Tournaments, First-Order Logic, and Twin-Width

Colin Geniet      Stéphan Thomassé

ENS Lyon

LoGAlg 2022
Simple classes for FO

Setting: $\mathcal{C}$ class of graphs (or relational structures), hereditary (= closed under induced subgraphs).

Is $\mathcal{C}$ simple with regards to first-order logic (FO)?
Simple classes for FO

Setting: \( \mathcal{C} \) class of graphs (or relational structures), hereditary (= closed under induced subgraphs).

Is \( \mathcal{C} \) simple with regards to first-order logic (FO)?

**Definition**

\( \mathcal{C} \) is *independent* if there is a FO transduction \( \Phi \) such that \( \Phi(\mathcal{C}) \) is the class of all graphs.

Otherwise, \( \mathcal{C} \) is dependent / NIP.
Simple classes for FO

Setting: \( \mathcal{C} \) class of graphs (or relational structures), hereditary (= closed under induced subgraphs).

Is \( \mathcal{C} \) simple with regards to first-order logic (FO)?

**Definition**

\( \mathcal{C} \) is **independent** if there is a FO transduction \( \Phi \) such that \( \Phi(\mathcal{C}) \) is the class of all graphs.
Otherwise, \( \mathcal{C} \) is dependent / NIP.

**Definition**

\( \mathcal{C} \) has **fixed-parameter tractable (FPT) FO model checking** if there is an algorithm to test \( G \models \phi \) for \( G \in \mathcal{C} \), in time

\[
f(\phi) \cdot \text{poly}(|G|)
\]
NIP and FO model checking

Conjecture (Gajarský et al., ’20)

A hereditary class $\mathcal{C}$ is NIP iff it has FPT model checking.
NIP and FO model checking

Conjecture (Gajarský et al., ’20)

A hereditary class $\mathcal{C}$ is NIP iff it has FPT model checking.

Theorem (Grohe, Kreutzer, Siebertz & Adler, Adler)

Let $\mathcal{C}$ class of graphs closed under subgraphs. TFAE:

- $\mathcal{C}$ is dependent,
- $\mathcal{C}$ has FPT model checking,
- $\mathcal{C}$ is nowhere dense.
Twin-width

Contracting vertices $x, y$ into $z$:
- if both $xu, yu$ are edges, then $zu$ is an edge
- if neither $xu, yu$ are edges, then $zu$ is not an edge
- if exactly one of $xu, yu$ are edges, then $zu$ is an error edge

Contraction sequence: start from $G$, contract until only one vertex is left.
Twin-width

Contracting vertices \( x, y \) into \( z \):
- if both \( xu, yu \) are edges, then \( zu \) is an edge
- if neither \( xu, yu \) are edges, then \( zu \) is not an edge
- if exactly one of \( xu, yu \) are edges, then \( zu \) is an error edge

![Diagram](image)

Contraction sequence: start from \( G \), contract until only one vertex is left.
Twin-width

Contracting vertices $x, y$ into $z$:

- if both $xu, yu$ are edges, then $zu$ is an edge
- if neither $xu, yu$ are edges, then $zu$ is not an edge
- if exactly one of $xu, yu$ are edges, then $zu$ is an error edge

Contraction sequence: start from $G$, contract until only one vertex is left.

width of the sequence = maximum red degree

twin-width $\text{tww}(G) = \text{minimum width of a sequence for } G$
Example: grids
FO and twin-width

**Theorem (Bonnet, Kim, Thomassé, Watrigant, '20)**

For any FO transduction $\Phi$, there is a function $f$ such that

$$\text{tww}(\Phi(G)) \leq f(\text{tww}(G))$$

**Corollary**

If $C$ has bounded twin-width, it is NIP.
**Theorem (Bonnet, Kim, Thomassé, Watrigant, ’20)**

For any FO transduction $\Phi$, there is a function $f$ such that

$$\text{tww}(\Phi(G)) \leq f(\text{tww}(G))$$

**Corollary**

If $C$ has bounded twin-width, it is NIP.

**Theorem (Bonnet, Kim, Thomassé, Watrigant, ’20)**

Given a graph $G$, a FO formula $\phi$, and a contraction sequence of width $t$, one can test $G \models \phi$ in time $f(\phi, t) \cdot n$.

effectively bounded twin-width $\Rightarrow$ FPT model checking
Converse results?

Cubic graphs:
- are NIP, 
- have FPT model checking, 
- but do *not* have bounded twin-width (counting argument).
Twin-width and ordered graphs

Ordered graph \((G, <)\): graph \(G\) with an order \(<\) on the vertices.

- FO logic can use the order:
  \[
  \forall x, y, z, x < y < z \land E(x, z) \implies E(x, y)
  \]

- Out of order contractions create errors for twin-width.
Twin-width and ordered graphs

Ordered graph \((G, <)\): graph \(G\) with an order \(<\) on the vertices.

- FO logic can use the order:

\[ \forall x, y, z, x < y < z \land E(x, z) \Rightarrow E(x, y) \]

- Out of order contractions create errors for twin-width.

**Theorem (BGOSTT, '21)**

Twin-width of ordered graphs can be approximated up to some function, and witnesses of twin-width can be computed.

Furthermore, for \(C\) a hereditary class of ordered graphs, TFAE:

- \(C\) is NIP,
- \(C\) has FPT FO model checking,
- \(C\) has bounded twin-width.
Tournaments

Tournament: clique with a choice of orientation of each edge.

Twin-width for tournaments: edges in opposite directions cause errors.
Tournaments

Tournament: clique with a choice of orientation of each edge.

Twin-width for tournaments: edges in opposite directions cause errors.
Tournaments

Tournament: clique with a choice of orientation of each edge.

Twin-width for tournaments: edges in opposite directions cause errors.
Twin-width and tournaments

Theorem (G., T.)

There is a function \( f \) and an FPT algorithm which given a tournament \( T \) and \( k \in \mathbb{N} \) answers

\[ \text{tww}(T) \geq k \quad \text{or} \quad \text{tww}(T) \leq f(k) \]

Theorem (G., T.)

Let \( C \) be a hereditary class of tournaments. TFAE:

- \( C \) is NIP,
- \( C \) has FPT FO model checking,
- \( C \) has bounded twin-width.
Transducing a total order?

Is there a FO transduction which gives a total order on any tournament?

- If yes, tournaments are FO-equivalent to ordered graphs.
- NIP, FPT model checking, and bounded twin-width go through transductions.
- So the result on tournaments reduces to the result on ordered graphs.
Transducing a total order?

Is there a FO transduction which gives a total order on any tournament?

Counter-example:
Binary search trees in tournaments

If $y$ is a left descendant of $x$, then

$$\begin{align*}
y &\rightarrow x \\
x &\rightarrow y
\end{align*}$$

If $y$ is a right descendant of $x$, then

$$\begin{align*}
y &\rightarrow x \\
x &\rightarrow y
\end{align*}$$

- Anything
- Anything
Binary search trees in tournaments

If $y$ is a descendant of $x$, then

If $y$ is a descendant of $x$, then

\[
\begin{align*}
\text{left} & \quad \text{right} \\
y \rightarrow x & \quad x \rightarrow y
\end{align*}
\]

$BST$ order: left-to-right order on a BST.
BST orders are good for twin-width

Lemma (G., T.)

There is a function $f$ such that if $T$ is a tournament, $<$ a BST order, then

\[ \text{tww}(T, <) \leq f(\text{tww}(T)) \]
BST orders are good for twin-width

**Lemma (G., T.)**

There is a function $f$ such that if $T$ is a tournament, $\prec$ a BST order, then

$$\text{tww}(T, \prec) \leq f(\text{tww}(T))$$

We will use:

**Theorem (Bonnet et al., '21)**

For an ordered graph $(G, \prec)$, TFAE:

- $(G, \prec)$ has large twin-width
- The matrix of $G$ in the order $\prec$ has a large rank minor
Theorem (G., T.)

A hereditary class of tournaments has unbounded twin-width if and only if it contains one of the obstructions $\mathcal{F}_\leq(\sigma)$, $\mathcal{F}_\geq(\sigma)$, where $\sigma$ is any permutation.
Obstruction to twin-width

Theorem (G., T.)

A hereditary class of tournaments has unbounded twin-width if and only if $C$ contains one of $F_=(\sigma), F_\leq(\sigma), F_\geq(\sigma)$ for any permutation $\sigma$. 
Lemma

The obstructions \( F_{\equiv}(\sigma), F_{\leq}(\sigma), F_{\geq}(\sigma) \) encode arbitrary graphs in first-order logic.
Lemma

The obstructions $F_{\equiv}(\sigma), F_{\leq}(\sigma), F_{\geq}(\sigma)$ encode arbitrary graphs in first-order logic.

Corollary

If $C$ is a (hereditary) class of tournaments with unbounded twin-width, then

- $C$ is independent, and
- FO model checking in $C$ is AW$[*]$-complete.
Generalisations

The results still hold

- for arbitrary binary relational structures, where one of the relations is a tournament, and
- when replacing tournaments with oriented graphs with bounded independence number.
Generalisations

The results still hold

- for arbitrary binary relational structures, where one of the relations is a tournament, and
- when replacing tournaments with oriented graphs with bounded independence number.

We also obtain an enumerative characterisation:

**Theorem**

If $C$ is a hereditary class of tournaments, TFAE:

- $C$ has bounded twin-width,
- $C$ has growth at most $c^n$
- $C$ has growth less than $(n/2 - 2)!$
Conclusion

For tournaments, bounded twin-width, NIP, and FPT FO model checking are equivalent + algorithm and characterisation by forbidden structures.

Based on similar results for ordered graphs [BGOSTT '21]
Main tool: BST order
Conclusion

For tournaments, bounded twin-width, NIP, and FPT FO model checking are equivalent + algorithm and characterisation by forbidden structures.

Based on similar results for ordered graphs [BGOSTT '21]
Main tool: BST order

Questions:

- NIP $\iff$ FPT model checking in general?
- Approximating twin-width in general?
- The equivalence ’NIP $\iff$ bounded twin-width’ is called delineation [BCKKLT '22].
  - Interval graphs are delineated.
  - Conjectured to be delineated: segment graphs, some visibility graphs.