(\Gamma, S) \).

**Lemma**

\( C \) is small.

**Proof.**

Any \( G \in C \) is characterized by a directed spanning tree, with edges labelled with \( S \cup S^{-1} \).
Cayley Graphs

Let $\Gamma$ group generated by $S$. Let $C$ be the class of finite induced subgraphs of $\text{Cay}(\Gamma, S)$.

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Any $G \in C$ is characterized by a directed spanning tree, with edges labelled with $S \cup S^{-1}$.

Suppose $\Gamma$ has infinite twin-width.

- $C$ is class of graphs with bounded degree and unbounded twin-width
- $C$ is a small class of graphs with unbounded twin-width
Group with Infinite Twin-Width

Theorem (Osajda, 2020)

Let \((G_n)_{n \in \mathbb{N}}\) be a sequence of graphs with

\[ \Delta(G_n) \leq D \]
\[ \text{diam}(G_n)/\text{girth}(G_n) \leq A \]
\[ \text{girth}(G_{n+1}) \geq \text{girth}(G_n) + 6 \]

There exists a group \(\Gamma\) finitely generated by \(S\) such that \(\text{Cay}(\Gamma, S)\) contains all \(G_n\) as isometric subgraphs.
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Lemma

There exists graphs \(G\) with arbitrarily large twin-width, and

- \(\Delta(G) \leq 6\)
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Sketch:
- Start from a random cubic graph.
- With probability $\frac{1}{2}$, there are not too many short ($< \log n$) cycles.
- Cut these short cycles (remove an edge in each).
- Choose a maximum packing $X$ of vertices at distance pairwise $> \log n$, and join them with a balanced cubic tree.
- The graph obtained satisfies the first 3 conditions.
- The above requires only $n^{1-\varepsilon}$ edge editions. This implies that the class of graphs satisfying the first 3 conditions is not small.
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There is a group with infinite twin-width.
We have no idea what it looks like.
It doesn’t help with constructing graphs of bounded degree and unbounded twin-width.
There is a small class of graphs with unbounded twin-width.
Grid Characterisation

A $k$-grid is a $k \times k$-division in which every zone has a ‘1’.

\[
\begin{array}{ccc}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 1 \\
1 & 0 & 0 \\
0 & 1 & 1 \\
\end{array}
\]

Grid number = maximum size of a grid.

Theorem

- A matrix $M$ has bounded twin-width if and only if it has bounded grid number.
- A graph $G$ has bounded twin-width if and only if there is an order $<$ on $V(G)$ such that the adjacency matrix of $G$ has bounded grid number.
For $x \in \Gamma$, $\prec$ order on $\Gamma$, $M^<_x$ is the permutation matrix of 

$$(y \in \Gamma) \mapsto y \cdot x$$

Claim

The adjacency matrix of $\text{Cay}(\Gamma, S)$ with order $\prec$ is

$$\bigvee_{s \in S \cup S^{-1}} M^<_s$$

Lemma

$\Gamma$ has finite twin-width if and only if there is an order $\prec$ on $\Gamma$ such that for every $x \in \Gamma$, $M^<_x$ has finite grid number.
Definition

$\Gamma$ has finite twin-width if there is an order $\prec$ on $\Gamma$ such that for every $x \in \Gamma$, $M_x^\prec$ has finite twin-width.

This definition works for non finitely generated groups.
Matrix Definition

Definition
Γ has finite twin-width if there is an order $<$ on $\Gamma$ such that for every $x \in \Gamma$, $M_x^<$ has finite twin-width.

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Definition
Uniform twin-width is

$$\text{utww}(\Gamma) = \inf_{< \text{ order on } \Gamma} \sup_{x \in \Gamma} \text{tww}(M_x^<)$$
Lemma

If $G$ is a group, $H \leq G$ a subgroup

$$\text{utww}(G) \leq \max(\text{utww}(H), \text{utww}(G/H))$$

Groups with finite uniform twin-width:

- Ordered groups
- Finitely generated abelian groups.
- Polycyclic groups
- Polynomial growth
Open Questions

- Explicit construction for groups (or graphs of bounded degree) with infinite twin-width?
- Separating twin-width and uniform twin-width for groups?
- Is there a universal bound on uniform twin-width of finite groups?
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3-dim. grid with diagonals has infinite stack number [Eppstein et. al.]
Stack number is not a group invariant, but queue number is!
- Matrix characterisation, uniform variants adapt to queue number.
- Separating queue number and twin-width?