Fibrations of Tree Automata

Colin Riba

ENS de Lyon, Université de Lyon, LIP*
colin.riba@ens-lyon.fr http://perso.ens-lyon.fr/colin.riba/

Abstract

We propose a notion of morphisms between tree automata based on game semantics. Morphisms are winning strategies on a synchronous restriction of the linear implication between acceptance games. This leads to split indexed categories, with substitution based on a suitable notion of synchronous tree function. By restricting to tree functions issued from maps on alphabets, this gives a fibration of tree automata. We then discuss the (fibrewise) monoidal structure issued from the synchronous product of automata. We also discuss how a variant of the usual projection operation on automata leads to an existential quantification in the fibered sense. Our notion of morphism is correctin the sense that it respects language inclusion, and in a weaker sense also complete.

1998 ACM Subject Classification F.3.2 Semantics of Programming Languages. F.4.1 Mathematical Logic. F.4.2 Formal Languages.

Keywords and phrases Tree automata, Game semantics, Categorical logic.

1 Introduction

This paper proposes a notion of morphism between tree automata based on game semantics. We follow the Curry-Howard-like slogan: *Automata as objects, Executions as morphisms.*

We consider general alternating automata on infinite ranked trees. These automata encompass Monadic Second-Order Logic (MSO) and thus most of the logics used in verification [8]. Tree automata are traditionally viewed as positive objects: one is primarily interested in satisfaction or satisfiability, and the primitive notion of quantification is existential. In contrast, Curry-Howard approaches tend to favor proof-theoretic oriented and negative approaches, *i.e.* approaches in which the predominant logical connective is the implication, and where the predominant form of quantification is universal. In order to handle quantifications, our categories are organized in fibrations.

We consider full infinite ranked trees, built from a non-empty finite set of directions D and labeled in non-empty finite alphabets Σ. The base category **Tree** has alphabets as objects and morphisms from Σ to Γ are $(\Sigma \to \Gamma)$-labeled D-ary trees.

The fibre categories are based on a generalization of the usual acceptance games, where for an automaton $\mathcal{A}$ on alphabet Γ (denoted $\Gamma \vdash \mathcal{A}$), input characters can be precomposed with a tree morphism $M \in \mathbf{Tree}[\Sigma, \Gamma]$, leading to substituted acceptance games of type $\Sigma \vdash \mathcal{G}(\mathcal{A}, M)$. Usual acceptance games, which correspond to the evaluation of $\Sigma \vdash \mathcal{A}$ on a Σ-labeled input tree, are substituted acceptance games $\mathbf{1} \vdash \mathcal{G}(\mathcal{A}, t)$ with $t \in \mathbf{Tree}[\mathbf{1}, \Sigma]$. Games of the form $\Sigma \vdash \mathcal{G}(\mathcal{A}, M)$ are the objects of the fibre category over Σ.

For morphisms, we introduce a notion of "synchronous" simple game between acceptance games. We rely on Hyland & Schalk's functor (denoted HS) from simple games to **Rel** [11]. A synchronous strategy $\Sigma \vdash \sigma : \mathcal{G}(\mathcal{A}, M) \multimap_{\circledast} \mathcal{G}(\mathcal{B}, N)$ is a strategy in the simple

* UMR 5668 CNRS ENS Lyon UCBL INRIA

Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

game $\mathcal{G}(\mathcal{A}, M) \multimap \mathcal{G}(\mathcal{B}, N)$ required to satisfy (in **Set**) a diagram of the form of (1) below, expressing that $\mathcal{A}$ and $\mathcal{B}$ are evaluated along the same path of the tree and read the same input characters:

$$\begin{tikzcd} \mathrm{HS}(\sigma) \arrow[r] \arrow[d] & \mathcal{G}(\mathcal{B}, N) \arrow[d] \\ \mathcal{G}(\mathcal{A}, M) \arrow[r] & (D+\Sigma)^* \end{tikzcd} \tag{1}$$

This gives a split fibration game of tree automata and acceptance games. When restricting the base to *alphabet* morphisms (*i.e.* functions $\Sigma \to \Gamma$), substitution can be internalized in automata. By change-of-base of fibrations, this leads to a split fibration aut. In the fibers of aut, the substituted acceptance games have finite-state winning strategies, whose existence can be checked by trivial adaptation of usual algorithms.

Each of these fibrations is monoidal in the sense of [21], by using a natural synchronous product of tree automata. We also investigate a linear negation, as well as existential quantifications, obtained by adapting the usual projection operation on non-deterministic automata to make it a left-adjoint to weakening, the adjunction satisfying the usual Beck-Chevalley condition.

Our linear implication of acceptance games seems to provide a natural notion of prenex universal quantification on automata not investigated before. As expected, if there is a synchronous winning strategy $\sigma \Vdash \mathcal{A} \multimap\!\circledast\, \mathcal{B}$, then $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$ (*i.e.* each input tree accepted by $\mathcal{A}$ is also accepted by $\mathcal{B}$). Under some assumptions on $\mathcal{A}$ and $\mathcal{B}$ the converse holds: $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$ implies $\sigma \Vdash \mathcal{A} \multimap\!\circledast\, \mathcal{B}$ for some σ.

At the categorical level, thanks to (1), the constructions mimic relations in slices categories $\mathbf{Set}/(D+\Sigma)^*$ of the co-domain fibration: substitution is given by a (well chosen) pullback, and the monoidal product of automata is issued from the Cartesian product of plays in $\mathbf{Set}/(D+\Sigma)^*$ (*i.e.* also by a well chosen pullback).

The paper is organized as follows. Section 2 presents notations for trees and tree automata. Our notions of substituted acceptance games and synchronous arrow games are then discussed in Sect. 3. Substitution functors and the corresponding fibrations are presented in Sect. 4, and Section 5 overviews the monoidal structure. We then state our main correctness results in Sect. 6. Section 7 presents existential quantifications and quickly discusses non-deterministic automata. A short Appendix A gives some definitions on simple games, and a long version of the paper with full proofs [20] can be found on the webpage of the author.

2 Preliminaries

Fix a singleton set $\mathbf{1} = \{\bullet\}$ and a finite non-empty set D of (tree) *directions*.

Alphabets and Trees. We write $\Sigma, \Gamma, \ldots$ for *alphabets*, *i.e.* finite non-empty sets. We let **Alph** be the category whose objects are alphabets and whose morphisms $\beta \in \mathbf{Alph}[\Sigma, \Gamma]$ are functions $\beta : \Sigma \to \Gamma$.

We let $\mathbf{Tree}[\Sigma]$ be the set of Σ-labeled full D-ary trees, *i.e.* the set of maps $T : D^* \to \Sigma$. Let **Tree** be the category with *alphabets* as objects and with morphisms $\mathbf{Tree}[\Sigma, \Gamma] := \mathbf{Tree}[(\Sigma \to \Gamma)]$, *i.e.* $(\Sigma \to \Gamma)$-labeled trees. Maps $M \in \mathbf{Tree}[\Sigma, \Gamma]$ and $L \in \mathbf{Tree}[\Gamma, \Delta]$ are composed as

$$L \circ M \quad : \quad p \in D^* \quad \mapsto \quad (a \in \Sigma \quad \mapsto \quad L(p)(M(p)(a)))$$

and the identity $\mathrm{Id}_\Sigma \in \mathbf{Tree}[\Sigma, \Sigma]$ is defined as $\mathrm{Id}_\Sigma(p)(a) := a$. Note that $\mathbf{Tree}[\mathbf{1}, \Sigma]$ is in bijection with $\mathbf{Tree}[\Sigma]$.

There is a faithful functor from $\mathbf{Alph}$ to $\mathbf{Tree}$, mapping $\beta \in \mathbf{Alph}[\Sigma, \Gamma]$ to the constant tree morphism $(\_ \mapsto \beta) \in \mathbf{Tree}[\Sigma, \Gamma]$ that we simply write β.

Tree Automata. Alternating tree automata [17] are finite state automata running on full infinite Σ-labeled D-ary trees. Their distinctive feature is that transitions are given by *positive Boolean* formulas with atoms pairs (q, d) of a state q and a tree direction $d \in D$ ((q, d) means that one copy of the automaton should start in state q from the d-th son of the current tree position).

Acceptance for alternating tree automata can be defined either *via* run trees or *via* the existence of winning strategies in *acceptance games* [17]. In both cases, we can w.l.o.g. restrict to transitions given by formulas in (irredundant) disjunctive normal form [18]. In our setting, it is quite convenient to follow the presentation of [23], in which disjunctive normal forms with atoms in $Q \times D$ are represented as elements of $\mathcal{P}(\mathcal{P}(Q \times D))$.

An alternating tree automaton $\mathcal{A}$ on alphabet Σ has the form $(Q, q^\imath, \delta, \Omega)$ where Q is the finite set of states, $q^\imath \in Q$ is the *initial state*, the *acceptance condition* is $\Omega \subseteq Q^\omega$ and following [23], the *transition function* δ has the form

$$\delta \quad : \quad Q \times \Sigma \quad \longrightarrow \quad \mathcal{P}(\mathcal{P}(Q \times D))$$

We write $\Sigma \vdash \mathcal{A}$ if $\mathcal{A}$ is a tree automaton on Σ. Usual acceptance games are described in Sec. 3.1. It is customary to put restrictions on the acceptance condition $\Omega \subseteq Q^\omega$, typically by assuming it is generated from a *Muller* family $\mathcal{F} \in \mathcal{P}(\mathcal{P}(Q))$ as the set of $\pi \in Q^\omega$ such that $\mathrm{Inf}(\pi) \in \mathcal{F}$. We call such automata *regular*[1]. They have decidable emptiness checking and the same expressive power as MSO on D-ary trees (see e.g. the survey [22]).

3 Categories of Acceptance Games and Automata

We present in this Section the categories $\mathbf{SAG}^{(\mathrm{W})}_\Sigma$ of *substituted acceptance games*. Their objects will be *substituted acceptance games* (to be presented in Sect. 3.1) and their morphisms will be strategies in corresponding *synchronous arrow games* (to be presented in Sect. 3.2). Substituted acceptance games and synchronous arrow games are the two main notions we introduce in this paper. Our categories of $\mathbf{Aut}^{(\mathrm{W})}_\Sigma$ of automata will be full subcategories of $\mathbf{SAG}^{(\mathrm{W})}_\Sigma$, while $\mathbf{SAG}^{(\mathrm{W})}_\Sigma$ and $\mathbf{Aut}^{(\mathrm{W})}_\Sigma$ will be the total categories of our fibrations

$$\mathsf{game}^{(\mathrm{W})} \quad : \quad \mathbf{SAG}^{(\mathrm{W})} \quad \longrightarrow \quad \mathbf{Tree} \qquad\qquad \mathsf{aut}^{(\mathrm{W})} \quad : \quad \mathbf{Aut}^{(\mathrm{W})} \quad \longrightarrow \quad \mathbf{Alph}$$

to be presented in Sect. 4. Appendix A summarizes the basic notion of games we are using.

3.1 Substituted Acceptance Games

Consider a tree automaton $\mathcal{A} = (Q, q^\imath, \delta, \Omega)$ on Γ and a morphism $M \in \mathbf{Tree}[\Sigma, \Gamma]$. The *substituted acceptance game* $\Sigma \vdash \mathcal{G}(\mathcal{A}, M)$ is the positive game

$$\mathcal{G}(\mathcal{A}, M) \quad := \quad (D^* \times (A_{\mathsf{P}} + A_{\mathsf{O}}), E, *, \lambda, \xi, \mathcal{W})$$

[1] By adding states to $\mathcal{A}$ if necessary, one can describe Ω by an equivalent *parity* condition.

whose positions are given by $A_{\mathsf{P}} := Q$ and $A_{\mathsf{O}} := \Sigma \times \mathcal{P}(Q \times D)$, whose polarized root is $* := (\varepsilon, q^\imath)$ with $\xi(*) = \mathsf{P}$, whose polarized moves (E, λ) are given by

$$\begin{array}{lrcll} \text{from } (D^* \times A_{\mathsf{P}}) \text{ to } (D^* \times A_{\mathsf{O}}) : & (p, q) & \xrightarrow{\mathsf{P}} & (p, a, \gamma) & \text{iff } \gamma \in \delta(q, M(p)(a)) \\ \text{from } (D^* \times A_{\mathsf{O}}) \text{ to } (D^* \times A_{\mathsf{P}}) : & (p, a, \gamma) & \xrightarrow{\mathsf{O}} & (p.d, q) & \text{iff } (q, d) \in \gamma \end{array}$$

and whose winning condition is given by

$$(\varepsilon, q_0) \cdot (\varepsilon, a_0, \gamma_0) \cdot (p_1, q_1) \cdot \ldots \cdot (p_n, q_n) \cdot (p_n, a_n, \gamma_n) \cdot \ldots \quad \in \mathcal{W} \qquad \text{iff} \quad (q_i)_{i \in \mathbb{N}} \in \Omega$$

The input alphabet of $\Gamma \vdash \mathcal{A}$ is Γ, and we use the tree morphism $M \in \mathbf{Tree}[\Sigma, \Gamma]$ in a contravariant way to obtain a game with "input alphabet" Σ, that we emphasize by writing $\Sigma \vdash \mathcal{G}(\mathcal{A}, M)$. Input characters $a \in \Sigma$ are chosen by P, directions $d \in D$ are chosen by O.

Write $\Sigma \vdash \sigma \Vdash \mathcal{G}(\mathcal{A}, M)$ if σ is a winning P-strategy on $\Sigma \vdash \mathcal{G}(\mathcal{A}, M)$, and $\Sigma \Vdash \mathcal{G}(\mathcal{A}, M)$ if $\Sigma \vdash \sigma \Vdash \mathcal{G}(\mathcal{A}, M)$ for some σ.

Correspondence with usual Acceptance Games. Usual acceptance games model the evaluation of automata $\Sigma \vdash \mathcal{A}$ on input trees $t \in \mathbf{Tree}[\Sigma]$. They correspond to games of the form $\mathbf{1} \vdash \mathcal{G}(\mathcal{A}, \dot{t})$, where $\dot{t} \in \mathbf{Tree}[\mathbf{1}, \Sigma]$ is the tree morphism corresponding to $t \in \mathbf{Tree}[\Sigma]$.

Note that in these cases, A_{O} is of the form $\mathbf{1} \times \mathcal{P}(Q \times D) \simeq \mathcal{P}(Q \times D)$, so that the games $\mathbf{1} \vdash \mathcal{G}(\mathcal{A}, \dot{t})$ are isomorphic to the acceptance games of [23].

▶ **Definition 3.1.** Let $\Sigma \vdash \mathcal{A}$.

(i) $\mathcal{A}$ *accepts* the tree $t \in \mathbf{Tree}[\Sigma]$ if there is a strategy σ such that $\mathbf{1} \vdash \sigma \Vdash \mathcal{G}(\mathcal{A}, \dot{t})$.
(ii) Let $\mathcal{L}(\mathcal{A}) \subseteq \mathbf{Tree}[\Sigma]$, the *language* of $\mathcal{A}$, be the set of trees accepted by $\mathcal{A}$.

3.2 Synchronous Arrow Games

Consider games $\Sigma \vdash \mathcal{G}(\mathcal{A}, M)$ and $\Sigma \vdash \mathcal{G}(\mathcal{B}, N)$ with $\mathcal{A} = (Q_{\mathcal{A}}, q^\imath_{\mathcal{A}}, \delta_{\mathcal{A}}, \Omega_{\mathcal{A}})$ and $\mathcal{B} = (Q_{\mathcal{B}}, q^\imath_{\mathcal{B}}, \delta_{\mathcal{B}}, \Omega_{\mathcal{B}})$. Similarly as in Sect. 3.1 above, write

$$A_{\mathsf{P}} := Q_{\mathcal{A}} \qquad A_{\mathsf{O}} := \Sigma \times \mathcal{P}(Q_{\mathcal{A}} \times D) \qquad B_{\mathsf{P}} := Q_{\mathcal{B}} \qquad B_{\mathsf{O}} := \Sigma \times \mathcal{P}(Q_{\mathcal{B}} \times D)$$

We define the *synchronous arrow game*

$$\Sigma \vdash \mathcal{G}(\mathcal{A}, M) \multimap_{\circledast} \mathcal{G}(\mathcal{B}, N)$$

as the negative game $(V, E, *, \lambda, \xi, \mathcal{W})$ whose positions are given by

$$V \quad := \quad (D^* \times A_{\mathsf{P}}) \times (D^* \times B_{\mathsf{P}}) \; + \; (D^* \times A_{\mathsf{O}}) \times (D^* \times B_{\mathsf{P}}) \; + \; (D^* \times A_{\mathsf{O}}) \times (D^* \times B_{\mathsf{O}})$$

whose polarized root is $* := ((\varepsilon, q^\imath_{\mathcal{A}}), (\varepsilon, q^\imath_{\mathcal{B}}))$ with $\xi(*) := \mathsf{O}$, whole polarized edges (E, λ) are given in Table 1, and whose winning condition is given by

$$\begin{array}{l} ((\varepsilon, q^0_{\mathcal{A}}) \,,\, (\varepsilon, q^0_{\mathcal{B}})) \;\cdot\; \ldots \;\cdot\; ((\varepsilon, q^n_{\mathcal{A}}) \,,\, (\varepsilon, q^n_{\mathcal{B}})) \;\cdot\; \ldots \quad \in \mathcal{W} \\ \qquad\qquad\qquad\qquad \text{iff} \quad \left((q^i_{\mathcal{A}})_{i \in \mathbb{N}} \in \Omega_{\mathcal{A}} \implies (q^i_{\mathcal{B}})_{i \in \mathbb{N}} \in \Omega_{\mathcal{B}}\right) \end{array}$$

Note that P-plays end in positions of the form

$$\begin{array}{lccc} & ((p, q_{\mathcal{A}}) \,,\, (p, q_{\mathcal{B}})) & \in & (D^* \times A_{\mathsf{P}}) \times (D^* \times B_{\mathsf{P}}) \\ \text{and} & ((p, a, \gamma_{\mathcal{A}}) \,,\, (p, a, \gamma_{\mathcal{B}})) & \in & (D^* \times A_{\mathsf{O}}) \times (D^* \times B_{\mathsf{O}}) \end{array}$$

λ	$\mathcal{G}(\mathcal{A}, M)$	$\mathrel{-\!\!\circledast}$	$\mathcal{G}(\mathcal{B}, N)$	
	$((p, q_{\mathcal{A}})$	,	$(p, q_{\mathcal{B}}))$	
O	$\downarrow$			
	$((p, a, \gamma_{\mathcal{A}})$	,	$(p, q_{\mathcal{B}}))$	if $\gamma_{\mathcal{A}} \in \delta_{\mathcal{A}}(q_{\mathcal{A}}, M(p)(a))$
P			$\downarrow$	
	$((p, a, \gamma_{\mathcal{A}})$	,	$(p, a, \gamma_{\mathcal{B}}))$	if $\gamma_{\mathcal{B}} \in \delta_{\mathcal{B}}(q_{\mathcal{B}}, N(p)(a))$
O			$\downarrow$	
	$((p, a, \gamma_{\mathcal{A}})$	,	$(p.d, q'_{\mathcal{B}}))$	if $(q'_{\mathcal{B}}, d) \in \gamma_{\mathcal{B}}$
P	$\downarrow$			
	$((p.d, q'_{\mathcal{A}})$	,	$(p.d, q'_{\mathcal{B}}))$	if $(q'_{\mathcal{A}}, d) \in \gamma_{\mathcal{A}}$

Figure 1 Moves of $\mathcal{G}(\mathcal{A}, M) \mathrel{-\!\!\circledast} \mathcal{G}(\mathcal{B}, N)$

Each of these position is of homogeneous type, and moreover in each case the D^* and Σ components coincide. On the other hand, O-plays end in positions of the form

$$\begin{array}{rlcl} & ((p, a, \gamma_{\mathcal{A}}) \ , \ (p, q_{\mathcal{A}})) & \in & (D^* \times A_{\mathsf{O}}) \ \times \ (D^* \times B_{\mathsf{P}}) \\ \text{and} & ((p, a, \gamma_{\mathcal{A}}) \ , \ (p \cdot d, q_{\mathcal{B}})) & \in & (D^* \times A_{\mathsf{O}}) \ \times \ (D^* \times B_{\mathsf{P}}) \end{array}$$

Each of these intermediate position is of heterogeneous type, and in the second one, the D^* components do not coincide.

We write $\Sigma \vdash \sigma : \mathcal{G}(\mathcal{A}, M) \mathrel{-\!\!\circledast} \mathcal{G}(\mathcal{B}, N)$ if σ is a P-strategy on $\mathcal{G}(\mathcal{A}, M) \mathrel{-\!\!\circledast} \mathcal{G}(\mathcal{B}, N)$, and $\Sigma \vdash \sigma \Vdash \mathcal{G}(\mathcal{A}, M) \mathrel{-\!\!\circledast} \mathcal{G}(\mathcal{B}, N)$ if σ is moreover winning. Finally, we write

$$\Sigma \Vdash \mathcal{G}(\mathcal{A}, M) \mathrel{-\!\!\circledast} \mathcal{G}(\mathcal{B}, N)$$

if there is a winning P-strategy σ on $\mathcal{G}(\mathcal{A}, M) \mathrel{-\!\!\circledast} \mathcal{G}(\mathcal{B}, N)$.

▶ Remark. Recall that if $\Omega_{\mathcal{A}}$ and $\Omega_{\mathcal{B}}$ are Borel sets, then $\mathcal{W}$ is a Borel set and by Martin's Theorem [14], either P or O has a winning strategy. Moreover, if the automata $\mathcal{A}$ and $\mathcal{B}$ are regular (in the sense of Sect. 2), then $\mathcal{W}$ is an ω-regular language. If in addition the trees M and N are regular (in the usual sense), then the game is equivalent to a finite regular game. By Büchi-Landweber Theorem, the existence of a winning strategy for a given player is decidable, and the winning player has *finite state* winning strategies (see e.g. [22]).

3.3 Characterization of the Synchronous Arrow Games

We now give a characterization of synchronous arrow games in traditional games semantics. Our characterization involve relations in slices categories $\mathbf{Set}/J$, that will give rise to a strong analogy between our fibrations $\mathsf{game}^{(\mathrm{W})}$ and $\mathsf{aut}^{(\mathrm{W})}$ and substitution (a.k.a *change-of-base*) in the codomain fibration $\mathsf{cod} : \mathbf{Set}^{\rightarrow} \to \mathbf{Set}$.

Simple Games. Recall the usual notion of *simple games* (see e.g. [1, 9]). Simple games are usually negative, but given positive games A and B, their *negative* linear arrow $A \multimap B$ can still be defined. Moreover, simple games, with linear arrows $A \multimap B$ between games A and B of the same polarity, form a category that we write $\mathbf{SGG}$. When equipped with winning conditions, winning strategies compose, giving rise to a category that we write $\mathbf{SGG}^{\mathrm{W}}$.

A P-strategy $\Sigma \vdash \sigma : \mathcal{G}(\mathcal{A}, M) \mathrel{-\!\!\circledast} \mathcal{G}(\mathcal{B}, N)$ is a morphism of $\mathbf{SGG}$ from the substituted acceptance game $\mathcal{G}(\mathcal{A}, M)$ to the substituted acceptance game $\mathcal{G}(\mathcal{B}, N)$. If σ is moreover winning, then it is a morphism of $\mathbf{SGG}^{\mathrm{W}}$.

The Hyland & Schalk Functor. Hyland & Schalk have presented in [11] a faithful functor, that we denote HS, from simple games to the category **Rel** of sets and relations. This functor can easily be extended to a functor $\mathrm{HS} : \mathbf{SGG}^{(\mathrm{W})} \to \mathbf{Rel}$.

Given a play $s \in \wp(A \multimap B)$ we let $s\restriction A \in \wp(A)$ be its projection on A and similarly for B,[2] so that $\mathrm{HS}(s) := (s\restriction A, s\restriction B)$. Given a P-strategy $\sigma : A \multimap B$ we have $\sigma \subseteq \wp^{\mathsf{P}}(A \multimap B)$ and thus

$$\mathrm{HS}(\sigma) \quad := \quad \{\mathrm{HS}(s) \mid s \in \sigma\} \quad \subseteq \quad \wp(A) \times \wp(B)$$

We write $\wp_\Sigma(\mathcal{A}, M)$ for the plays of the substituted acceptance game $\Sigma \vdash \mathcal{G}(\mathcal{A}, M)$. Given $\Sigma \vdash \sigma : \mathcal{G}(\mathcal{A}, M) \multimap\!\!\circledast\, \mathcal{G}(\mathcal{B}, N)$, we thus have

$$\mathrm{HS}(\sigma) \quad \subseteq \quad \wp_\Sigma(\mathcal{A}, M) \times \wp_\Sigma(\mathcal{B}, N)$$

Composition by pullbacks. An interesting of the faithful functor HS is that it allows to compose strategies as relations. Moreover, it is easy to check (and folklore) that composition of strategies, when seen as relations, is given by pullbacks: given $\sigma : A \multimap B$ and $\tau : B \multimap C$ we have, in **Set**:

$$\begin{tikzcd} \mathrm{HS}(\tau \circ \sigma) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{HS}(\tau) \arrow[d] \\ \mathrm{HS}(\sigma) \arrow[r] & \wp(B) \end{tikzcd} \qquad \text{where} \qquad \begin{tikzcd} \mathrm{HS}(\sigma) \arrow[r] \arrow[d] & \wp(B) \\ \wp(A) & \end{tikzcd} \qquad \begin{tikzcd} \mathrm{HS}(\tau) \arrow[r] \arrow[d] & \wp(C) \\ \wp(B) & \end{tikzcd} \tag{2}$$

Synchronous Relations. We will now see that P-strategies on a synchronous arrow game can be seen as relations in slice categories $\mathbf{Set}/J$. We call such relations *synchronous*.

Given a set J, define the category $\mathbf{Rel}(\mathbf{Set}/J)$ as follows:

Objects are indexed sets $A \xrightarrow{g} J$, written simply A when g is understood from the context.

Morphisms from $A \xrightarrow{g} J$ to $B \xrightarrow{h} J$ are given by relations $\mathring{R} : A \to\!\!\!\!\!+ B$ such that the following commutes:

$$\begin{tikzcd} & \mathring{R} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ A \arrow[dr, "g"'] & & B \arrow[dl, "h"] \\ & J & \end{tikzcd}$$

Traces. For the synchronous arrow games, synchronization is performed using the following notion of *trace*. Given $\Gamma \vdash \mathcal{A}$ and $M \in \mathbf{Tree}[\Gamma, \Sigma]$, define

$$\mathrm{tr} \quad : \quad \wp_\Sigma(\mathcal{A}, M) \quad \longrightarrow \quad (D + \Sigma)^*$$

inductively as follows

$$\mathrm{tr}(\varepsilon) \;:=\; \varepsilon \qquad \mathrm{tr}(s \to (p, a, \gamma)) \;:=\; \mathrm{tr}(s) \cdot a \qquad \mathrm{tr}(s \to (p \cdot d, q)) \;:=\; \mathrm{tr}(s) \cdot d$$

The image of tr is the set $\mathrm{Tr}_\Sigma \;:=\; (\Sigma \cdot D)^* + (\Sigma \cdot D)^* \cdot \Sigma$.

[2] We write $\wp(A)$ for the set of plays on A, and $\wp^{\mathsf{P}}(A)$ for the set of P-plays.

Characterization of the Synchronous Arrow. We can now characterize the synchronous arrow games. First, *via* the functor HS, synchronous strategies are synchronous relations.

▶ **Proposition 3.2.** *Strategies on the synchronous arrow game* $\mathcal{G}(\mathcal{A}, M) \multimap\!\!\circledast\ \mathcal{G}(\mathcal{B}, N)$ *are exactly the strategies* $\sigma : \mathcal{G}(\mathcal{A}, M) \multimap \mathcal{G}(\mathcal{B}, N)$ *such that*

$$\begin{tikzcd} \mathrm{HS}(\sigma) \arrow[r] \arrow[d] & \wp_\Sigma(\mathcal{B}, N) \arrow[d, "\mathrm{tr}"] \\ \wp_\Sigma(\mathcal{A}, M) \arrow[r, "\mathrm{tr}"'] & \mathrm{Tr}_\Sigma \end{tikzcd} \tag{3}$$

Second, plays on the synchronous arrow can be obtained in a canonical way from plays on its components.

▶ **Proposition 3.3.** *Let* $\Sigma \vdash \mathcal{G}(\mathcal{A}, M)$ *and* $\Sigma \vdash \mathcal{G}(\mathcal{B}, N)$*. The following is a pullback in* **Set***:*

$$\begin{tikzcd} \wp^{\mathsf{P}}_\Sigma(\mathcal{G}(\mathcal{A}, M) \multimap\!\!\circledast\ \mathcal{G}(\mathcal{B}, N)) \arrow[r, "(-)\restriction\mathcal{G}(\mathcal{B},N)"] \arrow[d, "(-)\restriction\mathcal{G}(\mathcal{A},M)"'] \arrow[dr, phantom, "\lrcorner", very near start] & \wp_\Sigma(\mathcal{B}, N) \arrow[d, "\mathrm{tr}"] \\ \wp_\Sigma(\mathcal{A}, M) \arrow[r, "\mathrm{tr}"'] & \mathrm{Tr}_\Sigma \end{tikzcd}$$

We write $\mathrm{tr}^{\multimap\circledast}$ for any of two equal maps

$$\mathrm{tr} \circ (-)\restriction\mathcal{G}(\mathcal{A}, M)\ ,\ \mathrm{tr} \circ (-)\restriction\mathcal{G}(\mathcal{B}, N) \quad : \quad \wp^{\mathsf{P}}_\Sigma(\mathcal{G}(\mathcal{A}, M) \multimap\!\!\circledast\ \mathcal{G}(\mathcal{B}, N)) \quad \longrightarrow \quad \mathrm{Tr}_\Sigma$$

3.4 Categories of Substituted Acceptance Games and Automata

We now define our categories $\mathbf{SAG}^{(\mathrm{W})}_\Sigma$ of substituted acceptance games and their full subcategories $\mathbf{Aut}^{(\mathrm{W})}_\Sigma$ of tree automata. That they indeed form categories follows from the characterization Prop. 3.2, together with the fact that $\mathbf{Rel}(\mathbf{Set}/J)$ and $\mathbf{SGG}^{(\mathrm{W})}$ are categories, and the fact that the identity strategies $\mathrm{id} : \mathcal{G}(\mathcal{A}, M) \multimap \mathcal{G}(\mathcal{B}, N)$ are synchronous.

The Categories $\mathbf{SAG}_\Sigma$ and $\mathbf{SAG}^{\mathrm{W}}_\Sigma$:
Objects of $\mathbf{SAG}_\Sigma$ and $\mathbf{SAG}^{\mathrm{W}}_\Sigma$ are games $\Sigma \vdash \mathcal{G}(\mathcal{A}, M)$,
Morphisms of $\mathbf{SAG}_\Sigma$ are synchronous strategies $\Sigma \vdash \sigma : \mathcal{G}(\mathcal{A}, M) \multimap\!\!\circledast\ \mathcal{G}(\mathcal{B}, N)$,
Morphisms of $\mathbf{SAG}^{\mathrm{W}}_\Sigma$ are synchronous winning strategies $\Sigma \vdash \sigma \Vdash \mathcal{G}(\mathcal{A}, M) \multimap\!\!\circledast\ \mathcal{G}(\mathcal{B}, N)$.

The Categories $\mathbf{Aut}_\Sigma$ and $\mathbf{Aut}^{\mathrm{W}}_\Sigma$:
Objects of $\mathbf{Aut}_\Sigma$ and $\mathbf{Aut}^{\mathrm{W}}_\Sigma$ are automata $\Sigma \vdash \mathcal{A}$,
Morphisms of $\mathbf{Aut}_\Sigma$ are synchronous strategies $\Sigma \vdash \sigma : \mathcal{G}(\mathcal{A}, \mathrm{Id}_\Sigma) \multimap\!\!\circledast\ \mathcal{G}(\mathcal{B}, \mathrm{Id}_\Sigma)$,
Morphisms of $\mathbf{Aut}^{\mathrm{W}}_\Sigma$ are synchronous winning strategies $\Sigma \vdash \sigma \Vdash \mathcal{G}(\mathcal{A}, \mathrm{Id}_\Sigma) \multimap\!\!\circledast\ \mathcal{G}(\mathcal{B}, \mathrm{Id}_\Sigma)$.

A Lifting Property. Among the useful consequences of Prop. 3.3, we state the following lifting property.

▶ **Proposition 3.4.** *Consider* $\Sigma \vdash \mathcal{G}(\mathcal{A}, M)$ *and* $\Sigma \vdash \mathcal{G}(\mathcal{B}, N)$*. Assume that, in* $\mathbf{Rel}(\mathbf{Set}/\mathrm{Tr}_\Sigma)$ *we have an isomorphism* $\mathring{R} : (\wp_\Sigma(\mathcal{A}, M) \xrightarrow{\mathrm{tr}} \mathrm{Tr}_\Sigma) \mathrel{+\!\!\rightarrow}_{/\mathrm{Tr}_\Sigma} (\wp_\Sigma(\mathcal{B}, N) \xrightarrow{\mathrm{tr}} \mathrm{Tr}_\Sigma)$*.*
There is a (unique, total) isomorphism $\sigma : \mathcal{G}(\mathcal{A}, M) \longrightarrow_{\mathbf{SAG}_\Sigma} \mathcal{G}(\mathcal{B}, N)$ *s.t.* $\mathrm{HS}(\sigma) = R$*.*

In general we can not ask σ to be winning, and in particular to be a morphism of $\mathbf{SAG}^{\mathrm{W}}_\Sigma$.

4 Fibrations of Acceptance Games and Automata

A tree morphism $L \in \mathbf{Tree}[\Sigma, \Gamma]$ defines a map L^* from the objects of $\mathbf{SAG}_\Gamma$ to the objects of $\mathbf{SAG}_\Sigma$: we let $L^*(\Gamma \vdash \mathcal{G}(\mathcal{A}, M)) := \Sigma \vdash \mathcal{G}(\mathcal{A}, M \circ L)$.

In this Section, we show that L^* extends to functors $L^* : \mathbf{SAG}_\Gamma^{(\mathrm{W})} \longrightarrow \mathbf{SAG}_\Sigma^{(\mathrm{W})}$ and that the operation $(-)^*$ is itself functorial and thus leads to split indexed categories $(-)^* : \mathbf{Tree}^{\mathrm{op}} \longrightarrow \mathbf{Cat}$. By applying Groethendieck completion, we obtain our split fibrations of acceptance games $\mathsf{game}^{(\mathrm{W})} : \mathbf{SAG}^{(\mathrm{W})} \longrightarrow \mathbf{Tree}$.

On the other hand, by restricting substitution to tree morphisms generated by alphabet morphisms $\beta \in \mathbf{Alph}[\Sigma, \Gamma]$, we obtain functors $\beta^* : \mathbf{Aut}_\Gamma^{(\mathrm{W})} \longrightarrow \mathbf{Aut}_\Sigma^{(\mathrm{W})}$ giving rise to split fibrations of tree automata $\mathsf{aut}^{(\mathrm{W})} : \mathbf{Aut}^{(\mathrm{W})} \longrightarrow \mathbf{Alph}$.

Our substitution functors L^* are build in strong analogy with change-of-base functors $\mathbf{Set}/\mathrm{Tr}_\Gamma \to \mathbf{Set}/\mathrm{Tr}_\Sigma$ of the codomain fibration $\mathsf{cod} : \mathbf{Set}^\to \to \mathbf{Set}$. We refer to [12] for basic material about fibrations.

4.1 Substitution Functors

Change-of-Base in $\mathbf{Set}^\to$. A morphism $L \in \mathbf{Tree}[\Sigma, \Gamma]$ induces a map $\mathrm{Tr}(L) : \mathrm{Tr}_\Sigma \longrightarrow \mathrm{Tr}_\Gamma$ inductively defined as follows (where $(-)_D$ is the obvious projection $\mathrm{Tr}_\Sigma \to D^*$):

$$\mathrm{Tr}(L)(\varepsilon) \;:=\; \varepsilon \qquad \mathrm{Tr}(L)(w \cdot a) \;:=\; \mathrm{Tr}(L)(w) \cdot L(w_D)(a) \qquad \mathrm{Tr}(L)(w \cdot d) \;:=\; \mathrm{Tr}(L)(w) \cdot d$$

The map $\mathrm{Tr}(L)$ gives rise to the usual change-of-base functor $L^\bullet : \mathbf{Set}/\mathrm{Tr}_\Gamma \to \mathbf{Set}/\mathrm{Tr}_\Sigma$, defined using chosen pullbacks in $\mathbf{Set}$:

$$\begin{tikzcd}
L^\bullet(\wp_\Gamma(\mathcal{A}, M)) \arrow[r] \arrow[d, "L^\bullet(\mathrm{tr})"'] \arrow[dr, phantom, "\lrcorner", very near start] & \wp_\Gamma(\mathcal{A}, M) \arrow[d, "\mathrm{tr}"] \\
\mathrm{Tr}_\Sigma \arrow[r, "\mathrm{Tr}(L)"] & \mathrm{Tr}_\Gamma
\end{tikzcd}$$

Substitution on Plays. The action of the substitution L^* on plays can be described, similarly as the action of $L^\bullet$ on objects of $\mathbf{Set}/\mathrm{Tr}_\Gamma$, by a pullback property.

Consider $\Gamma \vdash \mathcal{G}(\mathcal{A}, M)$, so that $\Sigma \vdash \mathcal{G}(\mathcal{A}, M \circ L)$. A position $(p, a, \gamma_\mathcal{A})$ of the game $\Sigma \vdash \mathcal{G}(\mathcal{A}, M \circ L)$ can be mapped to the position $(p, L(p)(a), \gamma_\mathcal{A})$ of the game $\Gamma \vdash \mathcal{G}(\mathcal{A}, M)$. Moreover, since $\delta_\mathcal{A}(q_\mathcal{A}, (M \circ L)(p)(a)) \;=\; \delta_\mathcal{A}(q_\mathcal{A}, M(p)(L(p)(a)))$, we have

$$(p, q_\mathcal{A}) \to (p, a, \gamma_\mathcal{A}) \qquad \text{if and only if} \qquad (p, q_\mathcal{A}) \to (p, L(p)(a), \gamma_\mathcal{A})$$

This gives a map

$$\wp(L) \quad : \quad \wp_\Sigma(\mathcal{A}, M \circ L) \quad \longrightarrow \quad \wp_\Gamma(\mathcal{A}, M)$$

If we are also given $\Gamma \vdash \mathcal{G}(\mathcal{B}, N)$, then we similarly obtain

$$\wp(L)_{\multimap\circledast} \quad : \quad \wp_\Sigma(\mathcal{G}(\mathcal{A}, M \circ L) \multimap\circledast \mathcal{G}(\mathcal{B}, N \circ L)) \quad \longrightarrow \quad \wp_\Gamma((\mathcal{G}(\mathcal{A}, M) \multimap\circledast \mathcal{G}(\mathcal{B}, N))$$

These two maps are related *via* HS as expected: $\mathrm{HS} \circ \wp(L)_{\multimap\circledast} = (\wp(L) \times \wp(L)) \circ \mathrm{HS}$. Moreover,

▶ **Proposition 4.1.** *We have, in* **Set***:*

$$
\begin{tikzcd}
\wp_\Sigma(\mathcal{A}, M \circ L) \arrow[r, "\wp(L)"] \arrow[d, "\mathrm{tr}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \wp_\Gamma(\mathcal{A}, M) \arrow[d, "\mathrm{tr}"] \\
\mathrm{Tr}_\Sigma \arrow[r, "\mathrm{Tr}(L)"] & \mathrm{Tr}_\Gamma
\end{tikzcd}
\qquad
\begin{tikzcd}
\wp^{\mathsf{P}}_\Sigma(\mathcal{G}(\mathcal{A}, M \circ L) \mathrel{-\!\!\circledast} \mathcal{G}(\mathcal{B}, N \circ L)) \arrow[d, "\mathrm{tr}^{-\circledast}"'] \arrow[dr, "\wp(L)_{-\circledast}"] \arrow[dr, phantom, "\lrcorner", very near start] & \\
\mathrm{Tr}_\Sigma \arrow[dr, "\mathrm{Tr}(L)"'] & \wp^{\mathsf{P}}_\Gamma(\mathcal{G}(\mathcal{A}, M) \mathrel{-\!\!\circledast} \mathcal{G}(\mathcal{B}, N)) \arrow[d, "\mathrm{tr}^{-\circledast}"] \\
& \mathrm{Tr}_\Gamma
\end{tikzcd}
$$

Substitution on Strategies. The action of L^* on strategies is defined using Prop. 4.1: Given $\Gamma \vdash \sigma : \mathcal{G}(\mathcal{A}, M) \mathrel{-\!\!\circledast} \mathcal{G}(\mathcal{B}, N)$, so that $\sigma \subseteq \wp^{\mathsf{P}}_\Gamma(\mathcal{G}(\mathcal{A}, M) \mathrel{-\!\!\circledast} \mathcal{G}(\mathcal{B}, N))$, we define

$$
L^*(\sigma) \quad := \quad \wp(L)^{-1}_{-\circledast}(\sigma) \quad \subseteq \quad \wp^{\mathsf{P}}_\Sigma(\mathcal{G}(\mathcal{A}, M \circ L) \mathrel{-\!\!\circledast} \mathcal{G}(\mathcal{B}, N \circ L))
$$

▶ **Proposition 4.2.** *$L^*(\sigma)$ is a strategy. If moreover σ is winning, then $L^*(\sigma)$ is also winning.*

Functoriality of Substitution. Proposition 4.1 can be formulated by saying that the maps $\langle \mathrm{tr}, \wp(L) \rangle$ and $\langle \mathrm{tr}^{-\circledast}, \wp(L)_{-\circledast} \rangle$ are bijections, respectively:

$$
\begin{array}{rcl}
\wp_\Sigma(\mathcal{A}, M \circ L) & \xrightarrow{\simeq} & \mathrm{Tr}_\Sigma \times_{\mathrm{Tr}_\Sigma} \wp_\Gamma(\mathcal{A}, M) \\
\wp^{\mathsf{P}}_\Sigma(\mathcal{G}(\mathcal{A}, M \circ L) \mathrel{-\!\!\circledast} \mathcal{G}(\mathcal{B}, N \circ L)) & \xrightarrow{\simeq} & \mathrm{Tr}_\Sigma \times_{\mathrm{Tr}_\Sigma} \wp^{\mathsf{P}}_\Gamma(\mathcal{G}(\mathcal{A}, M) \mathrel{-\!\!\circledast} \mathcal{G}(\mathcal{B}, N))
\end{array}
$$

These bijections are crucial to prove that

▶ **Proposition 4.3.** *L^* is a functor from $\mathbf{SAG}^{(\mathrm{W})}_\Gamma$ to $\mathbf{SAG}^{(\mathrm{W})}_\Sigma$.*

▶ Remark. Recall that $L^\bullet : \mathrm{Tr}_\Gamma \to \mathrm{Tr}_\Sigma$ has a left adjoint, and thus preserves limits. Since strategies can be seen as synchronous relations, which can moreover be composed by pullbacks (2), this suggests that the codomain fibration cod already provides enough categorical structure to obtain substitution functors on synchronous acceptance games. This seems however *a priori* not sufficient to obtain *strict* substitution functors, since the limits (2) may not be preserved on the noise. This motivated the finer description provided by the pullback properties of Prop. 4.1, in which all maps involved are specifically defined.

4.2 Fibrations of Acceptance Games

Consider now $L \in \mathbf{Tree}[\Sigma, \Gamma]$ and $K \in \mathbf{Tree}[\Gamma, \Delta]$. Since $\mathrm{Tr}(K \circ L) = \mathrm{Tr}(K) \circ \mathrm{Tr}(L)$ and $\wp(K \circ L)_{(-\circledast)} = \wp(K)_{(-\circledast)} \circ \wp(L)_{(-\circledast)}$ we immediately get

▶ **Proposition 4.4.** *The operations $(-)^* : \mathbf{Tree}^{\mathrm{op}} \to \mathbf{Cat}$, mapping Σ to $\mathbf{SAG}^{(\mathrm{W})}_\Sigma$, and mapping $L \in \mathbf{Tree}[\Sigma, \Gamma]$ to $L^* : \mathbf{SAG}^{(\mathrm{W})}_\Gamma \to \mathbf{SAG}^{(\mathrm{W})}_\Sigma$ are functors.*

By using Groethendieck completion (see e.g. [12, §1.10]), this gives us split fibrations of acceptance games $\mathsf{game}^{(\mathrm{W})} : \mathbf{SAG}^{(\mathrm{W})} \longrightarrow \mathbf{Tree}$ that we do not detail here by lack of space.

4.3 Fibrations of Automata

In order to obtain fibrations of automata, we restrict substitution to tree morphisms generated by alphabet morphisms $\beta \in \mathbf{Alph}[\Sigma, \Gamma]$. The crucial point is that these restricted substitutions can be internalized in automata.

Given $\Gamma \vdash \mathcal{A}$ with $\mathcal{A} = (Q, q^\imath, \delta, \Omega)$, and $\beta \in \mathbf{Alph}[\Sigma, \Gamma]$, define the automaton $\Sigma \vdash \mathcal{A}[\beta]$ as $\mathcal{A}[\beta] := (Q, q^\imath, \delta_\beta, \Omega)$ where $\delta_\beta(q, a) := \delta(q, \beta(a))$.

▶ **Proposition 4.5.** $\Sigma \vdash \mathcal{G}(\mathcal{A}[\beta], \mathrm{Id}_\Sigma) \;=\; \Sigma \vdash \mathcal{G}(\mathcal{A}, \beta)$.

It is easy to see that $(-)^*$ restricts to a functor from $\mathbf{Alph}^{\mathrm{op}}$ to $\mathbf{Cat}$, so that we get fibrations

$$\mathsf{aut}^{(\mathrm{W})} \quad : \quad \mathbf{Aut}^{(\mathrm{W})} \quad \longrightarrow \quad \mathbf{Alph}$$

5 Symmetric Monoidal Structure

We now consider a synchronous product of automata. When working on *complete* automata (to be defined in Sect. 5.1 below), it gives rise to split symmetric monoidal fibrations, in the sense of [21].

According to [21, Thm. 12.7], split symmetric monoidal fibrations can equivalently be obtained from split symmetric monoidal indexed categories. In our context, this means that the functors $(-)^*$ extend to

$$(-)^* \quad : \quad \mathbf{Tree}^{\mathrm{op}} \quad \longrightarrow \quad \mathbf{SymMonCat} \qquad\qquad (-)^* \quad : \quad \mathbf{Alph}^{\mathrm{op}} \quad \longrightarrow \quad \mathbf{SymMonCat}$$

where **SymMonCat** is the category of symmetric monoidal categories and strong monoidal functors. Hence, we equip our categories of (complete) acceptance games and automata with a symmetric monoidal structure. Substitution turns out to be *strict* symmetric monoidal.

We refer to [16] for background on symmetric monoidal categories.

5.1 Complete Tree Automata

An automaton $\mathcal{A}$ is *complete* if for every $(q, a) \in Q \times \Sigma$, the set $\delta(q, a)$ is not empty and moreover for every $\gamma \in \delta(q, a)$ and everydirection $d \in D$, we have $(q', d) \in \gamma$ for some $q' \in Q$.

Given an automaton $\mathcal{A} = (Q, q^\imath, \delta, \Omega)$ its *completion* is the automaton $\widehat{\mathcal{A}} := (\widehat{Q}, q^\imath, \widehat{\delta}, \widehat{\Omega})$ with states $\widehat{Q} := Q + \{\mathsf{true}, \mathsf{false}\}$, with acceptance condition $\widehat{\Omega} := \Omega + Q^* \cdot \mathsf{true} \cdot \widehat{Q}^\omega$, and with transition function $\widehat{\delta}$ defined as

$$\begin{array}{rcllrcl}
\widehat{\delta}(\mathsf{true}, q) & := & \{\{(\mathsf{true}, d) \mid d \in D\}\} & \qquad & \widehat{\delta}(\mathsf{false}, q) & := & \{\{(\mathsf{false}, d) \mid d \in D\}\} \\
\widehat{\delta}(q, a) & := & \{\{(\mathsf{false}, d) \mid d \in D\}\} & & \multicolumn{3}{l}{\text{if } q \in Q \text{ and } \delta(q, a) = \emptyset} \\
\widehat{\delta}(q, a) & := & \{\widehat{\gamma} \mid \gamma \in \delta(q, a)\} & & \multicolumn{3}{l}{\text{otherwise}}
\end{array}$$

where, given $\gamma \in \delta(q, a)$, we let $\widehat{\gamma} \;:=\; \gamma \cup \{(\mathsf{true}, d) \mid \text{there is no } q \in Q \text{ s.t. } (q, d) \in \gamma\}$.

▶ **Proposition 5.1.** $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\widehat{\mathcal{A}})$.

Restricting to complete automata gives rise to full subcategories $\widehat{\mathbf{SAG}}_\Sigma^{(\mathrm{W})}$ and $\widehat{\mathbf{Aut}}_\Sigma^{(\mathrm{W})}$ of resp. $\mathbf{SAG}_\Sigma^{(\mathrm{W})}$ and $\mathbf{Aut}_\Sigma^{(\mathrm{W})}$, and thus induces fibrations

$$\widehat{\mathsf{game}} \quad : \quad \widehat{\mathbf{SAG}}^{(\mathrm{W})} \quad \longrightarrow \quad \mathbf{Tree} \qquad\qquad \widehat{\mathsf{aut}} \quad : \quad \widehat{\mathbf{Aut}}^{(\mathrm{W})} \quad \longrightarrow \quad \mathbf{Alph}$$

5.2 The Synchronous Product

Assume given complete automata $\Sigma \vdash \mathcal{A}$ and $\Sigma \vdash \mathcal{B}$. Define $\Sigma \vdash \mathcal{A} \circledast \mathcal{B}$ as

$$\mathcal{A} \circledast \mathcal{B} \quad := \quad (Q_\mathcal{A} \times Q_\mathcal{B}, (q^\imath_\mathcal{A}, q^\imath_\mathcal{B}), \delta_{\mathcal{A}\circledast\mathcal{B}}, \Omega_{\mathcal{A}\circledast\mathcal{B}})$$

where $(q^n_\mathcal{A}, q^n_\mathcal{B})_{n\in\mathbb{N}} \in \Omega_{\mathcal{A}\circledast\mathcal{B}}$ iff $((q^n_\mathcal{A})_{n\in\mathbb{N}} \in \Omega_\mathcal{A}$ and $(q^n_\mathcal{B})_{n\in\mathbb{N}} \in \Omega_\mathcal{B})$, and where we let $\delta_{\mathcal{A}\circledast\mathcal{B}}((q_\mathcal{A}, q_\mathcal{B}), a)$ be the set of all the $\gamma_\mathcal{A} \circledast \gamma_\mathcal{B}$ for $\gamma_\mathcal{A} \in \delta_\mathcal{A}(q_\mathcal{A}, a)$ and $\gamma_\mathcal{B} \in \delta_\mathcal{B}(q_\mathcal{B}, a)$, with $\gamma_\mathcal{A} \circledast \gamma_\mathcal{B} \;:=\; \{((q'_\mathcal{A}, q'_\mathcal{B}), d) \mid d \in D \text{ and } (q'_\mathcal{A}, d) \in \gamma_\mathcal{A} \text{ and } (q'_\mathcal{B}, d) \in \gamma_\mathcal{B}\}$.

Note that since $\mathcal{A}$ and $\mathcal{B}$ are complete, each $\gamma_{\mathcal{A}\circledast\mathcal{B}} \in \delta_{\mathcal{A}\circledast\mathcal{B}}((q_\mathcal{A}, q_\mathcal{B}), a)$ uniquely decomposes as $\gamma_{\mathcal{A}\circledast\mathcal{B}} = \gamma_\mathcal{A} \circledast \gamma_\mathcal{B}$.

Action on Plays. The unique decomposition property of $\gamma_{\mathcal{A}\circledast\mathcal{B}}$ allows to define projections

$$\begin{array}{rcrcl} \varpi_i & : & \wp_\Sigma(\mathcal{A}_1 \circledast \mathcal{A}_2, M) & \longrightarrow & \wp_\Sigma(\mathcal{A}_i, M) \\ \varpi_i^{-\!\circledast} & : & \wp_\Sigma\left(\mathcal{G}(\mathcal{A}_1 \circledast \mathcal{B}_1, M) \mathrel{-\!\!\circledast} \mathcal{G}(\mathcal{A}_2 \circledast \mathcal{B}_2, N)\right) & \longrightarrow & \wp_\Sigma\left(\mathcal{G}(\mathcal{A}_i, M) \mathrel{-\!\!\circledast} \mathcal{G}(\mathcal{B}_i, N)\right) \end{array}$$

We write $\mathrm{SP} := \langle \varpi_1, \varpi_2 \rangle$ and $\mathrm{SP}_{-\circledast} := \langle \varpi_1^{-\circledast}, \varpi_2^{-\circledast} \rangle$.

▶ **Proposition 5.2.** *We have, in* **Set***:*

$$\begin{tikzcd} \wp_\Sigma(\mathcal{A} \circledast \mathcal{B}, M) \arrow[r, "\varpi_2"] \arrow[d, "\varpi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \wp_\Sigma(\mathcal{B}, M) \arrow[d, "\mathrm{tr}"] \\ \wp_\Sigma(\mathcal{A}, M) \arrow[r, "\mathrm{tr}"] & \mathrm{Tr}_\Sigma \end{tikzcd} \qquad \begin{tikzcd} \wp^{\mathsf{P}}_\Sigma(\mathcal{G}(\mathcal{A} \circledast \mathcal{B}, M) \mathrel{-\!\!\circledast} \mathcal{G}(\mathcal{C} \circledast D, N)) \arrow[dd, "\varpi_1^{-\circledast}"'] \arrow[dr, "\varpi_2^{-\circledast}"] \arrow[ddr, phantom, "\lrcorner", very near start] & \\ & \wp^{\mathsf{P}}_\Sigma(\mathcal{G}(\mathcal{B}, M) \mathrel{-\!\!\circledast} \mathcal{G}(\mathcal{D}, N)) \arrow[d, "\mathrm{tr}"] \\ \wp^{\mathsf{P}}_\Sigma(\mathcal{G}(\mathcal{A}, M) \mathrel{-\!\!\circledast} \mathcal{G}(\mathcal{C}, N)) \arrow[r, "\mathrm{tr}"'] & \mathrm{Tr}_\Sigma \end{tikzcd}$$

Action on Synchronous Games. The action of $\circledast$ on the objects of $\widehat{\mathbf{SAG}}_\Sigma^{(\mathrm{W})}$ is given by

$$(\Sigma \vdash \mathcal{G}(\mathcal{A}, M)) \circledast (\Sigma \vdash \mathcal{G}(\mathcal{B}, N)) \quad := \quad \Sigma \vdash \mathcal{G}(\mathcal{A}[\pi] \circledast \mathcal{B}[\pi'], \langle M, N \rangle)$$

where π and π' are suitable projections. For morphisms, let $\Sigma \vdash \sigma : \mathcal{G}(\mathcal{A}_0, M_0) \mathrel{-\!\!\circledast} \mathcal{G}(\mathcal{A}_1, M_1)$ and $\Sigma \vdash \tau : \mathcal{G}(\mathcal{B}_0, N_0) \mathrel{-\!\!\circledast} \mathcal{G}(\mathcal{B}_1, N_1)$. Then since $\Sigma \vdash \mathcal{G}(\mathcal{A}_i[\pi_i], \langle M_i, N_i \rangle) = \Sigma \vdash \mathcal{G}(\mathcal{A}_i, M_i)$ and $\Sigma \vdash \mathcal{G}(\mathcal{B}_i[\pi'_i], \langle M_i, N_i \rangle) = \Sigma \vdash \mathcal{G}(\mathcal{B}_i, N_i)$, thanks to Prop. 5.2 we can simply let $\sigma \circledast \tau := \mathrm{SP}_{-\circledast}^{-1}(\sigma, \tau)$.

▶ **Proposition 5.3.** *The product* $\_ \circledast \_$ *gives functors* $\widehat{\mathbf{SAG}}_\Sigma^{(\mathrm{W})} \times \widehat{\mathbf{SAG}}_\Sigma^{(\mathrm{W})} \longrightarrow \widehat{\mathbf{SAG}}_\Sigma^{(\mathrm{W})}$.

5.3 Symmetric Monoidal Structure

Thanks to Prop. 5.2 and Prop. 3.4 the symmetric monoidal structure of $\circledast$ in $\widehat{\mathbf{SAG}}_\Sigma^{(\mathrm{W})}$ can be directly obtained from the symmetric monoidal structure of the tensorial product of $\mathbf{Rel}(\mathbf{Set}/\mathrm{Tr}_\Sigma)$.

Symmetric Monoidal Structure in $\mathbf{Rel}(\mathbf{Set}/J)$. We define a product $\otimes$ in $\mathbf{Rel}(\mathbf{Set}/J)$:

On Objects: for (A, g) and (B, h) objects in $\mathbf{Rel}(\mathbf{Set}/J)$ the product $A \otimes B$ is $A \times_J B$ with the corresponding map, that is

$$A \otimes B \quad := \quad \{(a, b) \in A \times B \mid g(a) = h(b)\} \quad \xrightarrow{g \circ \pi_1 = h \circ \pi_2} \quad J$$

On Morphisms: given $R \in \mathbf{Rel}(\mathbf{Set}/J)[A, C]$ and $P \in \mathbf{Rel}(\mathbf{Set}/J)[B, D]$, we define $R \otimes P \in \mathbf{Rel}(\mathbf{Set}/J)[A \otimes B, C \otimes D]$ as

$$R \otimes P \quad := \quad \{((a, b), (c, d)) \in (A \otimes B) \times (C \otimes D) \mid (a, c) \in R \text{ and } (b, d) \in P\}$$

For the unit, we *choose* some $\mathbf{I} = (\jmath : I \xrightarrow{\simeq} J)$. Note that $\jmath$ is required to be a bijection. The natural isomorphisms are given by:

$$\begin{array}{rcl} \mathring{\alpha}_{A,B,C} & := & \{(((a, b), c)\ ,\ (a, (b, c))) \mid g_A(a) = g_B(b) = g_C(c)\} \\ \mathring{\lambda}_A & := & \{((e, a)\ ,\ a) \mid \jmath(e) = g_A(a)\} \\ \mathring{\rho}_A & := & \{((a, e)\ ,\ a) \mid g_A(a) = \jmath(e)\} \\ \mathring{\gamma}_{A,B} & := & \{((a, b)\ ,\ (b, a)) \mid g_A(a) = g_B(b)\} \end{array}$$

We easily get:

▶ **Proposition 5.4.** *The category* $\mathbf{Rel}(\mathbf{Set}/J)$*, equipped with the above data, is symmetric monoidal.*

Unit Automata. The requirement that the monoidal unit $\jmath : I \to J$ of $\mathbf{Rel}(\mathbf{Set}/J)$ should be a bijection leads us to the following unit automata. We let $\mathcal{I} := (Q_{\mathcal{I}}, q_{\mathcal{I}}, \delta_{\mathcal{I}}, \Omega_{\mathcal{I}})$ where $Q_{\mathcal{I}} := \mathbf{1}$, $q_{\mathcal{I}} := \bullet$, $\Omega_{\mathcal{I}} = Q_{\mathcal{I}}^{\omega}$ and $\delta_{\mathcal{I}}(q_{\mathcal{I}}, a) := \{\{(q_{\mathcal{I}}, d) \mid d \in D\}\}$.

Note that since $\delta_{\mathcal{I}}$ is constant, we have $\Sigma \vdash \mathcal{G}(\mathcal{I}, M) = \Sigma \vdash \mathcal{G}(\mathcal{I}, \mathrm{Id})$. Moreover,

▶ **Proposition 5.5.** *Given $M \in \mathbf{Tree}[\Sigma, \Gamma]$, we have, in* $\mathbf{Set}$*, a bijection*

$$\mathrm{tr} \quad : \quad \wp_{\Sigma}(\mathcal{I}, M) \quad \xrightarrow{\simeq} \quad \mathrm{Tr}_{\Sigma}$$

Symmetric Monoidal Structure. Using Prop. 3.4, the structure isos of $\mathbf{Rel}(\mathbf{Set}/\mathrm{Tr}_{\Sigma})$ can be lifted to $\widehat{\mathbf{SAG}}_{\Sigma}^{(\mathrm{W})}$ (winning is trivial). Moreover, the required equations (naturality and coherence) follows from Prop. 3.3, Prop 5.2, and the fact that $((\mathrm{SP} \times \mathrm{SP}) \circ \mathrm{HS})(\sigma \circledast \tau) = \mathrm{HS}(\sigma) \otimes \mathrm{HS}(\tau)$ (where composition on the left is in $\mathbf{Set}$, and the expression denotes the actions of the resulting function on the set of plays $(\sigma \circledast \tau)$).

All the symmetric monoidal structure restricts from $\widehat{\mathbf{SAG}}_{\Sigma}^{(\mathrm{W})}$ to $\widehat{\mathbf{Aut}}_{\Sigma}^{(\mathrm{W})}$.

▶ **Proposition 5.6.** *The categories $\widehat{\mathbf{SAG}}_{\Sigma}^{(\mathrm{W})}$ and $\widehat{\mathbf{Aut}}_{\Sigma}^{(\mathrm{W})}$ equipped with the above data, are symmetric monoidal.*

5.4 Symmetric Monoidal Fibrations

In order to obtain symmetric monoidal fibrations, by [21, Thm. 12.7], it remains to check that substitution is strong monoidal. It is actually *strict* monoidal: it directly commutes with $\circledast$ and preserves the unit, as well as all the structure maps.

▶ **Proposition 5.7.**

(i) Given $L \in \mathbf{Tree}[\Sigma, \Gamma]$, the functors $L^* : \widehat{\mathbf{SAG}}_{\Gamma}^{(\mathrm{W})} \to \widehat{\mathbf{SAG}}_{\Sigma}^{(\mathrm{W})}$ are strict monoidal.

(ii) Given $\beta \in \mathbf{Alph}[\Sigma, \Gamma]$, the functors $\beta^* : \widehat{\mathbf{Aut}}_{\Gamma}^{(\mathrm{W})} \to \widehat{\mathbf{Aut}}_{\Sigma}^{(\mathrm{W})}$ are strict monoidal.

6 Correctness w.r.t. Language Operations

This Section gathers several properties stating the correctness of our constructions w.r.t. operations on recognized languages. We begin in Sect. 6.1 by properties on the symmetric monoidal structure, the most important one being that the synchronous arrow is *correct*, in the sense that $\Sigma \vdash \mathcal{A} \multimap\!\!\circledast\; \mathcal{B}$ implies $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$. Then, in Sect. 6.2, we discuss complementation of automata, and its relation with the synchronous arrow.

6.1 Correctness of the Symmetric Monoidal Structure

We begin by a formal correspondence between acceptance games and synchronous games of a specific form. This allows to show that the synchronous arrow is *correct*, in the sense that $\Sigma \vdash \mathcal{A} \multimap\!\!\circledast\; \mathcal{B}$ implies $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$. We then briefly discuss the correctness of the synchronous product w.r.t. language intersection.

▶ **Proposition 6.1.** *Given $\Sigma \vdash \mathcal{A}$ and $t \in \mathbf{Tree}[\Sigma]$, there is a bijection:*

$$\{\sigma \mid \mathbf{1} \vdash \sigma \Vdash \mathcal{G}(\mathcal{A}, \dot{t})\} \qquad \simeq \qquad \{\theta \mid \mathbf{1} \vdash \theta \Vdash \mathcal{G}(\mathcal{I}, \mathrm{Id}_{\mathbf{1}}) \multimap\!\!\circledast\; \mathcal{G}(\mathcal{A}, \dot{t})\}$$

▶ Remark. The above correspondence is only possible for acceptance games over $\mathbf{1}$:

- In $\Sigma \vdash \sigma \Vdash \mathcal{G}(\mathcal{A}, M)$, σ is a positive P-strategy, hence chooses the input characters in Σ.

- In $\Sigma \vdash \theta \Vdash \mathcal{G}(\mathcal{I}_\Sigma, \mathrm{Id}_\Sigma) \multimap_{\circledast} \mathcal{G}(\mathcal{A}, M)$, the strategy θ is a negative. It plays positively in $\Sigma \vdash \mathcal{G}(\mathcal{A}, M)$, but must follow the input characters chosen by O in $\Sigma \vdash \mathcal{G}(\mathcal{I}_\Sigma, \mathrm{Id}_\Sigma)$.

We now check that the arrow $\mathcal{G}(\mathcal{A}, M) \multimap_{\circledast} \mathcal{G}(\mathcal{B}, N)$ is correct w.r.t. language inclusion:

▶ **Proposition 6.2** (Correctness of the Arrow). *Assume given $\Sigma \vdash \sigma \Vdash \mathcal{G}(\mathcal{A}, M) \multimap_{\circledast} \mathcal{G}(\mathcal{B}, N)$.*

(i) For all $t \in \mathbf{Tree}[\Sigma]$, we have $\dot{t}^*(\sigma) \Vdash \mathcal{G}(\mathcal{A}, M \circ \dot{t}) \multimap_{\circledast} \mathcal{G}(\mathcal{B}, N \circ \dot{t})$.
(ii) If $\mathbf{1} \Vdash \mathcal{G}(\mathcal{A}, M \circ \dot{t})$ then $\mathbf{1} \Vdash \mathcal{G}(\mathcal{B}, N \circ \dot{t})$.
(iii) For all tree $t \in \mathbf{Tree}[\Sigma]$, if $M(t) \in \mathcal{L}(\mathcal{A})$ then $N(t) \in \mathcal{L}(\mathcal{B})$.

The converse property will be discussed in Sect. 7. We finally check that the synchronous product is correct.

▶ **Proposition 6.3.** $\mathcal{L}(\widehat{\mathcal{A}} \circledast \widehat{\mathcal{B}}) = \mathcal{L}(\mathcal{A}) \cap \mathcal{L}(\mathcal{B})$.

6.2 Complementation and Falsity

Complementation. Given an automaton $\mathcal{A} = (Q, q^\imath, \delta, \Omega)$, following [23], we let its complement be $\sim\!\mathcal{A} := (Q, q^\imath, \delta_{\sim\mathcal{A}}, \Omega_{\sim\mathcal{A}})$, where $\Omega_{\sim\mathcal{A}} := Q^\omega \setminus \Omega$ and

$$\delta_{\sim\mathcal{A}}(q, a) \quad := \quad \{\gamma_\sim \in \mathcal{P}(Q \times D) \mid \forall \gamma \in \delta(q, a),\ \gamma_\sim \cap \gamma \neq \emptyset\}$$

The idea is that P on $\sim\!\mathcal{A}$ simulates O on $\mathcal{A}$, so that the correctness of $\sim\!\mathcal{A}$ relies on determinacy of acceptance games. In particular, thanks to Borel determinacy [14], we have:

▶ **Proposition 6.4** ([23]). *Given $\mathcal{A}$ with $\Omega_{\mathcal{A}}$ a Borel set, we have $\mathcal{L}(\sim\!\mathcal{A}) = \mathbf{Tree}[\Sigma] \setminus \mathcal{L}(\mathcal{A})$.*

Note that if $\mathcal{A}$ is complete, then $\sim\!\mathcal{A}$ is not necessarily complete, but $\delta_{\sim\mathcal{A}}$ is always not empty and so are the γ's in its image.

The Falsity Automaton $\perp\!\!\!\!\!\!\perp$. We let $\perp\!\!\!\!\!\!\perp := (Q_{\perp\!\!\!\perp}, q_{\perp\!\!\!\perp}, \delta_{\perp\!\!\!\perp}, \Omega_{\perp\!\!\!\perp})$ where $Q_{\perp\!\!\!\perp} := \mathbf{1}$, $q_{\perp\!\!\!\perp} := \bullet$, $\Omega_{\perp\!\!\!\perp} = \emptyset$ and $\delta_{\perp\!\!\!\perp}(q_{\perp\!\!\!\perp}, a) := \{\{(q_{\perp\!\!\!\perp}, d)\} \mid d \in D\}$. Note that $\mathcal{I} = \sim\!\perp\!\!\!\!\!\!\perp$. In particular, it is actually P who guides the evaluation of $\perp\!\!\!\!\!\!\perp$, by choosing the tree directions.

▶ **Proposition 6.5.** *Let $\mathcal{A}$ and $\mathcal{B}$ be complete. Then $\Sigma \Vdash \mathcal{A} \circledast \mathcal{B} \multimap_{\circledast} \widehat{\perp\!\!\!\perp}$ iff $\Sigma \Vdash \mathcal{A} \multimap_{\circledast} \widehat{\sim\!\mathcal{B}}$.*

▶ **Corollary 6.6.** *Let $\mathcal{A}$ be a complete automaton on Σ. Then $\mathbf{1} \Vdash \widehat{\sim\!\mathcal{A}}$ iff $\mathbf{1} \Vdash \mathcal{A} \multimap_{\circledast} \widehat{\perp\!\!\!\perp}$.*

7 Projection and Fibred Simple Coproducts

We now check that automata can be equipped with existential quantifications in the fibered sense. Namely, given a projection $\pi \in \mathbf{Alph}[\Sigma \times \Gamma, \Sigma]$, the induced weakening functor $\pi^* : \widehat{\mathbf{Aut}}^{(\mathrm{W})}_{\Sigma} \to \widehat{\mathbf{Aut}}^{(\mathrm{W})}_{\Sigma \times \Gamma}$ has a left-adjoint $\amalg_{\Sigma,\Gamma}$, and moreover this structure is preserved by substitution, in the sense of the Beck-Chevalley condition (see e.g. [12]). This will lead to a (weak) completeness property of the synchronous arrow on *non-deterministic* automata, to be discussed below.

Recall from [13, Thm. IV.1.2.(ii)] than an adjunction $\amalg_{\Sigma,\Gamma} \dashv \pi^*$, with π^* a functor, is completely determined by the following data: To each object $\Sigma \times \Gamma \vdash \mathcal{A}$, an object

$\Sigma \vdash \amalg_{\Sigma,\Gamma}\mathcal{A}$, and a map $\eta_{\mathcal{A}} : \Sigma \times \Gamma \vdash \mathcal{A} \longrightarrow \Sigma \times \Gamma \vdash (\amalg_{\Sigma,\Gamma}\mathcal{A})[\pi]$ satisfying the following universal lifting property:

For every
$\sigma \quad : \quad \Sigma \times \Gamma \vdash \mathcal{A} \quad \longrightarrow \quad \Sigma \times \Gamma \vdash \mathcal{B}[\pi]$
there is a unique
$\tau \quad : \quad \Sigma \vdash \amalg_{\Sigma,\Gamma}\mathcal{A} \quad \longrightarrow \quad \Sigma \vdash \mathcal{B}$

s.t.

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\eta_{\mathcal{A}}"] \arrow[dr, "\sigma"'] & (\amalg_{\Sigma,\Gamma}\mathcal{A})[\pi] \arrow[d, "\pi^*(\tau)"] \\ & \mathcal{B}[\pi] \end{tikzcd} \tag{4}$$

In our context, the Beck-Chevalley condition amounts to the equalities

$$\Delta \vdash (\amalg_{\Sigma,\Gamma}\mathcal{A})[\beta] \;=\; \Delta \vdash \amalg_{\Delta,\Gamma}(\mathcal{A}[\beta \times \mathrm{Id}_\Gamma]) \qquad \eta_{\mathcal{A}[\beta\times\mathrm{Id}_\Gamma]} \;=\; (\beta \times \mathrm{Id}_\Gamma)^*(\eta_{\mathcal{A}}) \tag{5}$$

It turns out that the usual projection operation on automata (see e.g. [23]) is not functorial. Surprisingly, this is independent from whether automata are non-deterministic or not[3]. We devise a *lifted* projection operation, which indeed leads to a fibered existential quantification, and which is correct, on non-deterministic automata, w.r.t. the recognized languages.

The Lifted Projection. Consider $\Sigma \times \Gamma \vdash \mathcal{A}$ with $\mathcal{A} = (Q, q^\imath, \delta, \Omega)$. Define $\Sigma \vdash \amalg_{\Sigma,\Gamma}\mathcal{A}$ as $\amalg_{\Sigma,\Gamma}\mathcal{A} \;:=\; (Q \times \Gamma + \{q^\imath\}, q^\imath, \delta_{\amalg\mathcal{A}}, \Omega_{\amalg\mathcal{A}})$ where

$$\begin{array}{rcl} \delta_{\amalg\mathcal{A}}(q^\imath, a) & := & \bigcup_{b\in\Gamma}\{\gamma^{+b} \mid \gamma \in \delta(q^\imath, (a,b))\} \\ \delta_{\amalg\mathcal{A}}((q, \_), a) & := & \bigcup_{b\in\Gamma}\{\gamma^{+b} \mid \gamma \in \delta(q, (a,b))\} \end{array}$$

and, given $\gamma \in \mathcal{P}(Q \times D)$ and $b \in \Gamma$, we let $\gamma^{+b} := \{((q^{+b}, d) \mid (q,d) \in \gamma\}$ with $q^{+b} := (q, b)$. For the acceptance condition, we let $q^\imath \cdot (q_0, b_0) \cdot \ldots \cdot (q_n, b_n) \cdot \ldots$ in $\Omega_{\amalg\mathcal{A}}$ iff $q^\imath \cdot q_0 \cdot \ldots \cdot q_n \cdot \ldots \;\in\; \Omega$.

Action on Plays of The Lifted Projection. The action on plays of $\amalg_{\Sigma,\Gamma}$ is characterized by the map $\wp(\amalg) : \wp_{\Sigma\times\Gamma}(\mathcal{A}) \longrightarrow \wp_\Sigma(\amalg_{\Sigma,\Gamma}\mathcal{A})$ inductively defined as $\wp(\amalg)(\varepsilon, q^\imath) := (\varepsilon, q^\imath)$ and

$$\begin{array}{rcl} \wp(\amalg)((\varepsilon, q^\imath) \to^* (p,q) \to (p,(a,b),\gamma)) & := & \wp(\amalg)((\varepsilon, q^\imath) \to^* (p,q)) \to (p, a, \gamma^{+b}) \\ \wp(\amalg)((\varepsilon, q^\imath) \to^* (p,(a,b),\gamma) \to (p.d, q)) & := & \wp(\amalg)((\varepsilon, q^\imath) \to^* (p,(a,b),\gamma)) \to (p.d, q^{+b}) \end{array}$$

▶ **Proposition 7.1.** *If $\mathcal{A}$ is a complete automaton, then $\wp(\amalg)$ is a bijection.*

The Unit Maps $\eta_{(-)}$. Consider the injection $\iota_{\Sigma,\Gamma} : \wp_\Sigma(\amalg_{\Sigma,\Gamma}\mathcal{A}) \longrightarrow \wp_{\Sigma\times\Gamma}((\amalg_{\Sigma,\Gamma}\mathcal{A})[\pi])$ inductively defined as $\iota_{\Sigma,\Gamma}((\varepsilon, q^\imath_{\mathcal{A}})) := (\varepsilon, q^\imath_{\mathcal{A}})$ and $\iota_{\Sigma,\Gamma}(s \to (p, q^{+b})) := \iota_{\Sigma,\Gamma}(s) \to (p, q^{+b})$ and $\iota_{\Sigma,\Gamma}(s \to (p, a, \gamma^{+b})) := \iota_{\Sigma,\Gamma}(s) \to (p, (a,b), \gamma^{+b})$.

If $\Sigma \times \Gamma \vdash \mathcal{A}$ is complete, we let the unit $\eta_{\mathcal{A}}$ be the unique strategy of $\widehat{\mathbf{SAG}}^{\mathrm{W}}_{\Sigma\times\Gamma}$ such that $\mathrm{HS}(\eta_{\mathcal{A}}) = \{(t, \iota_{\Sigma,\Gamma} \circ \wp(\amalg)(t)) \mid t \in \wp_{\Sigma\times\Gamma}(\mathcal{A})\}$. We do not detail the B.-C. condition (5).

The Unique Lifting Property (4). Consider some $\Sigma \times \Gamma \vdash \sigma : \mathcal{A} \mathrel{-\!\!\circledast} \mathcal{B}[\pi]$ with $\mathcal{A}$ complete. We let τ be the unique strategy such that $\mathrm{HS}(\tau) = \{(\wp(\amalg)(s), \wp(\pi)(t)) \mid (s,t) \in \mathrm{HS}(\sigma)\}$. It is easy to see that τ is winning whenever σ is winning. Moreover

▶ **Lemma 7.2.** $\sigma = \pi^*(\tau) \circ \eta_{\mathcal{A}}$.

For the unicity part of the lifting property of $\eta_{\mathcal{A}}$, it is sufficient to check:

▶ **Lemma 7.3.** *If $\pi^*(\theta) \circ \eta_{\mathcal{A}} = \pi^*(\theta') \circ \eta_{\mathcal{A}}$ then $\theta = \theta'$.*

[3] It is well-known that the projection operation is correct w.r.t. the recognized languages only on *non-deterministic automata*.

Non-Deterministic Tree Automata. An automaton $\mathcal{A}$ is *non-deterministic* if for every γ in the image of δ and every direction $d \in D$, there is at most one state q such that $(q, d) \in \gamma$.

▶ Remark. If $\mathcal{A}$ and $\mathcal{B}$ are non-deterministic, then so are $\mathcal{A} \circledast \mathcal{B}$ and $\amalg(\mathcal{A})$.

▶ **Proposition 7.4** ([6, 18, 23]). *For each regular automaton $\Sigma \vdash \mathcal{A}$ there is a complete non-deterministic automaton $\Sigma \vdash \mathrm{ND}(\mathcal{A})$ such that $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathrm{ND}(\mathcal{A}))$.*

▶ **Proposition 7.5.** *If $\Sigma \times \Gamma \vdash \mathcal{A}$ is non-deterministic and complete, then $\mathcal{L}(\amalg_{\Sigma,\Gamma}\mathcal{A}) = \pi_{\Sigma,\Gamma}(\mathcal{L}(\mathcal{A}))$ where $\pi_{\Sigma,\Gamma} \in \mathbf{Alph}[\Sigma \times \Gamma, \Sigma]$ is the first projection.*

▶ **Proposition 7.6.** *Consider complete regular automata $\Sigma \vdash \mathcal{A}$ and $\Sigma \vdash \mathcal{B}$.*
If $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$ then $\Sigma \Vdash \mathrm{ND}(\mathcal{A}) \multimap\!\!\circledast \widehat{\sim\mathcal{C}}$ with $\mathcal{C} := \mathrm{ND}(\sim\mathcal{B})$.

8 Conclusion

We presented monoidal fibrations of tree automata and acceptance games, in which the fibre categories are based on a synchronous restriction of linear simple games.

For technical simplicity, we did not yet consider monoidal closure, but strongly expect that it holds. One of the main question is whether suitable restrictions of these categories are Cartesian closed, so as to interpret proofs from intuitionistic variants of MSO. Among other questions are the status of non-determinization (*i.e.* whether it can be made functorial, or even co-monadic), as well as relation with the Dialectica interpretation (in the vein of e.g. [10]). Our result of weak completeness (Prop. 7.6) suggests strong connections with the notion of *guidable* non-deterministic automata of [4]. On a similar vein, connections with *game automata* [5, 7] might be relevant to investigate.

Acknowledgments. This work benefited from numerous discussions with Pierre Clairambault and Thomas Colcombet.

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A Simple Graph Games

We work on *simple graph games with winning*, of the form $G = (V, E, *, \lambda, \xi, \mathcal{W})$. They are played by *Opponent* (O) and *Proponent* (P) on the graph with vertices in V, edges in E, root $*$, edge labeling $\lambda : E \to \{\mathsf{O}, \mathsf{P}\}$, polarity $\xi : \{*\} \to \{\mathsf{O}, \mathsf{P}\}$ and winning condition $\mathcal{W} \subseteq V^\omega$. Vertices are game *positions*, while edges are *moves*: Opponent plays O-labeled moves and Proponent plays P-labeled moves. We write $v \to w$ if $(v, w) \in E$.

We assume that games are *alternating*, in the sense that $u \to v \to w$ implies $\lambda(u \to v) \neq \lambda(v \to w)$, and *polarized* in the sense that $\lambda(u \to v) = \lambda(u \to w)$ for all coinitial edges $u \to v$, $u \to w$, and moreover $\lambda(* \to u) = \xi(*)$ for all $* \to u$. A game is *positive* if $\xi(*) = \mathsf{P}$ and *negative* otherwise. A *play* is a finite path starting from the root $*$. It is a P-*play* (resp. an O-*play*) if it is either empty or ends with a P-move (resp. an O-move). A P-*strategy* is a non-empty set σ of P-plays which is

P-prefix-closed: if $s \to^* v \in \sigma$ and s is a P-play then $s \in \sigma$, and
P-deterministic: if $s \to w \in \sigma$ and $s \to w' \in \sigma$ then $w = w'$.

Consider a P-strategy σ and an O-play s. We say that s is an O-*interrogation* of σ if either $s = *$ and G is a positive game, or if $s = t \to u$ for some P-play $t \in \sigma$. We say that σ is *total* if for every O-interrogation s of σ, we have $s \to v \in \sigma$ for some v. A P-strategy σ is *winning* if it is total and moreover, for all infinite path $\pi \in V^\omega$, we have $\pi \in \mathcal{W}$ whenever $\pi(0) \to \ldots \to \pi(n) \in \sigma$ for infinitely many $n \in \mathbb{N}$.

2 Graph Games

We briefely discuss here graph games inspired from Melliès' presentation of Conway games (see e.g. [15, 16]). The two main differences are that we do not assume here that graphs are well-founded, nor that strategies are winning.

2.1 Basic Graph Games

Basic Graph Games are played by *Opponent* (O) and *Proponent* (P) on edge-labeled rooted graphs of the form $A = (V, E, *, \lambda)$, where V is the set of vertices, $E \subseteq V \times V$ is the edge relation, $* \in V$ is the root, and $\lambda : E \to \{\mathsf{O}, \mathsf{P}\}$ is the labelling function. Vertices are game *positions*, while edges are *moves*: Opponent plays O-labeled moves and Proponent plays P-labeled moves. We write $v \to w$ if $(v, w) \in E$.

A *path* from position v to position w is a non-empty sequence s of the form

$$s : v_0 \to v_1 \to \ldots \to v_n \qquad \text{where } v_0 = v \text{ and } v_n = w$$

In particular, for each position v, there is a unique empty path $\varepsilon : v$, that we also write v. We write $s : v \to^* w$ when s is a path from v to w or $v \to^* w$ when s is understood from the context.

Note that a basic graph game can have no edge, but must have at least one vertex, namely its root. Let $\mathbf{1}_{\mathbf{SGG}} := (\{*\}, \emptyset, *, \emptyset)$.

Dualization. The *dual* of the basic graph game $A = (V, E, *, \lambda)$ is

$$A^\perp := (V, E, *, \lambda^\perp) \qquad \text{where} \qquad \lambda^\perp(v \to w) = \mathsf{P} \quad \text{iff} \quad \lambda(v \to w) = \mathsf{O}$$

Plays. A finite *play* in $A = (V, E, *, \lambda)$ is a finite path $v_0 \to v_1 \to \ldots \to v_n$ which starts from the root ($v_0 = *$). A play as above is *alternating* if $\lambda(v_{i-1} \to v_i) = \lambda^\perp(v_i \to v_{i+1})$ whenever $n \geq 2$ and $1 \leq i < n$. Note that the empty path on the initial position $\varepsilon : *$ is a play. We let $\wp(A)$ be the set of plays on A.

We say that a play is a P-*play* (resp. O-*play*) if it is either empty or its last move is a P-move (resp. an O-move). We write $\wp^{\mathsf{P}}(A)$ and $\wp^{\mathsf{O}}(A)$ for the sets of resp. P-plays and O-plays on A.

Strategies. A P-*strategy* is a non-empty set σ of P-plays which is
P-prefix-closed: if $s \to^* v \in \sigma$ and s is a P-play then $s \in \sigma$, and
P-deterministic: if $s \to w \in \sigma$ and $s \to w' \in \sigma$ then $w = w'$.
An O-strategy is defined similarly, by exchanging P and O. Formally, an O-strategy on A is a P-strategy on $A^\perp$.

Tensor Product. The product of $A = (V_A, E_A, *_A, \lambda_A)$ and $B = (V_B, E_B, *_B, \lambda_B)$ is

$$A \times B \quad := \quad (V_A \times V_B, E_{A\times B}, (*_A, *_B), \lambda_{A\times B})$$

where $E_{A\times B}$ and $\lambda_{A\times B}$ are given by

$$\begin{array}{llllll} \text{if} & u \to_A u' & \text{then} & (u, v) \to_{A\times B} (u', v) & \text{with polarity} & \lambda_A(u \to u') \\ \text{if} & v \to_B v' & \text{then} & (u, v) \to_{A\times B} (u, v') & \text{with polarity} & \lambda_B(v \to v') \end{array}$$

Note that $(\lambda_{A\times B})^\perp = \lambda_{A^\perp \times B^\perp}$, so that dualization commutes with the tensor product:

$$(A \times B)^\perp = A^\perp \times B^\perp$$

Arrow Type. Given basic graph games A and B, we let $A \multimap B := (A \times B^\perp)^\perp = A^\perp \times B$.

2.2 Graph Games (with Legal Plays)

We discuss here a simple notion of *graph games*, which are basic graph games equipped with *legal plays*. This notion will be most useful to analyse composition in simple graph games (see Sect. 4), but it is conventient to introduce it here.

Games with Legal Plays. Formally, a *graph game with legal plays* (or *graph game* in short) has the form $A = (V, E, *, \lambda, L)$ where $(V, E, *, \lambda)$ is a basic graph game and $L \subseteq \wp(A)$ is a prefix-closed set *legal plays*. We write L^{P} and L^{O} for resp. the set of legal P-plays and O-plays of A.

Strategies σ on A are required to play only legal plays (*i.e.* $\sigma \subseteq L^{\mathsf{P}}$ for a P-strategy σ).

Projections. Given graph games $A_1, \dots, A_n$ and a play $t \in \wp(A_1 \times \dots \times A_n)$, the projection $t{\restriction}A_{i_1}, \dots, A_{i_k}$ is inductively defined as usual:

$$(*_{A_1}, \dots, *_{A_n}){\restriction}A_{i_1}, \dots, A_{i_k} \quad := \quad (*_{A_{i_1}}, \dots, *_{A_{i_k}})$$

and if $j \in \{i_1, \dots, i_k\}$ then

$$\begin{aligned}(t \to^* (u_1, \dots, u_j, \dots, u_n) \to (u_1, \dots, v_j, \dots, u_n)){\restriction}A_{i_1}, \dots, A_{i_k} \quad &:= \\ (t \to^* (u_1, \dots, u_n)){\restriction}A_{i_1}, \dots, A_{i_k} &\to (u_{i_1}, \dots, v_j, \dots, u_{i_k})\end{aligned}$$

and otherwise

$$\begin{aligned}(t \to^* (u_1, \dots, u_j, \dots, u_n) \to (u_1, \dots, v_j, \dots, u_n)){\restriction}A_{i_1}, \dots, A_{i_k} \quad &:= \\ (t \to^* (u_1, \dots, u_n)){\restriction}A_{i_1}, \dots, A_{i_k}&\end{aligned}$$

▶ **Lemma 2.1.** *If* $t \in \wp(A_1 \times \dots \times A_n)$ *then* $t{\restriction}A_{i_1}, \dots, A_{i_k} \in \wp(A_{i_1} \times \dots \times A_{i_k})$.

The Hyland-Schalk Map. Given graph games A and B, and following [11] (see also [2]), we let

$$\mathrm{HS} \quad := \quad \langle (-){\restriction}A, (-){\restriction}B \rangle \;:\; \wp(A \times B) \to \wp(A) \times \wp(B)$$

Polarized Plays and Strategies. The *polarity* of a proper play is that of its first move, namely: a proper play is *positive* (resp. *negative*) if it begins with a P-move (resp. an O-move).

Similarly, a strategy is *positive* (resp. *negative*) if all its proper plays are *positive* (resp. *negative*).

Arrow Type. Let $A = (V_A, E_A, *_A, \lambda_A, L_A)$ and $B = (V_B, E_B, *_B, \lambda_B, L_B)$ be graph games. We let:

$$A \multimap B \quad := \quad (V_{A\multimap B}, E_{A\multimap B}, *_{A\multimap B}, \lambda_{A\multimap B}, L_{A\multimap B})$$

where $L_{A\multimap B}$ is the set of *alternating* and *negative* plays $s \in \wp(A \multimap B)$ such that $\mathrm{HS}(s) \in L_A \times L_B$.

Hence, a legal strategy σ on $A \multimap B$ is alternating and negative: its plays begin with an O-move, end with a P-move and in between alternate polarities.

▶ **Lemma 2.2.** *$L_{A\multimap B}$ is closed under prefix.*

▶ **Definition 2.3.** Given graph games A and B, write $\sigma : A \to B$ if σ is a legal (hence negative) P-strategy on $A \multimap B$.

2.3 Conway-Like Games

We briefly review here Melliès' presentation of Conway-like games [15, 16].

A Conway-like game is a basic graph game, but a morphism of Conway-like games $\sigma : A \to B$, with $A = (V_A, E_A, *_A, \lambda_A)$ and $B = (V_B, E_B, *_B, \lambda_B)$ is a morphism of graph games $\sigma : A^0 \to B^0$, where $A^0 = (V_A, E_A, *_A, \lambda_A, \wp(A))$ and $B^0 = (V_B, E_B, *_B, \lambda_B, \wp(B))$.

A Conway game is a Conway-like game which is *well-founded*, in the sense that there is no infinite path starting from the root. A morphism of Conway games $\sigma : A \to B$ is a morphism of Conway-like games, which is moreover *total*, in the sense that:

- given $t = * \to^* u \in \sigma$, for every O-move $u \to v$ such that $t \to v$ is legal, there is a P-move $v \to w$ such that $t \to v \to w \in \sigma$.

We discuss in Sect. 5 (Prop. 5.3) the fact that totality is preserved by composition for Conway games.

3 Composition of Strategies on Graph Games (with Legal Plays)

We gather here some usual and useful results on the composition of strategies on graph games.

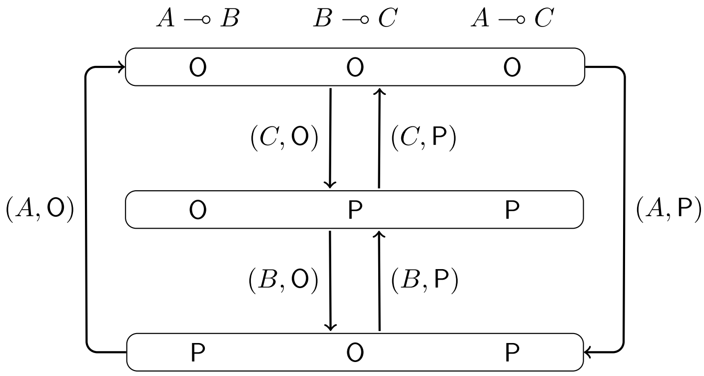
Consider graph games A, B and C and strategies $\sigma : A \to B$ and $\tau : B \to C$. Following the usual pattern we let:

$$\begin{array}{lcl} \sigma \parallel \tau & := & \{t \in \wp(A \times B \times C) \mid t{\restriction}A, B \in \sigma \ \wedge\ t{\restriction}B, C \in \tau\} \\ \tau \circ \sigma & := & \{s \in L^{\mathsf{P}}_{A\multimap C} \mid \exists t \in \sigma \parallel \tau.\ t{\restriction}A, C = s\} \end{array}$$

We now recall the usual way to show that $\tau \circ \sigma$ is a strategy on $A \multimap C$.

▶ **Lemma 3.1.** *Let $t \in \wp(A \times B \times C)$ be such that $t{\restriction}A, B \in L_{A\multimap B}$, $t{\restriction}B, C \in L_{B\multimap C}$ and $t{\restriction}A, C \in L_{A\multimap C}$.*

Then the word obtained from t by replacing each move by the name of its component (A, B, or C), together with its polarity in that component (beware that we take A and not $A^\perp$ etc) is accepted by the following automaton (with initial state (OOO) *and all states accepting), where the states correspond to the player allowed to play next in the corresponding components:*



▶ **Lemma 3.2** (Zipping). *Let $s, t \in \sigma \parallel \tau$ such that $s\restriction A, C$ and $t\restriction A, C$ have the same* O-*moves. Then $s = t$.*

Proof. Assume that $s \neq t$ and let $p = *_{A\times B\times C} \to^* u$ be their maximal comon prefix. Hence there are positions $v \neq w$ in $A \times B \times C$ such that $p \to v$ (resp. $p \to w$) is a prefix of s (resp. of t). We reason by cases on the last state on p of the diagram of Lem. 3.1.

(OOO) In this case, both $u \to v$ and $u \to w$ are O-moves in $A \multimap C$ and we are done.

(OPP) In this case, $u \to v$ and $u \to w$ are both P-moves in $B \multimap C$, so that $(p \to v)\restriction B, C$ and $(p \to u)\restriction B, C$ are plays of τ of the same length, hence $u = v$, a contradiction.

(POP) Similarly as in the case of (OPP), this case leads to a contradiction since $u \to v$ and $u \to w$ are both P-moves in $A \multimap B$. ◀

▶ **Proposition 3.3.** *Given graph games A, B and C and strategies $\sigma : A \to B$ and $\tau : B \to C$, the composite $\tau \circ \sigma$ is a strategy on $A \multimap B$.*

Proof. We have $\tau \circ \sigma \subseteq L^{\mathsf{P}}_{A\multimap C}$ by definition, and $\tau \circ \sigma$ contains the empty play since both σ and τ contain the empty play.

We now show that $\tau \circ \sigma$ is P-prefix-closed. Let $s \to u \to v \in \tau \circ \sigma$. If s is the empty play $\varepsilon : *_{A\multimap B}$ then we are done. Otherwise, let $t \in \sigma \parallel \tau$ such that

$$t\restriction A, C \quad = \quad s \to u \to v$$

Write $t := t' \to u \to^* v$, so that $t'\restriction A, C = s$. Now, since $s \in L^{\mathsf{P}}_{A\multimap C}$, by definition s ends with a P-move and is alternating, hence the last state of the diagram of Lem. 3.1 on t' is (OOO). By alternation in $L_{A\multimap B}$ and $L_{B\multimap C}$, it follows that $t'\restriction A, B \in L^{\mathsf{P}}_{A\multimap B}$ and $t'\restriction B, C \in L^{\mathsf{P}}_{B\multimap C}$. By P-prefix closure of strategies, it follows that $t'\restriction A, B$ and $t'\restriction B, C$ are plays of σ and τ respectively. Hence $t' \in \sigma \parallel \tau$ and $s \in \tau \circ \sigma$.

It remains to show that $\tau \circ \sigma$ is P-deterministic. Assume that $s \to u$ and $s \to v$ are two plays of $\tau \circ \sigma$. Then they have the same O-moves, and it follows from the Zipping Lemma 3.2 that $u = v$. ◀ ◀

4 Simple Graph Games

Simple graph games are graph games with legal plays, which are required to satisfy alternation and polarity conditions.

4.1 Simple Graph Games

4.1.1 Simple Graph Games.

A *simple graph game* is a graph game with legal plays $A = (V, E, *, \lambda, L)$ where L is subject to the following two additional requirements:
Alternance: all plays in L are alternating, and
Polarization: all plays in L have the same polarity.
The polarity of A is the polarity of the plays of L.

▶ **Example 4.1** (Linear Arrow of Graph Games with Legal Plays)**.** If A and B are graph games with legal plays, then the linear arrow game $A \multimap B$ in the sense of Sect. 3 is a simple graph game.

▶ **Example 4.2** (Substituted Acceptance Games)**.**

▶ **Definition 4.3** (Total Strategies)**.** Let σ be a P-strategy on a game A, and consider an O-play s. We say that s is an O-*interrogation* of σ if either $s = *_A$ and A is a polarized *positive* game, or if $s = t \to u$ for some P-play $t \in \sigma$.

We say that σ is total if for every O-interrogation s of σ, we have $s \overset{\mathsf{P}}{\to} v \in \sigma$ for some v.

4.1.2 The Category SGG of Simple Graph Games.

In the category **SGG** of simple graph games, we only consider morphisms between games of the same polarity.
Objects of **SGG** are simple graph games.
Morphisms in $\mathbf{SGG}[A, B]$, with A and B of the same polarity are negative legal P-strategies $\sigma : A \to B$.

4.1.3 The Hyland-Schalk Functor.

Given a strategy $\sigma : A \to B$, let

$$\mathrm{HS}(\sigma) \quad := \quad \{\mathrm{HS}(s) \mid s \in \sigma\}$$

Recall that by definition, HS restricts to a map from $L_{A \multimap B}$ to $L_A \times L_B$, and that $\sigma \subseteq L_{A \multimap B}$ by assumption. Hence $\mathrm{HS}(\sigma) \subseteq L_A \times L_B$.

▶ **Proposition 4.4** ([11])**.** HS *is a faithfull functor from* **SGG** *to* **Rel**, *the category of sets and relations.*

It follows that strategies are faithfully represented by the corresponding spans in **Set**:

$$\begin{tikzcd} & \sigma \arrow[dl] \arrow[dr] & \\ L_A & & L_B \end{tikzcd} \tag{6}$$

where the arrows $\sigma \longrightarrow L_A$ and $\sigma \longrightarrow L_B$ are given resp. by

$$\sigma \hookrightarrow L_{A \multimap B} \xrightarrow{(-)\restriction A} L_A \qquad \text{and} \qquad \sigma \hookrightarrow L_{A \multimap B} \xrightarrow{(-)\restriction B} L_B$$

We will detail the argument of Prop. 4.4, and show that composition in **SGG** is faithfully represented in **Set** by pullbacks of spans of the form (6).

4.2 Relational Decomposition of Strategies

We now discuss how the polarization and alternation assumptions in **SGG** imply that HS is faithful.

We first recall the following well-known basic fact about **SGG**.

▶ **Lemma 4.5** (Switching). *Let A and B be simple graph games of the same polarity.*

(i) *Consider a legal play $s = *_{A \multimap B} \to^* u \to v \to w \in L_{A \multimap B}$. If the moves $u \to_{A \multimap B} v$ and $v \to_{A \multimap B} w$ are not both in the same component, then $\lambda_{A \multimap B}(v \to w) = \mathsf{P}$.*

(ii) *Consider a legal play $s = *_{A \multimap B} \to^* (u, v) \in L_{A \multimap B}$. If there are moves $u \to_{A^\perp} u'$ and $v \to_B v'$ such that both plays $s \to (u', v)$ and $s \to (u, v')$ are legal in $A \multimap B$, then $\lambda_{A \multimap B}((u, v) \to_{A \multimap B} (u', v)) = \lambda_{A \multimap B}((u, v) \to_{A \multimap B} (u, v')) = \mathsf{P}$.*

(iii) *Consider two legal plays $s, t \in L_{A \multimap B}$:*

$$
\begin{array}{rcl}
s & = & *_{A \multimap B} \to u_1 \to \ldots \to u_n \\
t & = & *_{A \multimap B} \to v_1 \to \ldots \to v_n
\end{array}
$$

Assume that $s \neq t$ but $\mathrm{HS}(s) = \mathrm{HS}(t)$. Then for the least $i < n$ such that $u_{i+1} \neq v_{i+1}$, we have $\lambda_{A \multimap B}(u_i \to u_{i+1}) = \lambda_{A \multimap B}(v_i \to v_{i+1}) = \mathsf{P}$.

Proof. (i) Since s is alternating, the moves $u \to_{A \multimap B} v$ and $v \to_{A \multimap B} w$ have opposite polarity. Since moreover they are not in the same component, it follows that the projections $s{\restriction}A^\perp$ and $s{\restriction}B$ end with moves of opposite polarity. Hence $s{\restriction}A$ and $s{\restriction}B$ end with moves of the same polarity.
Since A and B have the same polarity, and since $s{\restriction}A$ and $s{\restriction}B$ are alternating, we get that the lengths of $s{\restriction}A$ and of $s{\restriction}B$ have the same parity. It follows that the length of s is even, and since s is a negative alternating play, it ends with a P-move.

(ii) Since the plays $s \to (u', v)$ and $s \to (u, v')$ are both alternating and negative, the moves $(u, v) \to_{A \multimap B} (u', v)$ and $(u, v) \to_{A \multimap B} (u, v')$ have the same polarity. It follows that $\lambda_A(u \to_A u') = \lambda_B(v \to_B v')^\perp$, and moreover that $s{\restriction}A$ and $s{\restriction}B$ end with moves of opposite polarity since $s{\restriction}A \to_A u'$ and $s{\restriction}B \to_B v'$ are both alternating.
Since A and B have the same polarity, and again since $s{\restriction}A$ and $s{\restriction}B$ are alternating, we get that the lengths of $s{\restriction}A$ and of $s{\restriction}B$ have opposite parity. It follows that the length of s is odd, and since s is a negative alternating play, it ends with an O-move. We conclude by alternation of $L_{A \multimap B}$.

(iii) Note that since s and t are both alternating and negative, the moves $u_i \to_{A \multimap B} u_{i+1}$ and $v_i \to_{A \multimap B} v_{i+1}$ have the same polarity. Since

$$
*_{A \multimap B} \to u_1 \to \ldots \to u_i \quad = \quad *_{A \multimap B} \to v_1 \to \ldots \to v_i
$$

we have

$$
\mathrm{HS}(*_{A \multimap B} \to u_1 \to \ldots \to u_i) \quad = \quad \mathrm{HS}(*_{A \multimap B} \to v_1 \to \ldots \to v_i)
$$

Since moreover $\mathrm{HS}(s) = \mathrm{HS}(t)$ by assumption, it follows that $u_i \to u_{i+1}$ and $v_i \to v_{i+1}$ can not be both in the same component, and the result follows from Lem. 4.5.(i). ◀

▶ Remark (Definition of $L_{A \multimap B}$ for graph games in Sect. 3). Note that a play $s \in \wp(A \multimap B)$ such that $s{\restriction}A \in L_A$ and $s{\restriction}B \in L_B$ needs not be negative nor alternating. Consider for instance the following plays where A and B are both negative:

$$
\begin{array}{c|ccc}
 & A & \longrightarrow & B \\
\hline
 & (*_A & , & *_B) \\
\mathsf{P} & \downarrow & & \\
 & (a_0 & , & *_B) \\
\mathsf{O} & \downarrow & & \\
 & (a_1 & , & *_B)
\end{array}
\qquad
\begin{array}{c|ccc}
 & A & \longrightarrow & B \\
\hline
 & (*_A & , & *_B) \\
\mathsf{O} & & & \downarrow \\
 & (*_A & , & b_0) \\
\mathsf{P} & \downarrow & & \\
 & (a_0 & , & b_0) \\
\mathsf{P} & & & \downarrow \\
 & (a_0 & , & b_1)
\end{array}
$$

▶ **Lemma 4.6.** *Let A and B be simple graph games of the same polarity.*

(i) Given $\sigma : A \to \mathrm{B}$, in **Set** *we have $\mathrm{HS}(\sigma) \simeq \sigma$.*
(ii) HS is injective on strategies: given $\sigma, \tau : A \to B$, if $\mathrm{HS}(\sigma) = \mathrm{HS}(\tau)$ then $\sigma = \tau$.

Proof. (i) By definition, we have in **Set** a surjective map

$$
\begin{array}{ccc}
\sigma & \longrightarrow & \mathrm{HS}(\sigma) \\
s & \mapsto & \mathrm{HS}(s) = (s{\upharpoonright}A, s{\upharpoonright}B)
\end{array}
$$

The injectivity of this map follows from the following property: Given legal plays $t \in L_A$ and $t' \in L_B$ there is at most one play $s \in \sigma$ such that $\mathrm{HS}(s) = (t, t')$. This property is a direct consequence of Lem. 4.5.(iii).

(ii) Let σ and τ be strategies on $A \to B$ such that $\mathrm{HS}(\sigma) = \mathrm{HS}(\tau)$. We show that $\sigma \subseteq \tau$ by induction on plays $s \in \sigma$. First, both σ and τ contain the empty play $\varepsilon : *_{A \multimap B}$. For the induction step, consider $s \in \sigma$ of the form

$$u_0 \to \ldots \to u_n \to u_{n+1} \to u_{n+2}$$

with $u_0 = *_{A \multimap B}$ and such that $u_0 \to \ldots \to u_n \in \sigma \cap \tau$.
By assumption, $\mathrm{HS}(s) = \mathrm{HS}(t)$ for some $t \in \tau$. Note that s and t have the same length. Hence t is of the form

$$v_0 \to \ldots \to v_n \to v_{n+1} \to v_{n+2} \qquad \text{with } v_0 = *_{A \multimap B}$$

If $s \neq t$, then by Lem. 4.5.(iii) they first differ at a P move, say $u_{i+1} \neq v_{i+1}$. Since $u_0 \to^* u_n$ and $v_0 \to^* v_n$ both belong to τ, we can not have $i+1 \leq n$, hence $i+1 = n+2$ since s and t are both alternating and negative.
But then

$$u_0 \to \ldots \to u_n \quad = \quad v_0 \to \ldots \to v_n$$

hence

$$\mathrm{HS}(u_0 \to \ldots \to u_n) \quad = \quad \mathrm{HS}(v_0 \to \ldots \to v_n)$$

and since $\mathrm{HS}(s) = \mathrm{HS}(t)$, it follows that $u_{n+2} = v_{n+2}$, contradicting $s \neq t$. ◀

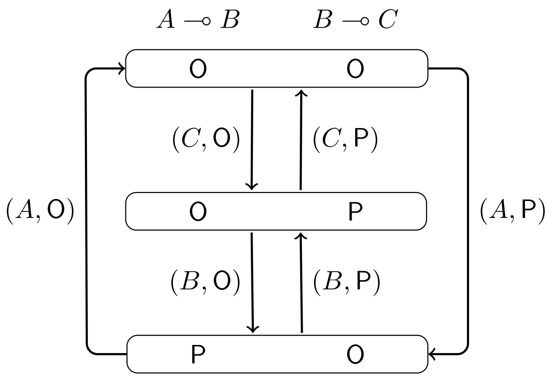
4.3 Relational Composition of Strategies

Let A, B and C be simple games of the same polarity. Given $\sigma : A \to B$ and $\tau : B \to C$, consider the following composite, in the category **Rel** of sets and relations:

$$\mathrm{HS}(\tau) \circ \mathrm{HS}(\sigma) \quad = \quad \{(s, s') \in L_A \times L_C \mid \exists t \in L_B.\ (s,t) \in \mathrm{HS}(\sigma) \ \wedge\ (t, s') \in \mathrm{HS}(\tau)\}$$

▶ **Lemma 4.7.** *Assume that A, B and C have the same polarity. Given $s \in L_{A \multimap B}$ and $s' \in L_{B \multimap C}$ with $s{\upharpoonright}B = s'{\upharpoonright}B$, there is $t \in \wp(A \times B \times C)$ such that $t{\upharpoonright}A, B = s$ and $t{\upharpoonright}B, C = s'$.*

Moreover, t is accepted by the state diagram:



Proof. We build t by induction on the sum of the lengths of s and s'.

In the base case, s and s' are both empty, and we let t be the empty play from $*_{A \times B \times C}$. By definition of $L_{A \multimap B}$ and $L_{B \multimap C}$, the diagram is in state (OO).

For the induction step, there are three cases according to wether either s or s' are empty or s and s' are both non-empty.

If s is non-empty and s' is empty with say $s = *_{A \multimap B} \to^* u \to v$. Then by induction hypothesis there is $t \in \wp(A \times B \times C)$ with $t{\upharpoonright}A, B = *_{A \multimap B} \to^* u$ and $t{\upharpoonright}B, C = s' = \varepsilon$. Since s' is empty, the move $u \to v$ must be in component A. Writing $t = *_{A \times B \times C} \to^* u'$, we extend it to t' with the corresponding move $u' \to v'$ in $A \times B \times C$.
Moreover, the diagram must be either in state (OO) or (PO) on t according to the polarity of the length of t, and it goes either to state (PO) or (OO) on t'.

If s is empty and s' is non-empty then we reason similarly, in component C instead of A.

Otherwise, both s and s' are non-empty. We first claim that at least one move of s and s' is in component B.

- **Proof.** Assume that no move of s (and thus of s') occurs in component B. Recall that s and s' are both non-empty. But their first moves are in component A and C respectively, contradicting that $s \in L_{A \multimap B}$ and $s' \in L_{B \multimap C}$ since A, B and C are of the same polarity. ◀

Let $b_0 \to b_1$ (resp. $b'_0 \to b'_1$) be the last move of s (resp. s') in component B. Note that by assumption, these two moves project to the same B-move. Hence s and s' are of the form:

$$
\begin{aligned}
s &: *_{A \multimap B} \to^* b_0 \to b_1 \to^\epsilon s_1 \\
s' &: *_{B \multimap C} \to^* b'_0 \to b'_1 \to^\epsilon s'_1
\end{aligned}
$$

Consider first the case where $b_1 \to^\epsilon s_1$ and $b'_1 \to^\epsilon s'_1$ are both empty:

$$
\begin{aligned}
s &: *_{A \multimap B} \to^* b_0 \to b_1 \\
s' &: *_{B \multimap C} \to^* b'_0 \to b'_1
\end{aligned}
$$

Now, we have $(*_{A \multimap B} \to^* b_0){\upharpoonright}B = (*_{B \multimap C} \to^* b'_0){\upharpoonright}B$ and by induction hypothesis, we get $t_0 \in \wp(A \times B \times C)$ such that $t_0{\upharpoonright}A, B = *_{A \multimap B} \to^* b_0$ and $t_0{\upharpoonright}B, C = *_{B \multimap C} \to^* b'_0$. Moreover t_0 is accepted by the state diagram. It can not be in state (OO) since the B-move corresponding to $b_0 \to b_1$ and $b'_0 \to b'_1$ has opposite polarities in $A \multimap B$ and

$B \multimap C$. Hence t_0 must either be in state (OP) or in state (PO), according to the polarity of $b_0 \to b_1$ (which is the opposite to that of $b'_0 \to b'_1$). We extend t_0 to t_1 with the B-move corresponding to $b_0 \to b_1$ and $b'_0 \to b'_1$. The state of the diagram is now either (PO) or (OP).

Consider now the case of $b_1 \to^{\epsilon} s_1$ and $b'_1 \to^{\epsilon} s'_1$ not both empty. We obtain t_1 as above. We claim that either $b_1 \to^{\epsilon} s_1$ or $b'_1 \to^{\epsilon} s'_1$ is empty.

- **Proof.** Assume that both are non-empty. Then they must respectively begin with an A-move and a C-move. But either $b_0 \to b_1$ or $b'_0 \to b'_1$ is a P-move, contradicting Switching (Lem. 4.5.(i)). ◀

We can thus extend t_1 by reasonning as in the case of s or s' empty above. ◀

▶ **Lemma 4.8.** *Let $t \in \wp(A \times B \times C)$ be such that $t{\restriction}A, B \in L_{A \multimap B}$ and $t{\restriction}B, C \in L_{B \multimap C}$. Then $t{\restriction}A, C$ is negative.*

Proof. We consider two cases, according to the polarity of A, B and C.

A, B and C are negative. In this case, the first move in A of t comes after its first move in B, which itself comes after the first move in C of t. It follows that $t{\restriction}A, C$ begins by an initial move in C, hence a negative move in $A \multimap B$.

A, B and C are positive. Similarly, the first move in C of t comes after its first move in B, which itself comes after the first move in A of t. It follows that $t{\restriction}A, C$ begins by an initial move in A, hence a negative move in $A \multimap C$. ◀

▶ **Corollary 4.9.** *Given plays $s_A \in L_A$, $s_B \in L_B$ and $s_C \in L_C$ such that $(s_A, s_B) \in \mathrm{HS}(\sigma)$ and $(s_B, s_C) \in \mathrm{HS}(\tau)$, there exists $t \in \sigma \parallel \tau$ such that $\mathrm{HS}(t{\restriction}A, B) = (s_A, s_B)$ and $\mathrm{HS}(t{\restriction}B, C) = (s_B, s_C)$, and moreover $t{\restriction}A, C \in L_{A \multimap C}$.*

Proof. Take $t \in \wp(A \times B \times C)$ obtained by Lem. 4.7 from $s \in \sigma$ and $s' \in \tau$ such that $\mathrm{HS}(s) = (s_A, s_B)$ and $\mathrm{HS}(s') = (s_B, s_C)$.

In order to show that $t{\restriction}A, C \in L^{\mathsf{P}}_{A \multimap C}$, we first get that $t{\restriction}A, C$ is negative by Lem. 4.8. Then, since the transition of the diagram in Lem. 4.7 respect the polarity in the $A \multimap C$ part, we obtain that t is accepted by the state diagram of Lem. 3.1. It then follows that $t{\restriction}A, C$ is alternating, since in the diagram of Lem. 3.1, all transitions in A and C preserve alternation in $A \multimap C$, and transitions in B preserve polarity in $A \multimap C$.

It remains to show that $t{\restriction}A, C$ ends by a P move. Note that both $t{\restriction}A, B$ and $t{\restriction}B, C$ have even length. Hence the lengths of $t{\restriction}A$, $t{\restriction}B$ and $t{\restriction}C$ have the same parity, and $t{\restriction}A, C$ has even length. It follows that it ends by a P-move since it is negative and alternating. ◀

▶ **Proposition 4.10** (Relational Composition of Strategies). *Let A, B and C be simple games of the same polarity. Given $\sigma : A \to B$ and $\tau : B \to C$, we have $\mathrm{HS}(\tau \circ \sigma) = \mathrm{HS}(\tau) \circ \mathrm{HS}(\sigma)$.*

Proof. The inclusion $\mathrm{HS}(\tau \circ \sigma) \subseteq \mathrm{HS}(\tau) \circ \mathrm{HS}(\sigma)$ is trivial.

For the other direction, let $(s_A, s_C) \in \mathrm{HS}(\tau) \circ \mathrm{HS}(\sigma)$, so that there is $s_B \in L_B$ such that $(s_A, s_B) \in \mathrm{HS}(\sigma)$ and $(s_B, s_C) \in \mathrm{HS}(\tau)$. By Cor. 4.9, there is $t \in \wp(A \times B \times C)$ such that $t{\restriction}A, B \in \sigma$, $t{\restriction}B, C \in \tau$, and $t{\restriction}A = s_A$, $t{\restriction}B = s_B$ and $t{\restriction}C = s_C$. We moreover get $t{\restriction}A, C \in \tau \circ \sigma$ since $t{\restriction}A, C \in L^{\mathsf{P}}_{A \multimap C}$. ◀

▶ **Proposition 4.11.** *Given a simple game A, there is a unique strategy id such that $\mathrm{HS}(\mathrm{id}) = L_A \times_{L_A} L_A$, where, in* **Set***, the following is a pullback:*

$$\begin{tikzcd}
\mathrm{id} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & L_A \arrow[d, "1"] \\
L_A \arrow[r, "1"'] & L_A
\end{tikzcd}$$

Proof. Consider the following pullback diagram in **Set**, where $L_A \times_{L_A} L_A \simeq L_A$:

$$\begin{tikzcd}
L_A \times_{L_A} L_A \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & L_A \arrow[d, "1"] \\
L_A \arrow[r, "1"'] & L_A
\end{tikzcd}$$

We define $\mathrm{id} \subseteq L^{\mathsf{P}}_{A \multimap A}$ such that $\mathrm{HS}(\mathrm{id}) = L_A \times_{L_A} L_A \simeq L_A$ by induction on $s \in L_A$ as follows:

- For $s = *_A$, we let $*_{A \multimap A} \in \mathrm{id}$, and we indeed have $\mathrm{HS}(*_{A \multimap A}) = (*_A, *_A)$ and $*_{A \multimap A} \in L^{\mathsf{P}}_{A \multimap A}$.
- Let now $s = s' \to a \in L_A$. By induction hypothesis, there is $t \in \mathrm{id}$ such that $\mathrm{HS}(t) = (s', s')$. Write $A \multimap A = A^{(1)} \multimap A^{(2)}$.
  If $s' \to a$ is an O-move in A, then we extend t with a in component $A^{(2)}$ and then by a in component $A^{(1)}$ (hence a P-move in $A^{(1)} \multimap A^{(2)}$). Otherwise, $s' \to a$ is a P-move in A, and we first extend t with a in component $A^{(1)}$ (hence an O-move in $A^{(1)} \multimap A^{(2)}$) and then with a in component $A^{(2)}$.
  Let t' be the obtained extension of t. In both cases, we have $\mathrm{HS}(t') = (s' \to a, s' \to a)$ and t' ends by a P-move by construction. Alternation is preserved in $A \multimap A$, hence by induction hypothesis t' is alternating. Moreover, t' is negative since either $t = *_{A \multimap A}$ and t' begins with an O-move, or $t \neq *_{A \multimap A}$ is negative by induction hypothesis. We thus get $t' \in L^{\mathsf{P}}_{A \multimap A}$ since $s' \to a \in L_A$ by assumption.

We now check that id is a strategy. First, id is P-prefix-closed by construction. Moreover, id is P-deterministic since its P-moves are uniquely determined by their immediately preceding O-moves.

The fact that the diagram is a pullback as well as the unicity of id follows from the fact that in **Set**, we have $\mathrm{HS}(\mathrm{id}) \simeq \mathrm{id}$ thanks to Lem. 4.6.(i). ◀

▶ Remark. A direct definition of id as follows (where $L_A \times_{L_A} L_A \simeq L_A$)

$$\mathrm{id} \quad := \quad \mathrm{HS}^{-1}(L_A \times_{L_A} L_A) \cap L^{\mathsf{P}}_{A \multimap A}$$

does *not* work since there might be $t \neq t'$ in $L^{\mathsf{P}}_{A \multimap A}$ such that $\mathrm{HS}(t) = \mathrm{HS}(t')$.

▶ Remark (Associativiy of Composition in **SGG**). Proposition 4.10 can be read in two direction. The original one [11], is that given the categories **SGG** and **Rel**, the map $\mathrm{HS} : \mathbf{SGG} \to \mathbf{Rel}$ is functorial.

The other one, is that together with the injectivity of the map HS (Lem 4.6.(ii)), Prop. 4.10, can be used to show that composition in **SGG** is associative, and in particular that **SGG** is a category.

Indeed, from Prop. 4.10 and the associativity of composition in **Rel** we get:

$$\mathrm{HS}(\tau \circ (\sigma \circ \theta)) = \mathrm{HS}(\tau) \circ (\mathrm{HS}(\sigma) \circ \mathrm{HS}(\theta)) = (\mathrm{HS}(\tau) \circ \mathrm{HS}(\sigma)) \circ \mathrm{HS}(\theta) = \mathrm{HS}((\tau \circ \sigma) \circ \theta)$$

and it follows from Lem 4.6.(ii) that

$$\tau \circ (\sigma \circ \theta) \quad = \quad (\tau \circ \sigma) \circ \theta$$

4.4 Composition by Pullbacks

We now show that composition of strategies form pullback squares in **Set** based on the the repsentation of strategies as spans (6):

$$\begin{tikzcd} \tau \circ \sigma \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \tau \arrow[r] \arrow[d] & L_C \\ \sigma \arrow[r] \arrow[d] & L_B & \\ L_A & & \end{tikzcd}$$

where the arrows

$$\tau \circ \sigma \longrightarrow \sigma \qquad \text{and} \qquad \tau \circ \sigma \longrightarrow \tau$$

are obtained thanks to the following unicity property (which follows form the usual Zipping Lemma 3.2):

▶ **Lemma 4.12** (Relational Zipping). *Given $s_A \in L_A$ and $s_C \in L_C$, there is at most one $s_B \in L_B$ such that $(s_A, s_B) \in \mathrm{HS}(\sigma)$ and $(s_B, s_C) \in \mathrm{HS}(\tau)$.*

Proof. Let $s, s' \in L_B$ such that $(s_A, s), (s_A, s') \in \mathrm{HS}(\sigma)$ and $(s, s_C), (s', s_C) \in \mathrm{HS}(\tau)$.

Let $t, t' \in \wp(A \times B \times C)$ the corresponding plays obtained by Cor. 4.9. Then $t{\restriction}A, C$ and $t'{\restriction}A, C$ are two plays of $\tau \circ \sigma$ with the same image under $\mathrm{HS}(-)$. It follows from Lem. 4.6.(i) that $t{\restriction}A, C = t'{\restriction}A, C$ and by Zipping (Lem. 3.2) that $t = t'$. Hence $s = t{\restriction}B = t'{\restriction}B = s'$. ◀

Let now $\tau \circ \sigma \longrightarrow \sigma$ map $s \in \tau \circ \sigma$ to $(s{\restriction}A, s')$ where s' is by Lem 4.12 unique in L_B such that $(s{\restriction}A, s') \in \mathrm{HS}(\sigma)$ and $(s', s{\restriction}C) \in \mathrm{HS}(\tau)$. The map $\tau \circ \sigma \longrightarrow \tau$ is defined similarly.

Note that this immediately implies the commutation of the diagram

$$\begin{tikzcd} \tau \circ \sigma \arrow[r] \arrow[d] & \tau \arrow[d] \\ \sigma \arrow[r] & L_B \end{tikzcd}$$

▶ **Proposition 4.13** (Composition as Pullback). *The following is a pullback in* **Set***:*

$$\begin{tikzcd} \tau \circ \sigma \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \tau \arrow[r] \arrow[d] & L_C \\ \sigma \arrow[r] \arrow[d] & L_B & \\ L_A & & \end{tikzcd}$$

Proof. We only have to show that $\tau \circ \sigma$ is in bijection with

$$\sigma \times_{L_B} \tau \quad = \quad \{(s, t) \in \sigma \times \tau \mid s{\restriction}B = t{\restriction}B\}$$

But in **Set**, we have:

$$\begin{array}{rcll} \sigma \times_{L_B} \tau & \simeq & \{((s_A, s_B), (t_B, t_C)) \in \mathrm{HS}(\sigma) \times \mathrm{HS}(\tau) \mid s_B = t_B\} & \text{(by Lem. 4.6.(i))} \\ & = & \{(s_A, s_B, s_C) \mid (s_A, s_B) \in \mathrm{HS}(\sigma) \ \wedge \ (s_B, s_C) \in \mathrm{HS}(\tau)\} & \\ & \simeq & \{(s_A, s_C) \mid \exists s_B \in L_B.\ (s_A, s_B) \in \mathrm{HS}(\sigma) \ \wedge \ (s_B, s_C) \in \mathrm{HS}(\tau)\} & \text{(by Lem. 4.12)} \\ & = & \mathrm{HS}(\tau) \circ \mathrm{HS}(\sigma) & \\ & = & \mathrm{HS}(\tau \circ \sigma) & \text{(by Prop. 4.10)} \\ & \simeq & \tau \circ \sigma & \text{(by Lem. 4.6.(i))} \end{array}$$

◀

5 Simple Graph Games with Winning

5.1 Graph Games with Winning

We discuss a notion of winning conditions on infinite plays in graph games. The basic mechanism of this notion is very simple and well-known [1, 9], as well as the fact that winning (including totality) is preserved by composition.

5.1.1 Graph Games with Winning

have the form $A = (V, E, *, \lambda, L, \mathcal{W})$ where $(V, E, *, \lambda, L)$ is a graph game and $\mathcal{W} \subseteq V^\omega$ is a *winning condition*.

We let $\mathcal{W}^+$ be the union of $\mathcal{W}$ with the set of finite legal P-plays on A.

▶ Remark (Finite Winning). The finite part of winning conditions (here the finite part of the sets $\mathcal{W}^+$) must be formulated in terms of the *polarities* of plays, and not in terms of their lengths since we want to handle, in the composition of strategies (see Prop. 5.3) the case of *positive* constituent games. This contrasts with [3], which is based on "negative" (*i.e.* O-starting) HO-games.

5.1.2 Winning Strategies.

A P-strategy σ on $A = (V, E, *, \lambda, L, \mathcal{W})$ is *winning* if

it is total in the sense of Def. 4.3, and

all its infinite plays are winning: if $(t_n)_{n\in\mathbb{N}}$ is a sequence of pairwise compatible plays of σ such that $\bigcup_{n\in\mathbb{N}} t_n$ is infinite, then $\bigcup_{n\in\mathbb{N}} t_n \in \mathcal{W}$.

Note that in the second condition above it is equivalent to require $t_n \in \sigma$ for infinitely many $n \in \mathbb{N}$, instead of for all $n \in \mathbb{N}$.

5.1.3 Arrow Type with Winning.

Given games with winning A and B, the game with winning $A \multimap B$ is the graph game $A \multimap B$ equipped with the winning condition $\mathcal{W}_{A\multimap B} \subseteq (V_A \times V_B)^\omega$ defined as follows. Given an infinite sequence $\rho \in (V_A \times V_B)^\omega$, let

$$\rho\restriction A \quad := \quad \bigcup_{n\in\mathbb{N}} (\rho(0) \to \ldots \to \rho(n))\restriction A$$

Let now $\mathcal{W}_{A\multimap B}$ be the set of $\rho \in V^\omega_{A\multimap B}$ such that $\rho\restriction A \in \mathcal{W}^+_A$ implies $\rho\restriction B \in \mathcal{W}^+_B$.

▶ **Lemma 5.1.** *Given simple graph games with winning A and B of the same polarity, and a legal (hence negative) strategy σ on $A \multimap B$, let $(t_n)_{n\in\mathbb{N}}$ be a sequence of pairwise compatible plays of σ, and let $\rho := \bigcup_{n\in\mathbb{N}} t_n$.*

Then $\rho \in \mathcal{W}^+_{A\multimap B}$ iff ($\rho\restriction A \in \mathcal{W}^+_A$ implies $\rho\restriction B \in \mathcal{W}^+_B$).

Proof. If ρ is infinite, then $\rho \in \mathcal{W}^+_{A\multimap B}$ iff $\mathcal{W}_{A\multimap B}$ and the result follows by definition of $\mathcal{W}_{A\multimap B}$. If ρ is finite, then $\rho \in \sigma$, hence $\rho \in \mathcal{W}^+_{A\multimap B}$. Moreover, ρ has even-length since it is a negative alternating P-play, hence the lengths of $\rho\restriction A$ and $\rho\restriction B$ have the same parity, and $\rho\restriction A \in \mathcal{W}^+_A$ implies $\rho\restriction B \in \mathcal{W}^+_B$ since A and B are simple games of the same polarity. ◀

▶ **Definition 5.2.** Given graph games with winning A and B, write $\sigma \Vdash A \to B$ if σ is a winning legal (hence negative) P-strategy on $A \multimap B$.

▶ **Proposition 5.3.** *Assume that A, B and C are graph games with winning which are either all well-founded or all simple and of the same polarity.*

If $\sigma \Vdash A \to B$ and $\tau \Vdash B \to C$ then $\tau \circ \sigma \Vdash A \to C$.

Proof. Assume that $\tau \circ \sigma$ is not total. In this case, there is some $t \in \sigma \parallel \tau$ and a position v in $A \times B \times C$ such that $(t \to v)\restriction A, C$ is a legal O-play in $A \multimap C$ and $t\restriction A, C$ is maximal in $\tau \circ \sigma$. The last move of $t \to v$ is either an O-move in component $A \multimap B$ or an O-move in component $B \multimap C$. In either cases, since σ and τ are total, for some u_0 we have $(t \to v \to u_0)\restriction A, B \in \sigma$ or $(t \to v \to u_0)\restriction B, C \in \tau$. In the first case, $v \to u_0$ is a P-move in component B, hence an O-move in component $B \multimap C$, and by totality of τ there is a move $u_0 \to u_1$ in component B such that $(t \to v \to u_0 \to u_1)\restriction B, C \in \tau$. Similarly, the second case leads to a move $u_0 \to u_1$ in component B such that $(t \to v \to u_0 \to u_1)\restriction A, B \in \sigma$. By induction on $n \in \mathbb{N}$ we thus obtain a sequence of moves $u_n \to u_{n+1}$ in component B. This leads to a contradiction in case B is well-founded. Otherwise, let

$$t_n \quad := \quad (t \to v \to u_0 \to u_1 \to \ldots \to u_n)$$

Note that $t_n\restriction A, B$ and $t_n\restriction B, C$ are plays for all $n \in \mathbb{N}$ and $t_n\restriction A, B \in \sigma$ (resp. $t_n\restriction B, C \in \tau$) for infinitely many $n \in \mathbb{N}$. Consider the infinite sequence of positions $\rho := \bigcup_{n\in\mathbb{N}} t_n$. Note that $\rho\restriction B$ is infinite, so that $\rho\restriction A, B$ and $\rho\restriction B, C$ are also infinite. Since σ and τ are winning and

$$\rho\restriction A, B = \bigcup_{n\in\mathbb{N}} t_n\restriction A, B \qquad \text{and} \qquad \rho\restriction B, C = \bigcup_{n\in\mathbb{N}} t_n\restriction B, C$$

it follows that $\rho\restriction A, B \in \mathcal{W}_{A\multimap B}$ and $\rho\restriction B, C \in \mathcal{W}_{B\multimap C}$. We therefore have

$$\rho\restriction A = (t \to u_0)\restriction A \in \mathcal{W}^+_A \qquad \Longrightarrow \qquad \rho\restriction C = (t \to u_0)\restriction C \in \mathcal{W}^+_C$$

On the other hand, $\rho\restriction A, C = (t \to u_0)\restriction A, C$ is a finite O-play in $A \multimap C$ directly extending $t\restriction A, C$, and its last move is either a P-move in component A or an O-move in component C. Since $t\restriction A, C$ has even length, the projections $t\restriction A$ and $t\restriction C$ have length of the same parity. Hence $t\restriction A \in \mathcal{W}^+_A$ iff $t\restriction C \in \mathcal{W}^+_C$ since A and B are simple games of the same polarity. Now, if the last move of $(t \to u_0)\restriction A, C$ is a P-move in component A, then $(t \to u_0)\restriction A \in \mathcal{W}^+_A$. Since A is a simple game, by **alternation** we get $t\restriction A \notin \mathcal{W}^+_A$, hence $t\restriction C = (t \to u_0)\restriction C \notin \mathcal{W}^+_C$, a contradiction. Hence the last move of $(t \to u_0)\restriction A, C$ must be an O-move in component C, hence $(t \to u_0)\restriction C \notin \mathcal{W}^+_C$. But by **alternation** again we have $t\restriction C \in \mathcal{W}^+_C$, hence $t\restriction A = (t \to u_0)\restriction A \in \mathcal{W}^+_A$, a contradiction again.

Let now $(s_n)_{n\in\mathbb{N}}$ be an infinite sequence of pairwise compatible plays of $\tau \circ \sigma$ such that $\pi := \bigcup_{n\in\mathbb{N}} s_n$ is infinite. Note that we are in the case of A, B and C simple games of the same polarity. Then there is a sequence $(t_n)_{n\in\mathbb{N}} \in \sigma \parallel \tau$ such that $t_n\restriction A, C = s_n$, and the projections of t_n on $A \multimap B$ and $B \multimap C$ are resp. plays of σ and τ. By Zipping (Lem 3.2), t_i is compatible with t_j for all $i, j \in \mathbb{N}$. Consider the sequence $\rho := \bigcup_{n\in\mathbb{N}} t_n$. Note that $\rho\restriction A, C = \pi$. Since $\rho\restriction A, B = \bigcup_{n\in\mathbb{N}} t_n\restriction A, B$ and since σ is winning we have $\rho\restriction A, B \in \mathcal{W}^+_{A\multimap B}$. Similarly, $\rho\restriction B, C \in \mathcal{W}^+_{B\multimap C}$ since τ is winning. It then follows from Lem 5.1 that

$$\rho\restriction A = \pi\restriction A \in \mathcal{W}^+_A \quad \Longrightarrow \quad \rho\restriction B \in \mathcal{W}^+_B \quad \Longrightarrow \quad \rho\restriction C = \pi\restriction C \in \mathcal{W}^+_C$$

◀

▶ **Proposition 5.4.** *Given a simple graph game A, we have* $\mathrm{id} \Vdash A \to A$.

Proof. Totality directly follows from the definition of id (see Prop. 4.11).

Consider now a sequence $(t_n)_{n\in\mathbb{N}}$ of pairwise compatible plays of id such that $\rho := \bigcup_{n\in\mathbb{N}} t_n$ is infinite. Write $A \multimap A$ as $A^{(1)} \multimap A^{(2)}$. For all $n \in \mathbb{N}$, by Prop. 4.11 we have

$$(\rho(0) \to \ldots \to \rho(2n)\restriction A^{(1)}\ ,\ \rho(0) \to \ldots \to \rho(2n)\restriction A^{(2)}) \quad \in \quad \mathrm{HS}(\mathrm{id})$$

hence

$$\rho(0) \to \ldots \to \rho(2n)\restriction A^{(1)} \quad = \quad \rho(0) \to \ldots \to \rho(2n)\restriction A^{(2)}$$

It follows that $\rho\restriction A^{(1)}$ are both infinite $\rho\restriction A^{(2)}$ and moreover that

$$\rho\restriction A^{(1)} \quad = \quad \rho\restriction A^{(2)}$$

hence

$$\rho\restriction A^{(1)} \in W^+_{A^{(1)}} \qquad \Longleftrightarrow \qquad \rho\restriction A^{(2)} \in W^+_{A^{(2)}}$$

◀

5.1.4 The Category $\mathrm{SGG}^{\mathrm{W}}$ of Simple Graph Games with Winning.

Objects of $\mathbf{SGG}^{\mathrm{W}}$ are simple graph games with winning.

Morphisms in $\mathbf{SGG}^{\mathrm{W}}[A, B]$, with A and B of the same polarity are winning P-strategies $\sigma \Vdash A \to B$.

5.2 Subgames

Consider a graph game with winning $A = (V, E, *, \lambda, L, \mathcal{W})$, and consider a set of legal plays $L_0 \subseteq L$. The graph game with winning $A_0 = (V, E, *, \lambda, L_0, \mathcal{W})$ is a P-*imposed* subgame of A if for all P-play $s = * \to^* u \in L_0^{\mathsf{P}}$ and all O-move $u \to v$ such that $s \to v$ is legal in A, we have $s \to v \in L_0$.

O-imposed subgames are defined similarly. The main point of these notions is the following:

▶ **Lemma 5.5.** *If A_0 is a* P*-imposed subgame of A and σ be a winning* P*-strategy on A_0, then*

(i) σ is total on A,
(ii) σ is winning on A.

Proof. (i) Consider a play $s = * \to^* u \in \sigma$ and an O-move $u \to v$ such that $s \to v$ is legal in A. Since $s \to v$ is also legal in A_0 and since σ is total on A_0, there is a P-move $v \to w$ such that $s \to v \to w \in \sigma$.

(ii) Since the winning condition of A is the same as that of A_0. ◀

▶ **Example 5.6.** Synchronized arrow games

$$\mathcal{G}(\mathcal{A}, M) \multimap_{\circledast} \mathcal{G}(\mathcal{B}, N)$$

are P-imposed subgames of

$$\mathcal{G}(\mathcal{A}, M) \multimap \mathcal{G}(\mathcal{B}, N)$$

5.3 Reduction Games

In this Section we consider games equipped with a symmetric notion of winning. These games are meant to be used for reductions (see Sect. 6.3).

5.3.1 Arrow Type with Symmetric Winning.

Given games with winning A and B, the game with symmetric winning $A \multimap_{\Leftrightarrow} B$ is the graph game $A \multimap B$ equipped with the winning condition $\mathcal{W}_{A \multimap_{\Leftrightarrow} B} \subseteq (V_A \times V_B)^\omega$ defined as follows. Given an infinite sequence $\rho \in (V_A \times V_B)^\omega$, let

$$\rho{\restriction}A \quad := \quad \bigcup_{n \in \mathbb{N}} (\rho(0) \to \ldots \to \rho(n)){\restriction}A$$

Let now $\mathcal{W}_{A \multimap_{\Leftrightarrow} B}$ be the set of $\rho \in V^\omega_{A \multimap B}$ such that

$$\rho{\restriction}A \in \mathcal{W}^+_A \qquad \Longleftrightarrow \qquad \rho{\restriction}B \in \mathcal{W}^+_B$$

Note that $\mathcal{W}_{A \multimap_{\Leftrightarrow} B} \subseteq \mathcal{W}_{A \multimap B}$.

We are now going to show that simple reduction games form a category. The method is the same as in Sect. 5.1 above.

▶ **Lemma 5.7.** *Given simple graph games with symmetric winning A and B of the same polarity, and a legal (hence negative) strategy σ on $A \multimap B$, let $(t_n)_{n \in \mathbb{N}}$ be a sequence of pairwise compatible plays of σ, and let $\rho := \bigcup_{n \in \mathbb{N}} t_n$.*

Then $\rho \in \mathcal{W}^+_{A \multimap_{\Leftrightarrow} B}$ iff ($\rho{\restriction}A \in \mathcal{W}^+_A$ iff $\rho{\restriction}B \in \mathcal{W}^+_B$).

The proof is a direct and straightforward adaptation of Lem. 5.1.

Proof. If ρ is infinite, then $\rho \in \mathcal{W}^+_{A \multimap B}$ iff $\mathcal{W}_{A \multimap_{\Leftrightarrow} B}$ and the result follows by definition of $\mathcal{W}_{A \multimap_{\Leftrightarrow} B}$. If ρ is finite, then $\rho \in \sigma$, hence $\rho \in \mathcal{W}^+_{A \multimap_{\Leftrightarrow} B}$. Moreover, ρ has even-length since it is a negative alternating P-play, hence the lengths of $\rho{\restriction}A$ and $\rho{\restriction}B$ have the same parity, and $\rho{\restriction}A \in \mathcal{W}^+_A$ iff $\rho{\restriction}B \in \mathcal{W}^+_B$ since A and B are simple games of the same polarity. ◀

▶ **Definition 5.8.** Given graph games with winning A and B, write $\sigma \Vdash A \multimap_{\Leftrightarrow} B$ if σ is a winning legal (hence negative) P-strategy on $A \multimap_{\Leftrightarrow} B$.

▶ **Proposition 5.9.** *Assume that A, B and C are simple graph games with symmetric winning which are of the same polarity.*

If $\sigma \Vdash A \multimap_{\Leftrightarrow} B$ and $\tau \Vdash B \multimap_{\Leftrightarrow} C$ then $\tau \circ \sigma \Vdash A \multimap_{\Leftrightarrow} C$.

Proof. Since $\mathcal{W}_{A \multimap_{\Leftrightarrow} B} \subseteq \mathcal{W}_{A \multimap B}$, the totality of $\tau \circ \sigma$ can be proved exactly as for Prop. 5.3.

As for the winning condition, we again reason as in the proof of Prop. 5.3. Consider an infinite sequence $(s_n)_{n \in \mathbb{N}}$ of pairwise compatible plays of $\tau \circ \sigma$ such that $\pi := \bigcup_{n \in \mathbb{N}} s_n$ is infinite. Similarly as in Prop. 5.3, we obtain an infinite sequence ρ such that $\rho{\restriction}A, C = \pi$ and $\rho{\restriction}A, B \in \mathcal{W}^+_{A \multimap_{\Leftrightarrow} B}$ and $\rho{\restriction}B, C \in \mathcal{W}^+_{B \multimap_{\Leftrightarrow} C}$. It then follows from Lem 5.7 that

$$\rho{\restriction}A = \pi{\restriction}A \in \mathcal{W}^+_A \qquad \Longleftrightarrow \qquad \rho{\restriction}B \in \mathcal{W}^+_B \qquad \Longleftrightarrow \qquad \rho{\restriction}C = \pi{\restriction}C \in \mathcal{W}^+_C$$

◀

▶ **Proposition 5.10.** *Given a simple graph game A, we have* $\mathrm{id} \Vdash A \Longleftrightarrow A$.

Proof. The proof of Prop. 5.4 actually gives the result. ◀

5.3.2 The Category GR of Reduction Games.

Objects of $\mathbf{GR}$ are simple graph games with winning.

Morphisms in $\mathbf{GR}[A, B]$, with A and B of the same polarity are winning P-strategies σ on the symmetric arrow: $\sigma \Vdash A \multimap_{\Leftrightarrow} B$.

6 Categories of Substituted Acceptance Games

We define the categories $\mathbf{SAG}_\Sigma$, $\mathbf{SAG}^{\mathrm{W}}_\Sigma$ and $\mathbf{SAG}^{\mathrm{R}}_\Sigma$. for each alphabet Σ. The categories $\mathbf{SAG}_\Sigma$ and $\mathbf{SAG}^{\mathrm{W/R}}_\Sigma$ will have as objects substituted acceptances games $\Sigma \vdash \mathcal{G}(\mathcal{A}, M)$. Morphisms in $\mathbf{SAG}_\Sigma$ will be *synchronous* strategies $\sigma : \mathcal{G}(\mathcal{A}, M) \multimap \mathcal{G}(\mathcal{B}, M)$, while morphisms in $\mathbf{SAG}^{\mathrm{W}}_\Sigma$ will synchronous strategies which are moreover *total* and *winning* in the sense of Sect. 5.

As for $\mathbf{SAG}^{\mathrm{R}}_\Sigma$, we consider an notion of *symmetric* winning, which leads to a notion of *synchronous reduction*, while winning in $\mathbf{SAG}^{\mathrm{W}}_\Sigma$ correspond to a form of implication.

6.1 The Categories SAG_Σ

6.1.1 Traces.

Given $\Gamma \vdash \mathcal{A}$ and $M \in \mathbf{Tree}[\Gamma, \Sigma]$, define

$$\mathrm{tr} \quad : \quad \wp_\Sigma(\mathcal{A}, M) \quad \longrightarrow \quad (D + \Sigma)^*$$

follows:

$$\begin{array}{rcl} \mathrm{tr}(\varepsilon : *_{\mathcal{G}(\mathcal{A},M)}) & := & \varepsilon \\ \mathrm{tr}(s \to (p, a, \gamma)) & := & \mathrm{tr}(s) \cdot a \\ \mathrm{tr}(s \to (p \cdot d, q)) & := & \mathrm{tr}(s) \cdot d \end{array}$$

The image of tr is the set

$$\mathrm{Tr}_\Sigma \quad := \quad (\Sigma \cdot D)^* + (\Sigma \cdot D)^* \cdot \Sigma$$

Write $(-)_D$ and $(-)_\Sigma$ for the projections

$$\mathrm{Tr}_\Sigma \longrightarrow D^* \qquad \text{and} \qquad \mathrm{Tr}_\Sigma \longrightarrow \Sigma^*$$

and let

$$\mathrm{tr}_D \quad := \quad (-)_D \circ \mathrm{tr} \qquad \text{and} \qquad \mathrm{tr}_\Sigma \quad := \quad (-)_\Sigma \circ \mathrm{tr}$$

▶ **Lemma 6.1.**

$$\mathrm{tr}_D(*_{\mathcal{G}(\mathcal{A},M)} \to^* (p, a, \gamma)) = p \qquad \textit{and} \qquad \mathrm{tr}_D(*_{\mathcal{G}(\mathcal{A},M)} \to^* (p, q)) = p$$

Proof. We show by induction on the length of $s \in \wp_\Sigma(\mathcal{A}, M)$ that

$$\begin{array}{rl} & \left[s = *_{\mathcal{G}(\mathcal{A},M)} \to^* (p, a, \gamma) \quad \Longrightarrow \quad \mathrm{tr}_D(s) = p\right] \\ \text{and} & \left[s = *_{\mathcal{G}(\mathcal{A},M)} \to^* (p, q) \quad \Longrightarrow \quad \mathrm{tr}_D(s) = p\right] \end{array}$$

In the base case, we have

$$s \quad = \quad \varepsilon : *_{\mathcal{G}(\mathcal{A},M)} \quad = \quad \varepsilon : (\varepsilon, q^\imath)$$

and we are done since

$$\mathrm{tr}_D(s) \quad = \quad \varepsilon_D \quad = \quad \varepsilon$$

For the induction step we consider two cases:

- If

$$s \quad = \quad * \to^* (p,q) \to (p,a,\gamma)$$

then

$$\mathrm{tr}_D(s) \quad = \quad \mathrm{tr}_D(* \to^* (p,q)) \cdot \mathrm{tr}_D(a) \quad = \quad \mathrm{tr}_D(* \to^* (p,q))$$

and we are done since by induction hypothesis

$$\mathrm{tr}_D(* \to^* (p,q)) \quad = \quad p$$

- Otherwise

$$s \quad = \quad * \to^* (p,a,\gamma) \to (p.d,q')$$

and we have

$$\mathrm{tr}_D(s) \quad = \quad \mathrm{tr}_D(* \to^* (p,a,\gamma)) \cdot d$$

and we are done since by induction hypothesis

$$\mathrm{tr}_D(* \to^* (p,a,\gamma)) \quad = \quad p$$

◀

6.1.2 Synchronous Strategies.

Given $\Sigma \vdash \mathcal{G}(\mathcal{A},M)$ and $\Sigma \vdash \mathcal{G}(\mathcal{B},N)$, and a strategy

$$\sigma : \mathcal{G}(\mathcal{A},M) \multimap \mathcal{G}(\mathcal{B},N)$$

consider the following span:

$$\begin{tikzcd}
& \sigma \arrow[dl] \arrow[dr] & \\
\wp_\Sigma(\mathcal{A},M) & & \wp_\Sigma(\mathcal{B},N)
\end{tikzcd}$$

where $\wp_\Sigma(\mathcal{A},M) := \wp(\Sigma \vdash \mathcal{G}(\mathcal{A},M))$ and, writting A for $\mathcal{G}(\mathcal{A},M)$, the arrow $\sigma \longrightarrow \wp_\Sigma(\mathcal{A},M)$ is

$$\begin{tikzcd}
\sigma \arrow[r, hook] & \wp_\Sigma(\mathcal{G}(\mathcal{A},M) \multimap \mathcal{G}(\mathcal{B},N)) \arrow[r, "(-)\restriction A"] & \wp_\Sigma(\mathcal{A},M)
\end{tikzcd}$$

and similarly for $\wp_\Sigma(\mathcal{B},N)$ and $\sigma \longrightarrow \wp_\Sigma(\mathcal{B},N)$.

We say that σ is *synchronous*, and write

$$\Sigma \vdash \sigma : \mathcal{G}(\mathcal{A},M) \multimap_{\circledast} \mathcal{G}(\mathcal{B},N) \qquad \text{(or simply} \quad \sigma : \mathcal{G}(\mathcal{A},M) \multimap_{\circledast} \mathcal{G}(\mathcal{B},N))$$

when the following diagram commutes:

$$\begin{tikzcd}
& \sigma \arrow[dl] \arrow[dr] & \\
\wp_\Sigma(\mathcal{A},M) \arrow[dr, "\mathrm{tr}"'] & & \wp_\Sigma(\mathcal{B},N) \arrow[dl, "\mathrm{tr}"] \\
& \mathrm{Tr}_\Sigma &
\end{tikzcd} \tag{7}$$

6.1.3 Relations in Slice Categories Set/J.

Given a set J, define the category $\mathbf{Rel}(\mathbf{Set}/J)$ as follows:

Objects are indexed sets $A \overset{g}{\to} J$, written simply A when g is understood from the context.

Morphisms from $A \overset{g_A}{\to} J$ to $B \overset{g_B}{\to} J$ are given by relations $R : A \twoheadrightarrow B$ such that the following commutes:

$$
\begin{tikzcd}
 & R \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
A \arrow[dr, "g_A"'] & & B \arrow[dl, "g_B"] \\
 & J &
\end{tikzcd}
\tag{8}
$$

Write

$$
R \quad : \quad (A, g_A) \quad \twoheadrightarrow_{/J} \quad (B, g_B)
$$

when $R \in \mathbf{Rel}(\mathbf{Set}/J)[(A, g_A), (B, g_B)]$

For the identity, note that

$$
1_A \quad = \quad \{(a, a) \mid a \in A\} \quad \in \quad \mathbf{Rel}(\mathbf{Set}/J)[(A, g), (A, g)]
$$

since $g(a) = g(a)$.
For composition, note that given $R \in \mathbf{Rel}(\mathbf{Set}/J)[A, B]$ and $P \in \mathbf{Rel}(\mathbf{Set}/J)[B, C]$ with $C \overset{g_C}{\to} J$, we have $P \circ R \in \mathbf{Rel}(\mathbf{Set}/J)[A, C]$ since given $(a, c) \in P \circ R$, by definition there is $b \in B$ such that $(a, b) \in R$ and $(b, c) \in P$, hence $g_A(a) = g_B(b) = g_C(c)$.

▶ Remark. We will see below that the relational structure in $\mathbf{Set}/\mathrm{Tr}_\Sigma$ issued from $\mathbf{SAG}_\Sigma$ *via* $\mathrm{HS}(-)$ satisfies the stronger property:

$$
\begin{tikzcd}
P \circ R \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & P \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & C \arrow[d, "k"] \\
R \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & B \arrow[r, "h"] \arrow[d, "h"'] & J \arrow[d, equal] \\
A \arrow[r, "g"'] & J \arrow[r, equal] & J
\end{tikzcd}
$$

6.1.4 Composition and Identities by Pullback in Set.

Substituted acceptance games are simple positive games, in the sense of Sect. 4. We now recall some properties of the category $\mathbf{SGG}$ of simple games discussed.

First, Prop. 4.11 tells us that identities satisfy the following pullback square:

$$
\begin{tikzcd}
\mathrm{id}_{\mathcal{G}(\mathcal{A},M)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \wp_\Sigma(\mathcal{A}, M) \arrow[d, "1"] \\
\wp_\Sigma(\mathcal{A}, M) \arrow[r, "1"'] & \wp_\Sigma(\mathcal{A}, M)
\end{tikzcd}
\tag{9}
$$

Moreover, we know from Prop. 4.13 (see App. 4.4) that composition in $\mathbf{SGG}$ is given by the following pullback square in **Set**:

$$\begin{tikzcd}
\theta \circ \sigma \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \tau \arrow[r] \arrow[d] & \wp_\Sigma(\mathcal{C}, P) \\
\sigma \arrow[r] \arrow[d] & \wp_\Sigma(\mathcal{B}, N) & \\
\wp_\Sigma(\mathcal{A}, M) & &
\end{tikzcd} \tag{10}$$

where $\theta \circ \sigma \longrightarrow \sigma$ map $s \in \theta \circ \sigma$ to $(s{\upharpoonright}(\mathcal{G}(\mathcal{A}, M)), s')$ where s' is by Lem 4.12 unique in $\wp_\Sigma(\mathcal{B}, N)$ such that $(s{\upharpoonright}(\mathcal{G}(\mathcal{A}, M)), s') \in \mathrm{HS}(\sigma)$ and the map $\theta \circ \sigma \longrightarrow \theta$, is defined similarly.

6.1.5 The Categories $\mathbf{SAG}_\Sigma$:

Objects are games $\Sigma \vdash \mathcal{G}(\mathcal{A}, M)$
Morphisms from $\mathcal{G}(\mathcal{A}, M)$ to $\mathcal{G}(\mathcal{B}, N)$ are synchronous strategies $\sigma : \mathcal{G}(\mathcal{A}, M) \mathrel{-\!\circledast} \mathcal{G}(\mathcal{B}, N)$.

We now discuss identities and composition in $\mathbf{SAG}_\Sigma$.
Write $\mathrm{id}_{(\mathcal{A},M)}$ for $\mathrm{id}_{\mathcal{G}(\mathcal{A},M)} : \mathcal{G}(\mathcal{A}, M) \multimap \mathcal{G}(\mathcal{A}, M)$. It immediately follows from (9) that

$$\begin{tikzcd}
\mathrm{id}_{(\mathcal{A},M)} \arrow[r] \arrow[d] & \wp_\Sigma(\mathcal{A}, M) \arrow[d, "\mathrm{tr}"] \\
\wp_\Sigma(\mathcal{A}, M) \arrow[r, "\mathrm{tr}"'] & \mathrm{Tr}_\Sigma
\end{tikzcd} \tag{11}$$

▶ Remark. We actually here only need the commutation of the diagram, not the fact that it is a pullback. Hence the assumption that $\mathcal{G}(\mathcal{A}, M)$ is a simple game is not necessary.

Consider now $\Sigma \vdash \sigma : \mathcal{G}(\mathcal{A}, M) \mathrel{-\!\circledast} \mathcal{G}(\mathcal{B}, N)$ and $\Sigma \vdash \theta : \mathcal{G}(\mathcal{B}, N) \mathrel{-\!\circledast} \mathcal{G}(\mathcal{C}, P)$, so that

$$\begin{tikzcd}
\sigma \arrow[r] \arrow[d] & \wp_\Sigma(\mathcal{B}, N) \arrow[d, "\mathrm{tr}"] \\
\wp_\Sigma(\mathcal{A}, M) \arrow[r, "\mathrm{tr}"'] & \mathrm{Tr}_\Sigma
\end{tikzcd}
\qquad \text{and} \qquad
\begin{tikzcd}
\theta \arrow[r] \arrow[d] & \wp_\Sigma(\mathcal{C}, P) \arrow[d, "\mathrm{tr}"] \\
\wp_\Sigma(\mathcal{B}, N) \arrow[r, "\mathrm{tr}"'] & \mathrm{Tr}_\Sigma
\end{tikzcd}$$

It follows from (10) that

$$\begin{tikzcd}
\theta \circ \sigma \arrow[r] \arrow[d] & \tau \arrow[r] \arrow[d] & \wp_\Sigma(\mathcal{C}, P) \arrow[d, "\mathrm{tr}"] \\
\sigma \arrow[r] \arrow[d] & \wp_\Sigma(\mathcal{B}, N) \arrow[r, "\mathrm{tr}"] \arrow[d, "\mathrm{tr}"] & \mathrm{Tr}_\Sigma \arrow[d, equal] \\
\wp_\Sigma(\mathcal{A}, M) \arrow[r, "\mathrm{tr}"'] & \mathrm{Tr}_\Sigma \arrow[r, equal] & \mathrm{Tr}_\Sigma
\end{tikzcd}$$

and by definition of $\theta \circ \sigma \longrightarrow \sigma$ and $\theta \circ \sigma \longrightarrow \theta$ we deduce that

$$\theta \circ \sigma : \mathcal{G}(\mathcal{A}, M) \mathrel{-\!\!-\!\circledast} \mathcal{G}(\mathcal{C}, P)$$

We thus have shown:

▶ **Proposition 6.2.** *(i)* $\Sigma \vdash \mathrm{id}_{(\mathcal{A},M)} : \mathcal{G}(\mathcal{A}, M) \mathrel{-\!\circledast} \mathcal{G}(\mathcal{B}, N)$.
(ii) If $\Sigma \vdash \sigma : \mathcal{G}(\mathcal{A}, M) \mathrel{-\!\circledast} \mathcal{G}(\mathcal{B}, N)$ *and* $\Sigma \vdash \theta : \mathcal{G}(\mathcal{B}, N) \mathrel{-\!\circledast} \mathcal{G}(\mathcal{C}, P)$ *then*

$$\Sigma \vdash \theta \circ \sigma : \mathcal{G}(\mathcal{A}, M) \mathrel{-\!\circledast} \mathcal{G}(\mathcal{C}, P)$$

6.1.6 The HS Functor.

By definition of strategies in $\mathbf{SAG}_\Sigma$, given

$$\Sigma \vdash \sigma : \mathcal{G}(\mathcal{A}, M) \multimap\!\!\circledast\, \mathcal{G}(\mathcal{B}, N)$$

we have

$$\mathrm{HS}(\sigma) \subseteq \wp_\Sigma(\mathcal{A}, M) \times_{\mathrm{Tr}_\Sigma} \wp_\Sigma(\mathcal{C}, N)$$

It follows that the HS functor from **SGG** to **Rel** (see Sect. 4.3) restricts to a functor from $\mathbf{SAG}_\Sigma$ to $\mathbf{Rel}(\mathbf{Set}/\mathrm{Tr}_\Sigma)$ (see Sect. 10.1):

▶ **Proposition 6.3.** HS *restricts to a functor from* $\mathbf{SAG}_\Sigma$ *to* $\mathbf{Rel}(\mathbf{Set}/\mathrm{Tr}_\Sigma)$.

6.2 Concrete Description of the Synchronous Arenas

We now concretely define the synchronous game arenas

$$\Sigma \vdash \mathcal{G}(\mathcal{A}, M) \multimap\!\!\circledast\, \mathcal{G}(\mathcal{B}, N)$$

as subgames of

$$\mathcal{G}(\mathcal{A}, M) \multimap \mathcal{G}(\mathcal{B}, N)$$

so that strategies on

$$\Sigma \vdash \mathcal{G}(\mathcal{A}, M) \multimap\!\!\circledast\, \mathcal{G}(\mathcal{B}, N)$$

are exactly the synchronous strategies on

$$\mathcal{G}(\mathcal{A}, M) \multimap \mathcal{G}(\mathcal{B}, N)$$

6.2.1 The Synchronous Arrow $\_ \multimap\!\!\circledast \_$.

Consider

$$\Sigma \vdash \mathcal{G}(\mathcal{A}, M) \qquad \text{and} \qquad \Sigma \vdash \mathcal{G}(\mathcal{B}, N)$$

where

$$\Gamma_\mathcal{A} \vdash \mathcal{A} \qquad \Gamma_\mathcal{B} \vdash \mathcal{B} \qquad \Sigma \vdash M : \Gamma_\mathcal{A} \qquad \Sigma \vdash N : \Gamma_\mathcal{B}$$

Write

$$\begin{array}{rrcccl} & A & := & \mathcal{G}(\mathcal{A}, M) & = & (V_A, E_A, *_A, \lambda_A, \mathcal{W}_A) \\ \text{and} & B & := & \mathcal{G}(\mathcal{B}, N) & = & (V_B, E_B, *_B, \lambda_B, \mathcal{W}_B) \end{array}$$

where

$$\begin{array}{rcl@{\qquad\qquad}rcl} V_A & := & D^* \times (A_\mathsf{P} + A_\mathsf{O}) & *_A & := & (\varepsilon, q^\imath_A) \\ V_B & := & D^* \times (B_\mathsf{P} + B_\mathsf{O}) & *_B & := & (\varepsilon, q^\imath_B) \\ A_\mathsf{P} & := & Q_A & A_\mathsf{O} & := & \Sigma \times \mathcal{P}(Q_A \times D) \\ B_\mathsf{P} & := & Q_B & B_\mathsf{O} & := & \Sigma \times \mathcal{P}(Q_B \times D) \end{array}$$

The game

$$\Sigma \vdash \mathcal{G}(\mathcal{A}, M) \multimap\!\!\circledast\, \mathcal{G}(\mathcal{B}, N)$$

is the subgame of $A \multimap B$

$$\mathcal{G}(\mathcal{A}, M) \multimap\!\!\circledast\, \mathcal{G}(\mathcal{B}, N) \quad := \quad (V_{A\multimap B}, E, *_{A\multimap B}, \lambda_{A\multimap B}, \mathcal{W}_{A\multimap B})$$

where $E \subseteq E_{A\multimap B}$ is defined in Fig. 2.

	$\mathcal{G}(\mathcal{A}, M)$	$\mathrel{-\!\!\circledast}$	$\mathcal{G}(\mathcal{B}, N)$	
	$((p, q_{\mathcal{A}})$	,	$(p, q_{\mathcal{B}}))$	
O	$\downarrow$			
	$((p, a, \gamma_{\mathcal{A}})$	,	$(p, q_{\mathcal{B}}))$	if $\gamma_{\mathcal{A}} \in \delta_{\mathcal{A}}(q_{\mathcal{A}}, M(p)(a))$
P			$\downarrow$	
	$((p, a, \gamma_{\mathcal{A}})$	,	$(p, a, \gamma_{\mathcal{B}}))$	if $\gamma_{\mathcal{B}} \in \delta_{\mathcal{B}}(q_{\mathcal{B}}, N(p)(a))$
O			$\downarrow$	
	$((p, a, \gamma_{\mathcal{A}})$	,	$(p.d, q'_{\mathcal{B}}))$	if $(q'_{\mathcal{B}}, d) \in \gamma_{\mathcal{B}}$
P	$\downarrow$			
	$((p.d, q'_{\mathcal{A}})$	,	$(p.d, q'_{\mathcal{B}}))$	if $(q'_{\mathcal{A}}, d) \in \gamma_{\mathcal{A}}$

Figure 2 Moves of $\mathcal{G}(\mathcal{A}, M) \mathrel{-\!\!\circledast} \mathcal{G}(\mathcal{B}, N)$

6.2.2 Characterization of $\_ \mathrel{-\!\!\circledast} \_$.

We are now going to see that in **Set**,

$$\begin{tikzcd}
\wp^{\mathsf{P}}_{\Sigma}(\mathcal{G}(\mathcal{A}, M) \mathrel{-\!\!\circledast} \mathcal{G}(\mathcal{B}, N)) \arrow[r, "(-)\restriction\mathcal{G}(\mathcal{B},N)"] \arrow[d, "(-)\restriction\mathcal{G}(\mathcal{A},M)"'] \arrow[dr, phantom, "\lrcorner", very near start] & \wp_{\Sigma}(\mathcal{B}, N) \arrow[d, "\mathrm{tr}"] \\
\wp_{\Sigma}(\mathcal{A}, M) \arrow[r, "\mathrm{tr}"'] & \mathrm{Tr}_{\Sigma}
\end{tikzcd}$$

First, an easy induction on $t \in \wp^{\mathsf{P}}_{\Sigma}(\mathcal{G}(\mathcal{A}, M) \mathrel{-\!\!\circledast} \mathcal{G}(\mathcal{B}, N))$ shows that

▶ **Proposition 6.4.**

$$\begin{tikzcd}
\wp^{\mathsf{P}}_{\Sigma}(\mathcal{G}(\mathcal{A}, M) \mathrel{-\!\!\circledast} \mathcal{G}(\mathcal{B}, N)) \arrow[r, "(-)\restriction\mathcal{G}(\mathcal{B},N)"] \arrow[d, "(-)\restriction\mathcal{G}(\mathcal{A},M)"'] & \wp_{\Sigma}(\mathcal{B}, N) \arrow[d, "\mathrm{tr}"] \\
\wp_{\Sigma}(\mathcal{A}, M) \arrow[r, "\mathrm{tr}"'] & \mathrm{Tr}_{\Sigma}
\end{tikzcd}$$

It follows that

$$\mathrm{HS} \quad : \quad \wp_{\Sigma}(\mathcal{G}(\mathcal{A}, M) \mathrel{-\!\!\circledast} \mathcal{G}(\mathcal{B}, N)) \quad \longrightarrow \quad \wp_{\Sigma}(\mathcal{A}, M) \times \wp_{\Sigma}(\mathcal{B}, N)$$

restricts to

$$\mathrm{HS} \quad : \quad \wp^{\mathsf{P}}_{\Sigma}(\mathcal{G}(\mathcal{A}, M) \mathrel{-\!\!\circledast} \mathcal{G}(\mathcal{B}, N)) \quad \longrightarrow \quad \wp_{\Sigma}(\mathcal{A}, M) \times_{\mathrm{Tr}_{\Sigma}} \wp_{\Sigma}(\mathcal{B}, N)$$

▶ **Lemma 6.5.** *The map*

$$\mathrm{HS} \quad : \quad \wp^{\mathsf{P}}_{\Sigma}(\mathcal{G}(\mathcal{A}, M) \mathrel{-\!\!\circledast} \mathcal{G}(\mathcal{B}, N)) \quad \longrightarrow \quad \wp_{\Sigma}(\mathcal{A}, M) \times_{\mathrm{Tr}_{\Sigma}} \wp_{\Sigma}(\mathcal{B}, N)$$

is surjective.

Proof. Note that $(s, t) \in \wp_{\Sigma}(\mathcal{A}, M) \times_{\mathrm{Tr}_{\Sigma}} \wp_{\Sigma}(\mathcal{B}, N)$ implies $|s| = |t|$.

By induction on $|s| = |t|$ for $(s, t) \in \wp_{\Sigma}(\mathcal{A}, M) \times \wp_{\Sigma}(\mathcal{B}, N)$ with $\mathrm{tr}(s) = \mathrm{tr}(t)$, we show that there is $w \in \wp^{\mathsf{P}}_{\Sigma}(\mathcal{G}(\mathcal{A}, M) \mathrel{-\!\!\circledast} \mathcal{G}(\mathcal{B}, N))$ such that $\mathrm{HS}(w) = (s, t)$.

In the base case $|s| = |t| = 0$, and we take $w := \varepsilon : *$. In the inductive step, by definition of tr there are two cases:

- Assume $s = s' \to (p, a, \gamma_\mathcal{A})$ and $t = t' \to (p', a', \gamma_\mathcal{B})$.
  Since $\mathrm{tr}(s) = \mathrm{tr}(t)$, we have $a = a'$ and $\mathrm{tr}(s') = \mathrm{tr}(t')$. By Lem 6.1, we get $p = p'$. Moreover, s' and t' are of the form:

  $$s' \quad = \quad * \to^* (p, q_\mathcal{A}) \qquad \text{and} \qquad t' \quad = \quad * \to^* (p, q_\mathcal{B})$$

  Let w' be obtained by induction hypothesis applied to s' and t'. Then we are done by taking

  $$w \quad = \quad w' \quad \to \quad ((p, a, \gamma_\mathcal{A}), (p, q_\mathcal{B})) \quad \to \quad ((p, a, \gamma_\mathcal{A}), (p', a', \gamma_\mathcal{B}))$$

- Otherwise $s = s' \to (p.d, q_\mathcal{A})$ and $t = t' \to (p'.d', q_\mathcal{B})$.
  Since $\mathrm{tr}(s) = \mathrm{tr}(t)$, we have $p = p'$, $d = d'$ and $\mathrm{tr}(s') = \mathrm{tr}(t')$. Moreover, s' and t' are of the form:

  $$s' \quad = \quad * \to^* (p, a, \gamma_\mathcal{A}) \qquad \text{and} \qquad t' \quad = \quad * \to^* (p, a, \gamma_\mathcal{B})$$

  Let w' be obtained by induction hypothesis applied to s' and t'. Then we are done by taking

  $$w \quad = \quad w' \quad \to \quad ((p, a, \gamma_\mathcal{A}), (p.d, q_\mathcal{B})) \quad \to \quad ((p.d, q_\mathcal{A}), (p.d, q_\mathcal{B}))$$

◀

▶ **Lemma 6.6.** *The map*

$$\mathrm{HS} \quad : \quad \wp_\Sigma(\mathcal{G}(\mathcal{A}, M) \multimap\!\!\circledast\, \mathcal{G}(\mathcal{B}, N)) \quad \longrightarrow \quad \wp_\Sigma(\mathcal{A}, M) \times \wp_\Sigma(\mathcal{B}, N)$$

is injective.

Proof. We have to show that:

$$\forall s, t \in \wp_\Sigma(\mathcal{G}(\mathcal{A}, M) \multimap\!\!\circledast\, \mathcal{G}(\mathcal{B}, N)), \quad \mathrm{HS}(s) = \mathrm{HS}(t) \quad \Longrightarrow \quad s = t$$

Note that if $\mathrm{HS}(s) = \mathrm{HS}(t)$, then $|s| = |t|$. We reason by induction on $|s| = |t|$.

In the base case, $|s| = |t| = 0$, hence $s = t = \varepsilon : *$ and we are done.

Consider now the inductive step. First, since $|s| = |t|$, and since $\_ \multimap\!\!\circledast \_$ is negative, either both s and t end with an O-move or they both end with a P-move. Moreover, by definition of $\_ \multimap\!\!\circledast \_$, $|s| = |t|$ implies that s and t end by a move in the same component, and hence by the same move since $\mathrm{HS}(s) = \mathrm{HS}(t)$. But then, writing

$$s \quad = \quad s' \to m \qquad \text{and} \qquad t \quad = \quad t' \to m$$

we must have $\mathrm{HS}(s') = \mathrm{HS}(t')$, hence $s' = t'$ by induction hypothesis and we are done. ◀

▶ **Corollary 6.7** (Prop. 3.3)**.** *In* **Set***,*

$$\begin{tikzcd}
\wp^{\mathsf{P}}_\Sigma(\mathcal{G}(\mathcal{A}, M) \multimap\!\!\circledast\, \mathcal{G}(\mathcal{B}, N)) \arrow[r, "(-)\upharpoonright\mathcal{G}(\mathcal{B},N)"] \arrow[d, "(-)\upharpoonright\mathcal{G}(\mathcal{A},M)"'] \arrow[dr, phantom, "\lrcorner", very near start] & \wp_\Sigma(\mathcal{B}, N) \arrow[d, "\mathrm{tr}"] \\
\wp_\Sigma(\mathcal{A}, M) \arrow[r, "\mathrm{tr}"'] & \mathrm{Tr}_\Sigma
\end{tikzcd}$$

Proof. Commutation is given by Prop. 6.4. Moreover, thanks to Lemmas 6.6 and 6.5, HS is a bijection in **Set**:

$$\mathrm{HS} \quad : \quad \wp^{\mathsf{P}}_{\Sigma}(\mathcal{G}(\mathcal{A}, M) \multimap\!\!\circledast\, \mathcal{G}(\mathcal{B}, N)) \quad \xrightarrow{\simeq} \quad \wp_{\Sigma}(\mathcal{A}, M) \times_{\mathrm{Tr}_\Sigma} \wp_{\Sigma}(\mathcal{B}, N)$$

◀

Since moreover by definition

$$\wp^{\mathsf{P}}_{\Sigma}(\mathcal{G}(\mathcal{A}, M) \multimap\!\!\circledast\, \mathcal{G}(\mathcal{B}, N)) \quad \subseteq \quad \wp^{\mathsf{P}}(\mathcal{G}(\mathcal{A}, M) \multimap \mathcal{G}(\mathcal{B}, N))$$

it follows that we get:

▶ **Corollary 6.8.** *Strategies on* $\Sigma \vdash \mathcal{G}(\mathcal{A}, M) \multimap\!\!\circledast\, \mathcal{G}(\mathcal{B}, N)$ *are exactly the synchronous strategies on* $\mathcal{G}(\mathcal{A}, M) \multimap \mathcal{G}(\mathcal{B}, N)$*.*

6.2.3 Traces on the Synchronous Arrow.

Thanks to Prop. 6.4, that is commutation of

$$\begin{tikzcd}
\wp^{\mathsf{P}}_{\Sigma}(\mathcal{G}(\mathcal{A}, M) \multimap\!\!\circledast\, \mathcal{G}(\mathcal{B}, N)) \arrow[r, "(-)\restriction\mathcal{G}(\mathcal{B},N)"] \arrow[d, "(-)\restriction\mathcal{G}(\mathcal{A},M)"'] & \wp_{\Sigma}(\mathcal{B}, N) \arrow[d, "\mathrm{tr}"] \\
\wp_{\Sigma}(\mathcal{A}, M) \arrow[r, "\mathrm{tr}"'] & \mathrm{Tr}_\Sigma
\end{tikzcd}$$

we extend traces of plays on acceptance games (given by $\mathrm{tr} : \wp_{\Sigma}(\mathcal{A}, M) \to \mathrm{Tr}_\Sigma$) to traces of P-*plays* on synchronous arrow games.

▶ **Definition 6.9.** Define

$$\mathrm{tr}^{\multimap\!\circledast} \quad : \quad \wp^{\mathsf{P}}_{\Gamma}(\mathcal{G}(\mathcal{A}, M) \multimap\!\!\circledast\, \mathcal{G}(\mathcal{B}, N)) \quad \longrightarrow \quad \mathrm{Tr}_\Gamma$$

as any of two following maps, which coincide by Prop. 6.4:

$$\begin{array}{ccccc}
\wp^{\mathsf{P}}_{\Sigma}(\mathcal{G}(\mathcal{A}, M) \multimap\!\!\circledast\, \mathcal{G}(\mathcal{B}, N)) & \xrightarrow{\restriction\mathcal{G}(\mathcal{A},M)} & \wp_{\Sigma}(\mathcal{A}, M) & \xrightarrow{\mathrm{tr}} & \mathrm{Tr}_\Sigma \\
\wp^{\mathsf{P}}_{\Sigma}(\mathcal{G}(\mathcal{A}, M) \multimap\!\!\circledast\, \mathcal{G}(\mathcal{B}, N)) & \xrightarrow{\restriction\mathcal{G}(\mathcal{B},N)} & \wp_{\Sigma}(\mathcal{B}, N) & \xrightarrow{\mathrm{tr}} & \mathrm{Tr}_\Sigma
\end{array}$$

▶ Remark. Note that $\mathrm{tr}^{\multimap\!\circledast}$ can only be defined on P-*plays* of $\mathcal{G}(\mathcal{A}, M) \multimap\!\!\circledast\, \mathcal{G}(\mathcal{B}, N)$ since the following diagram *does not* commute:

$$\begin{tikzcd}
\wp_{\Sigma}(\mathcal{G}(\mathcal{A}, M) \multimap\!\!\circledast\, \mathcal{G}(\mathcal{B}, N)) \arrow[r, "(-)\restriction\mathcal{G}(\mathcal{B},N)"] \arrow[d, "(-)\restriction\mathcal{G}(\mathcal{A},M)"'] & \wp_{\Sigma}(\mathcal{B}, N) \arrow[d, "\mathrm{tr}"] \\
\wp_{\Sigma}(\mathcal{A}, M) \arrow[r, "\mathrm{tr}"'] & \mathrm{Tr}_\Sigma
\end{tikzcd}$$

6.3 The Categories $\mathrm{SAG}^{\mathrm{W}}_{\Sigma}$ and $\mathrm{SAG}^{\mathrm{R}}_{\Sigma}$

6.3.1 Winning and Total Synchronous Strategies.

Consider

$$\Sigma \vdash \sigma : \mathcal{G}(\mathcal{A}, M) \multimap\!\!\circledast\, \mathcal{G}(\mathcal{B}, N)$$

Write

$$\Sigma \vdash \sigma \Vdash \mathcal{G}(\mathcal{A}, M) \multimap_{\circledast} \mathcal{G}(\mathcal{B}, N) \qquad \text{(or simply} \quad \sigma \Vdash \mathcal{G}(\mathcal{A}, M) \multimap_{\circledast} \mathcal{G}(\mathcal{B}, N))$$

if σ is total and winning in the game

$$\mathcal{G}(\mathcal{A}, M) \quad \multimap \quad \mathcal{G}(\mathcal{B}, N)$$

(*i.e.* in sense of Sect.5.1), and write

$$\Sigma \vdash \sigma \Vdash \mathcal{G}(\mathcal{A}, M) \circledast\!\!-\!\!\circledast \mathcal{G}(\mathcal{B}, N) \qquad \text{(or simply} \quad \sigma \Vdash \mathcal{G}(\mathcal{A}, M) \circledast\!\!-\!\!\circledast \mathcal{G}(\mathcal{B}, N))$$

if σ is total and winning in the game

$$\mathcal{G}(\mathcal{A}, M) \quad \multimap_{\Leftrightarrow} \quad \mathcal{G}(\mathcal{B}, N)$$

(*i.e.* in sense of Sect.5.3).

6.3.2 The Categories $\mathbf{SAG}_\Sigma^{\mathrm{W}}$:

Objects are games $\Sigma \vdash \mathcal{G}(\mathcal{A}, M)$

Morphisms from $\mathcal{G}(\mathcal{A}, M)$ to $\mathcal{G}(\mathcal{B}, N)$ are synchronous total and winning strategies $\sigma \Vdash \mathcal{G}(\mathcal{A}, M) \multimap_{\circledast} \mathcal{G}(\mathcal{B}, N)$.

Since $\mathrm{id}_{(\mathcal{A},M)}$ is total and winning (Prop. 5.4), and since winning total strategies are preserved by composition (Prop. 5.3), we obtain:

▶ **Corollary 6.10.** *(i)* $\mathrm{id}_{(\mathcal{A},M)} \Vdash \mathcal{G}(\mathcal{A}, M) \multimap_{\circledast} \mathcal{G}(\mathcal{B}, N)$

(ii) If $\sigma \Vdash \mathcal{G}(\mathcal{A}, M) \multimap_{\circledast} \mathcal{G}(\mathcal{B}, N)$ *and* $\theta \Vdash \mathcal{G}(\mathcal{B}, N) \multimap_{\circledast} \mathcal{G}(\mathcal{C}, P)$ *then*

$$\theta \circ \sigma \Vdash \mathcal{G}(\mathcal{A}, M) \multimap_{\circledast} \mathcal{G}(\mathcal{C}, P)$$

6.3.3 The Categories $\mathbf{SAG}_\Sigma^{\mathrm{R}}$:

Objects are games $\Sigma \vdash \mathcal{G}(\mathcal{A}, M)$

Morphisms from $\mathcal{G}(\mathcal{A}, M)$ to $\mathcal{G}(\mathcal{B}, N)$ are synchronous total and winning strategies $\sigma \Vdash \mathcal{G}(\mathcal{A}, M) \circledast\!\!-\!\!\circledast \mathcal{G}(\mathcal{B}, N)$.

Since $\mathrm{id}_{(\mathcal{A},M)}$ is total and winning for $\_ \circledast\!\!-\!\!\circledast \_$ (Prop. 5.10), and since $\_ \circledast\!\!-\!\!\circledast \_$-winning total strategies are preserved by composition (Prop. 5.9), we obtain:

▶ **Corollary 6.11.** *(i)* $\mathrm{id}_{(\mathcal{A},M)} \Vdash \mathcal{G}(\mathcal{A}, M) \circledast\!\!-\!\!\circledast \mathcal{G}(\mathcal{B}, N)$

(ii) If $\sigma \Vdash \mathcal{G}(\mathcal{A}, M) \circledast\!\!-\!\!\circledast \mathcal{G}(\mathcal{B}, N)$ *and* $\theta \Vdash \mathcal{G}(\mathcal{B}, N) \circledast\!\!-\!\!\circledast \mathcal{G}(\mathcal{C}, P)$ *then*

$$\theta \circ \sigma \Vdash \mathcal{G}(\mathcal{A}, M) \circledast\!\!-\!\!\circledast \mathcal{G}(\mathcal{C}, P)$$

6.4 Relational Lifting

We now describe how synchronous relational isomorphisms (*i.e.* isos of $\mathbf{Rel}(\mathbf{Set}/J)$) can be lifted to strategies.

▶ **Proposition 6.12** (Prop 3.4). *Consider $\Sigma \vdash \mathcal{G}(\mathcal{A}, M)$ and $\Sigma \vdash \mathcal{G}(\mathcal{B}, N)$. Assume that, in $\mathbf{Rel}(\mathbf{Set}/\mathrm{Tr}_\Sigma)$ we have an isomorphism*

$$R \quad : \quad \wp_\Sigma(\mathcal{A}, M) \quad \twoheadrightarrow\!\!\!\!+_{/\mathrm{Tr}_\Sigma} \quad \wp_\Sigma(\mathcal{B}, N)$$

Then there is a (unique, total) isomorphism

$$\sigma \quad : \quad \mathcal{G}(\mathcal{A}, M) \quad \longrightarrow_{\mathbf{SAG}_\Sigma} \quad \mathcal{G}(\mathcal{B}, N)$$

such that $\mathrm{HS}(\sigma) = R$*.*

Here, $\wp_\Sigma(\mathcal{A}, M)$ and $\wp_\Sigma(\mathcal{B}, N)$ are understood, as objects of $\mathbf{Rel}(\mathbf{Set}/\mathrm{Tr}_\Sigma)$, as resp. $\wp_\Sigma(\mathcal{A}, M) \xrightarrow{\mathrm{tr}} \mathrm{Tr}_\Sigma$ and $\wp_\Sigma(\mathcal{B}, N) \xrightarrow{\mathrm{tr}} \mathrm{Tr}_\Sigma$.

In general we can not ask σ to be winning, and in particular to be a morphism of $\mathbf{SAG}_\Sigma^{\mathrm{W/R}}$

Proof. First, note that since R is synchronous, for all $(s, t) \in R$ we have $\mathrm{tr}(s) = \mathrm{tr}(t)$, which implies that s and t have the same length, and finish by the same kind of moves.

By induction on $|s| = |t|$ for $(s, t) \in R$ we define a strategy σ such that $(s, t) \in \mathrm{HS}(\sigma)$.

In the base case, we have

$$s \quad = \quad (\varepsilon, q^\imath_\mathcal{A}) \quad \text{and} \quad t \quad = \quad (\varepsilon, q^\imath_\mathcal{B})$$

and we put

$$((\varepsilon, q^\imath_\mathcal{A}) \ , \ (\varepsilon, q^\imath_\mathcal{B})) \in \sigma$$

For the induction step, there are two cases:

Case of $s = * \to^* (p, q_\mathcal{A}) \xrightarrow{\mathsf{P}} (p, a, \gamma_\mathcal{A})$ and $t = * \to^* (p', q_\mathcal{B}) \xrightarrow{\mathsf{P}} (p', a', \gamma_\mathcal{B})$.

Since $\mathrm{tr}(s) = \mathrm{tr}(t)$ we have $a = a'$ and it moreover follows from Lem. 6.1 that $p = p'$. By induction hypothesis there is $u' \in \sigma$ such that $\mathrm{HS}(u') = (s', t')$. Hence we have

$$u' \quad = \quad * \to^* ((p, q_\mathcal{A}) \ , \ (p, q_\mathcal{B}))$$

We can then extend u' to

$$u \quad := \quad * \to^* ((p, q_\mathcal{A}) \ , \ (p, q_\mathcal{B})) \xrightarrow{\mathsf{O}} ((p, a, \gamma_\mathcal{A}) \ , \ (p, q_\mathcal{B})) \xrightarrow{\mathsf{P}} ((p, a, \gamma_\mathcal{A}) \ , \ (p, a, \gamma_\mathcal{B}))$$

and we indeed have $\mathrm{HS}(u) = (s, t)$.

Case of $s = * \to^* (p, a, \gamma_\mathcal{A}) \xrightarrow{\mathsf{O}} (p \cdot d, q_\mathcal{A})$ and $t = * \to^* (p', a', \gamma_\mathcal{B}) \xrightarrow{\mathsf{O}} (p' \cdot d', q_\mathcal{B})$.

Since $\mathrm{tr}(s) = \mathrm{tr}(t)$ we have $a = a'$ and $d = d'$ and it moreover follows from Lem. 6.1 that $p = p'$. By induction hypothesis there is $u' \in \sigma$ such that $\mathrm{HS}(u') = (s', t')$. Hence we have

$$u' \quad = \quad * \to^* ((p, a, \gamma_\mathcal{A}) \ , \ (p, a, \gamma_\mathcal{B}))$$

We can then extend u' to

$$u \quad := \quad * \to^* ((p, a, \gamma_\mathcal{A}) \ , \ (p, a, \gamma_\mathcal{B})) \xrightarrow{\mathsf{O}} ((p, a, \gamma_\mathcal{A}) \ , \ (p \cdot d, q_\mathcal{B})) \xrightarrow{\mathsf{P}} ((p \cdot d, q_\mathcal{A}) \ , \ (p \cdot d, q_\mathcal{B}))$$

and we indeed have $\mathrm{HS}(u) = (s, t)$.

We now have to check that σ is indeed a strategy. P-prefix-closure follows from the definition. P-determinism follows from the fact that R is an iso. Consider P-plays of σ:

$$\begin{array}{l} u \;:\; * \to^* ((p, q_{\mathcal{A}})\;,\;(p, q_{\mathcal{B}})) \xrightarrow{\mathsf{O}} ((p, a, \gamma_{\mathcal{A}})\;,\;(p, q_{\mathcal{B}})) \xrightarrow{\mathsf{P}} ((p, a, \gamma_{\mathcal{A}})\;,\;(p, a, \gamma_{\mathcal{B}})) \\ u' \;:\; * \to^* ((p, q_{\mathcal{A}})\;,\;(p, q_{\mathcal{B}})) \xrightarrow{\mathsf{O}} ((p, a, \gamma_{\mathcal{A}})\;,\;(p, q_{\mathcal{B}})) \xrightarrow{\mathsf{P}} ((p, a, \gamma_{\mathcal{A}})\;,\;(p, a, \gamma'_{\mathcal{B}})) \end{array}$$

Then by construction we have $\mathrm{HS}(u) = \mathrm{HS}(u') \in R$. But $\mathrm{HS}(u) = (s, t)$ and $\mathrm{HS}(u') = (s', t')$ with $s = s'$, hence $t = t'$ since R is an iso. It follows that $u = u'$ by Lem. ii.(ii).

For plays:

$$\begin{array}{l} u \;:\; * \to^* ((p, a, \gamma_{\mathcal{A}})\;,\;(p, a, \gamma_{\mathcal{B}})) \xrightarrow{\mathsf{O}} ((p, a, \gamma_{\mathcal{A}})\;,\;(p \cdot d, q_{\mathcal{B}})) \xrightarrow{\mathsf{P}} ((p \cdot d, q_{\mathcal{A}})\;,\;(p \cdot d, q_{\mathcal{B}})) \\ u' \;:\; * \to^* ((p, a, \gamma_{\mathcal{A}})\;,\;(p, a, \gamma_{\mathcal{B}})) \xrightarrow{\mathsf{O}} ((p, a, \gamma_{\mathcal{A}})\;,\;(p \cdot d, q_{\mathcal{B}})) \xrightarrow{\mathsf{P}} ((p \cdot d, q_{\mathcal{A}})\;,\;(p \cdot d, q'_{\mathcal{B}})) \end{array}$$

Similarly as above, we have $\mathrm{HS}(u) = (s, t) \in R$ and $\mathrm{HS}(u') = (s', t') \in R$ with $t = t'$ hence $s = s'$ since R is an iso, from which it follows that $u = u'$.

We now show that σ is total. Let $u \in \sigma$ of the form

$$u \;:\; * \to^* ((p, q_{\mathcal{A}})\;,\;(p, q_{\mathcal{B}}))$$

and consider

$$u \xrightarrow{\mathsf{O}} ((p, a, \gamma_{\mathcal{A}})\;,\;(p, q_{\mathcal{B}}))$$

By construction, we have $\mathrm{HS}(u) = (s, t) \in R$. Since $t \to (p, a, \gamma_{\mathcal{A}}) \in \wp_{\Sigma}(\mathcal{A}, M)$ and since R is a synchronous iso, for some $\gamma_{\mathcal{B}} \in \delta_{\mathcal{B}}(q_{\mathcal{B}}, N(p)(a))$ we have

$$(s \to (p, a, \gamma_{\mathcal{A}})\;,\; t \to (p, a, \gamma_{\mathcal{B}})) \quad \in R = \mathrm{HS}(\sigma)$$

and follows that

$$u \xrightarrow{\mathsf{O}} ((p, a, \gamma_{\mathcal{A}})\;,\;(p, q_{\mathcal{B}})) \xrightarrow{\mathsf{P}} ((p, a, \gamma_{\mathcal{A}})\;,\;(p, a, \gamma_{\mathcal{B}})) \quad \in \sigma$$

Similarly, if

$$u \;:\; * \to^* ((p, a, \gamma_{\mathcal{A}})\;,\;(p, a, \gamma_{\mathcal{B}})) \quad \text{and} \quad u \xrightarrow{\mathsf{O}} ((p, a, \gamma_{\mathcal{A}})\;,\;(p \cdot d, q_{\mathcal{B}}))$$

then there is some $q_{\mathcal{A}}$ such that $(q_{\mathcal{A}}, d) \in \gamma_{\mathcal{A}}$ and

$$(s \to (p \cdot d, q_{\mathcal{A}})\;,\; t \to (p \cdot d, q_{\mathcal{B}})) \quad \in \quad R = \mathrm{HS}(\sigma)$$

hence

$$u \xrightarrow{\mathsf{O}} ((p, a, \gamma_{\mathcal{A}})\;,\;(p \cdot d, q_{\mathcal{B}})) \xrightarrow{\mathsf{P}} ((p \cdot d, d_{\mathcal{A}})\;,\;(p \cdot d, q_{\mathcal{B}})) \quad \in \sigma$$

Finally, we chek that σ is an isomorphism. First, since R is a morphism, reasoning as above we obtain from R^{-1} a strategy

$$\Sigma \vdash \sigma^{-1} : \mathcal{G}(\mathcal{B}, N) \multimap\!\!\circledast\; \mathcal{G}(\mathcal{A}, M)$$

such that $\mathrm{HS}(\sigma^{-1}) = R^{-1}$. Now, by functoriality of HS (Prop. 4.10), it follows that

$$\mathrm{HS}(\sigma \circ \sigma^{-1}) \quad = \quad 1_{\wp(\mathcal{A}, M) \xrightarrow{\mathrm{tr}} \mathrm{Tr}_{\Sigma}} \qquad \text{and} \qquad \mathrm{HS}(\sigma^{-1} \circ \sigma) \quad = \quad 1_{\wp(\mathcal{B}, N) \xrightarrow{\mathrm{tr}} \mathrm{Tr}_{\Sigma}}$$

and Prop. 4.11 together with (11) give

$$\sigma \circ \sigma^{-1} \quad = \quad 1_{\mathcal{G}(\mathcal{A}, M)} \qquad \text{and} \qquad \sigma^{-1} \circ \sigma \quad = \quad 1_{\mathcal{G}(\mathcal{B}, N)}$$

◀

7 Change-of-Base

In this Section, we discuss the basic machinery behind to substitution functors to be define in Sect. 8.

The main idea is that substitutions functors on strategies will be obtained from the effect of substitution on plays, which, thanks to the synchronous setting, can be expressed in a way very similar to usual change-of-base in slice categories.

Our notion of change-of-base on plays is obtained by pullbacks along maps on traces extending tree morphisms:

▶ **Definition 7.1** (Trace Lifting). Consider a tree morphism $L \in \mathbf{Tree}[\Sigma, \Gamma] = \mathbf{Tree}[\Sigma \to \Gamma]$, and define:

$$\mathrm{Tr}(L) \quad : \quad \mathrm{Tr}_\Sigma \longrightarrow \mathrm{Tr}_\Gamma$$

as

$$\begin{array}{rcl} \mathrm{Tr}(L)(\varepsilon) & := & \varepsilon \\ \mathrm{Tr}(L)(w \cdot a) & := & \mathrm{Tr}(L)(w) \cdot L(w_D)(a) \\ \mathrm{Tr}(L)(w \cdot d) & := & \mathrm{Tr}(L)(w) \cdot d \end{array}$$

On the other hand, write $L^\bullet$ for the change-of-base functor $\mathbf{Set}/\mathrm{Tr}_\Gamma \to \mathbf{Set}/\mathrm{Tr}_\Sigma$ induced by $\mathrm{Tr}(L)$.

Recall that $L^\bullet$ is given by pullbacks. In particular:

$$\begin{tikzcd} L^\bullet(\wp_\Gamma(\mathcal{A}, M)) \arrow[r] \arrow[d, "L^\bullet(\mathrm{tr})"'] \arrow[dr, phantom, "\lrcorner", very near start] & \wp_\Gamma(\mathcal{A}, M) \arrow[d, "\mathrm{tr}"] \\ \mathrm{Tr}_\Sigma \arrow[r, "\mathrm{Tr}(L)"] & \mathrm{Tr}_\Gamma \end{tikzcd}$$

Our notion of change-of-base will satisfy a similar property. In particuler we will get (Cor. 7.14):

$$L^\bullet(\wp_\Gamma(\mathcal{A}, M) \xrightarrow{\mathrm{tr}} \mathrm{Tr}_\Gamma) \quad \simeq_{\mathbf{Set}/\mathrm{Tr}_\Sigma} \quad \wp_\Sigma(\mathcal{A}, M \circ L) \xrightarrow{\mathrm{tr}} \mathrm{Tr}_\Sigma$$

We briefly come back on this point in Sect. 7.4.

7.1 Change-of-Base on Acceptance Games

Consider $\Delta \vdash \mathcal{A}$ and $M \in \mathbf{Tree}[\Gamma, \Delta]$, so that $\Gamma \vdash \mathcal{G}(\mathcal{A}, M)$. Given a morphism $L \in \mathbf{Tree}[\Sigma, \Gamma]$ as above (so that $\Sigma \vdash \mathcal{G}(\mathcal{A}, M \circ L)$), a move

$$(p, q) \quad \xrightarrow{\mathsf{P}} \quad (p, a, \gamma) \qquad \text{in } \Sigma \vdash \mathcal{G}(\mathcal{A}, M \circ L) \quad \text{with } \gamma \in \delta_\mathcal{A}(q, (M \circ L)(p)(a))$$

can be mapped to a move

$$(p, q) \quad \xrightarrow{\mathsf{P}} \quad (p, L(p)(a), \gamma) \qquad \text{in } \Gamma \vdash \mathcal{G}(\mathcal{A}, M) \quad \text{with } \gamma \in \delta_\mathcal{A}(q, M(p)(L(p)(a)))$$

Similarly, a move

$$(p, a, \gamma) \quad \xrightarrow{\mathsf{O}} \quad (p.d, q) \qquad \text{in } \Sigma \vdash \mathcal{G}(\mathcal{A}, M \circ L) \quad \text{with } (q, d) \in \gamma$$

can be mapped to a move

$$(p, L(p)(a), \gamma) \quad \xrightarrow{\mathsf{O}} \quad (p.d, q) \qquad \text{in } \Gamma \vdash \mathcal{G}(\mathcal{A}, M) \quad \text{with } (q, d) \in \gamma$$

We homomorphically extend this to a map

$$\wp(L) \quad : \quad \wp_\Sigma(\mathcal{A}, M \circ L) \quad \longrightarrow \quad \wp_\Gamma(\mathcal{A}, M)$$

The map $\wp(L)$ is formally defined as:

$$\begin{array}{rcl} \wp(L)(\varepsilon) & := & \varepsilon \\ \wp(L)(s \to (p, q)) & := & \wp(L)(s) \to (p, q) \\ \wp(L)(s \to (p, a, \gamma)) & := & \wp(L)(s) \to (p, L(p)(a), \gamma) \end{array}$$

▶ Remark. For the correctness of the last case, remember that in $\Sigma \vdash \mathcal{G}(\mathcal{A}, M \circ L)$ and $\Gamma \vdash \mathcal{G}(\mathcal{A}, M)$, P-moves are respectively of the form:

$$\begin{array}{ccccl} (p, q) & \xrightarrow{\mathsf{P}} & (p, a, \gamma) & \text{iff} & \gamma \in \delta_{\mathcal{A}}(q, (M \circ L)(p)(a)) \\ (p, q) & \xrightarrow{\mathsf{P}} & (p, b, \gamma) & \text{iff} & \gamma \in \delta_{\mathcal{A}}(q, M(p)(b)) \end{array}$$

where $(M \circ L)(p)(a) = M(p)(L(p)(a))$.

In this Section, we elaborate on the connection between the map

$$\wp(L) \quad : \quad \wp_\Sigma(\mathcal{A}, M \circ L) \quad \longrightarrow \quad \wp_\Gamma(\mathcal{A}, M)$$

and the usual change-of-base in $\mathbf{Set}^{\to}$:

$$L^\bullet : \mathbf{Set}/\mathrm{Tr}_\Gamma \longrightarrow \mathbf{Set}/\mathrm{Tr}_\Sigma$$

defined by pullbacks, which in particular satisfies:

$$\begin{tikzcd} L^\bullet(\wp_\Gamma(\mathcal{A}, M)) \arrow[r] \arrow[d, "L^\bullet(\mathrm{tr})"'] \arrow[dr, phantom, "\lrcorner", very near start] & \wp_\Gamma(\mathcal{A}, M) \arrow[d, "\mathrm{tr}"] \\ \mathrm{Tr}_\Sigma \arrow[r, "\mathrm{Tr}(L)"] & \mathrm{Tr}_\Gamma \end{tikzcd}$$

A crucial property, given by Prop. 7.7 is the following pullback:

$$\begin{tikzcd} \wp_\Sigma(\mathcal{A}, M \circ L) \arrow[r, "\wp(L)"] \arrow[d, "\mathrm{tr}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \wp_\Gamma(\mathcal{A}, M) \arrow[d, "\mathrm{tr}"] \\ \mathrm{Tr}_\Sigma \arrow[r, "\mathrm{Tr}(L)"] & \mathrm{Tr}_\Gamma \end{tikzcd}$$

This leads in particluar to the lifting property of Lem. 7.12, which is crucial for the functoriality of substitution (Prop. 8.4).

7.2 Change-of-Base on the Synchronous Arrow

We now extend the map $\wp(L)$ on Acceptance Games to a map $\wp(L)_{\multimap\circledast}$ on Synchronous Arrow Games.

Given $L \in \mathbf{Tree}[\Sigma, \Gamma]$, the map

$$\wp(L)_{\multimap\circledast} \quad : \quad \wp_\Sigma(\mathcal{G}(\mathcal{A}, M \circ L) \multimap\circledast \mathcal{G}(\mathcal{B}, N \circ L)) \quad \longrightarrow \quad \wp_\Gamma(\mathcal{G}(\mathcal{A}, M) \multimap\circledast \mathcal{G}(\mathcal{B}, N))$$

is defined inductivelly as follows:

$$\begin{array}{rcl}
\wp(L)_{\multimap\circledast}(\varepsilon : *) & := & \varepsilon \\
\wp(L)_{\multimap\circledast}(s \to ((p, a, \gamma_\mathcal{A}), (p, q_\mathcal{B}))) & := & \\
& & \wp(L)_{\multimap\circledast}(s) \to ((p, L(p)(a), \gamma_\mathcal{A}), (p, q_\mathcal{B})) \\
\wp(L)_{\multimap\circledast}(s \to ((p, a, \gamma_\mathcal{A}), (p, a, \gamma_\mathcal{B}))) & := & \\
& & \wp(L)_{\multimap\circledast}(s) \to ((p, L(p)(a), \gamma_\mathcal{A}), (p, L(p)(a), \gamma_\mathcal{B})) \\
\wp(L)_{\multimap\circledast}(s \to ((p, a, \gamma_\mathcal{A}), (p.d, q_\mathcal{B}))) & := & \\
& & \wp(L)_{\multimap\circledast}(s) \to ((p, L(p)(a), \gamma_\mathcal{A}), (p.d, q_\mathcal{B})) \\
\wp(L)_{\multimap\circledast}(s \to ((p.d, q_\mathcal{A}), (p.d, q_\mathcal{B}))) & := & \wp(L)_{\multimap\circledast}(s) \to ((p.d, q_\mathcal{A}), (p.d, q_\mathcal{B}))
\end{array}$$

▶ **Lemma 7.2.**

$$\begin{tikzcd}
\wp_\Sigma(\mathcal{G}(\mathcal{A}, M \circ L) \multimap\circledast \mathcal{G}(\mathcal{B}, N \circ L)) \arrow[r, "\wp(L)_{\multimap\circledast}"] \arrow[d, "\mathrm{HS}"'] & \wp_\Gamma(\mathcal{G}(\mathcal{A}, M) \multimap\circledast \mathcal{G}(\mathcal{B}, N)) \arrow[d, "\mathrm{HS}"] \\
\wp_\Sigma(\mathcal{A}, M \circ L) \times \wp_\Sigma(\mathcal{B}, N \circ L) \arrow[r, "\wp(L) \times \wp(L)"] & \wp_\Gamma(\mathcal{A}, M) \times \wp_\Sigma(\mathcal{B}, N)
\end{tikzcd}$$

Proof. Let $s \in \wp_\Sigma(\mathcal{G}(\mathcal{A}, M \circ L) \multimap\circledast \mathcal{G}(\mathcal{B}, N \circ L))$ be the sequence:

	$\mathcal{G}(\mathcal{A}, M \circ L)$	$\multimap\circledast$	$\mathcal{G}(\mathcal{B}, N \circ L)$	where
	$((p, q_\mathcal{A})$	,	$(p, q_\mathcal{B}))$	
O	$\downarrow$			
	$((p, a, \gamma_\mathcal{A})$	,	$(p, q_\mathcal{B}))$	$\gamma_\mathcal{A} \in \delta_\mathcal{A}(q_\mathcal{A}, (M \circ L)(p)(a))$
P			$\downarrow$	
	$((p, a, \gamma_\mathcal{A})$	,	$(p, a, \gamma_\mathcal{B}))$	$\gamma_\mathcal{B} \in \delta_\mathcal{B}(q_\mathcal{B}, (N \circ L)(p)(a))$
O			$\downarrow$	
	$((p, a, \gamma_\mathcal{A})$	,	$(p.d, q'_\mathcal{B}))$	$(q'_\mathcal{B}, d) \in \gamma_\mathcal{B}$
P	$\downarrow$			
	$((p.d, q'_\mathcal{A})$	,	$(p.d, q'_\mathcal{B}))$	$(q'_\mathcal{A}, d) \in \gamma_\mathcal{A}$

We have $\mathrm{HS}(s)$ is the pair

$$((p, q_\mathcal{A}) \to (p, a, \gamma_\mathcal{A}) \to (p.d, q'_\mathcal{A}),\ (p, q_\mathcal{B}) \to (p, a, \gamma_\mathcal{B}) \to (p.d, q'_\mathcal{B}))$$

where

$$\gamma_\mathcal{A} \in \delta_\mathcal{A}(q_\mathcal{A}, (M \circ L)(p)(a)) \quad \text{and} \quad \gamma_\mathcal{B} \in \delta_\mathcal{B}(q_\mathcal{B}, (N \circ L)(p)(a)) \quad \text{and} \quad (q'_\mathcal{A}, d) \in \gamma_\mathcal{A} \quad \text{and} \quad (q'_\mathcal{B}, d) \in \gamma_\mathcal{B}$$

It follows that $(\wp(L) \times \wp(L)) \circ \mathrm{HS}(s)$ is

$$((p, q_{\mathcal{A}}) \to (p, L(p)(a), \gamma_{\mathcal{A}}) \to (p.d, q'_{\mathcal{A}}),\ (p, q_{\mathcal{B}}) \to (p, L(p)(a), \gamma_{\mathcal{B}}) \to (p.d, q'_{\mathcal{B}}))$$

where

$$\gamma_{\mathcal{A}} \in \delta_{\mathcal{A}}(q_{\mathcal{A}}, M(p)(L(p)(a))) \quad \text{and} \quad \gamma_{\mathcal{B}} \in \delta_{\mathcal{B}}(q_{\mathcal{B}}, N(p)(L(p)(a))) \quad \text{and} \quad (q'_{\mathcal{A}}, d) \in \gamma_{\mathcal{A}} \quad \text{and} \quad (q'_{\mathcal{B}}, d) \in \gamma_{\mathcal{B}}$$

On the other hand, $\wp(L)_{-\circledast}(s)$ is

	$\mathcal{G}(\mathcal{A}, M)$	$-\!\circledast$	$\mathcal{G}(\mathcal{B}, N)$	where
	$((p, q_{\mathcal{A}})$	,	$(p, q_{\mathcal{B}}))$	
O	$\downarrow$			
	$((p, L(p)(a), \gamma_{\mathcal{A}})$	,	$(p, q_{\mathcal{B}}))$	$\gamma_{\mathcal{A}} \in \delta_{\mathcal{A}}(q_{\mathcal{A}}, M(p)(L(p)(a)))$
P			$\downarrow$	
	$((p, L(p)(a), \gamma_{\mathcal{A}})$	,	$(p, L(p)(a), \gamma_{\mathcal{B}}))$	$\gamma_{\mathcal{B}} \in \delta_{\mathcal{B}}(q_{\mathcal{B}}, N(p)(L(p)(a)))$
O			$\downarrow$	
	$((p, L(p)(a), \gamma_{\mathcal{A}})$	,	$(p.d, q'_{\mathcal{B}}))$	$(q'_{\mathcal{B}}, d) \in \gamma_{\mathcal{B}}$
P	$\downarrow$			
	$((p.d, q'_{\mathcal{A}})$	,	$(p.d, q'_{\mathcal{B}}))$	$(q'_{\mathcal{A}}, d) \in \gamma_{\mathcal{A}}$

It follows that we indeed have

$$\mathrm{HS} \circ \wp(L)_{-\circledast}(s) \quad = \quad (\wp(L) \times \wp(L)) \circ \mathrm{HS}(s)$$

and same holds for prefixes of s. ◀

7.3 Universal Properties of Change-of-Base on Plays

7.3.1 Acceptance Games.

▶ **Lemma 7.3.**

$$\begin{tikzcd}
\wp_{\Sigma}(\mathcal{A}, M \circ L) \arrow[rr, "\wp(L)"] \arrow[dr, "\mathrm{tr}_D"'] & & \wp_{\Gamma}(\mathcal{A}, M) \arrow[dl, "\mathrm{tr}_D"] \\
& D^* &
\end{tikzcd}$$

Proof. By induction on $s \in \wp_{\Sigma}(\mathcal{A}, M \circ L)$. In the base case $s = \varepsilon : *$, we have $\wp(L)(s) = s = \varepsilon$ and we are done.

For the induction step, we distinguish two cases:

- If $s = s' \to (p, a, \gamma)$, then $\wp(L)(s) = \wp(L)(s') \to (p, L(p)(a), \gamma)$. By induction hypothesis we have

  $$\mathrm{tr}_D(s') \quad = \quad \mathrm{tr}_D(\wp(L)(s'))$$

  and we are done since

  $$\mathrm{tr}_D(s) \quad = \quad \mathrm{tr}_D(s') \qquad \text{and} \qquad \mathrm{tr}_D(\wp(L)(s)) \quad = \quad \mathrm{tr}_D(\wp(L)(s'))$$

- Otherwise, $s = s' \to (p.d, q)$. Then $\wp(L)(s) = \wp(L)(s') \to (p.d, q)$. By induction hypothesis we have

  $$\mathrm{tr}_D(s') \quad = \quad \mathrm{tr}_D(\wp(L)(s'))$$

and we are done since

$$\mathrm{tr}_D(s) \quad = \quad \mathrm{tr}_D(s') \cdot d \qquad \text{and} \qquad \mathrm{tr}_D(\wp(L)(s)) \quad = \quad \mathrm{tr}_D(\wp(L)(s')) \cdot d$$

◀

▶ **Lemma 7.4.**

$$\begin{tikzcd} \mathrm{Tr}_\Sigma \arrow[rr, "\mathrm{Tr}(L)"] \arrow[dr, "(-)_D"'] & & \mathrm{Tr}_\Gamma \arrow[dl, "(-)_D"] \\ & D^* & \end{tikzcd}$$

Proof. By induction on $t \in \mathrm{Tr}_\Sigma$.

In the base case, we have $t = \varepsilon$ and we are done since

$$(\varepsilon)_D \quad = \quad \varepsilon \quad = \quad \mathrm{Tr}(L)(\varepsilon)_D$$

For the induction step, there are two cases.

Consider first the case of $t = t' \cdot a$ for some $a \in \Sigma$. In this case we are done by induction hypothesis since

$$(t' \cdot a)_D \quad = \quad (t')_D \quad = \quad (\mathrm{Tr}(L)(t'))_D \quad = \quad (\mathrm{Tr}(L)(t') \cdot L((t')_D)(a))_D \quad = \quad (\mathrm{Tr}(L)(t' \cdot a))_D$$

The other case is when $t = t' \cdot d$ with $d \in D$. In this case, we are also done by induction hypothesis since:

$$(t' \cdot d)_D \quad = \quad (t')_D \cdot d \quad = \quad (\mathrm{Tr}(L)(t')_D \cdot d \quad = \quad (\mathrm{Tr}(L)(t' \cdot d))_D$$

◀

▶ **Lemma 7.5.**

$$\begin{tikzcd} \wp_\Sigma(\mathcal{A}, M \circ L) \arrow[r, "\wp(L)"] \arrow[d, "\mathrm{tr}"'] & \wp_\Gamma(\mathcal{A}, M) \arrow[d, "\mathrm{tr}"] \\ \mathrm{Tr}_\Sigma \arrow[r, "\mathrm{Tr}(L)"] & \mathrm{Tr}_\Gamma \end{tikzcd}$$

Proof. We reason by induction on $s \in \wp_\Sigma(\mathcal{A}, M \circ L)$.

In the base case, $s = \varepsilon : *$ and we are done since

$$\mathrm{Tr}(L)(\mathrm{tr}(\varepsilon)) \quad = \quad \varepsilon \quad = \quad \mathrm{tr}(\wp(L)(\varepsilon))$$

For the induction step, there are two cases:

- If $s = s' \to (p, a, \gamma)$ the we have

$$\mathrm{Tr}(L)(\mathrm{tr}(s)) \quad = \quad \mathrm{Tr}(L)(\mathrm{tr}(s')) \cdot L(\mathrm{tr}_D(s))(a)$$

(recall that $\mathrm{tr}_D(s) = \mathrm{tr}(s)_D$) and

$$\mathrm{tr}(\wp(L)(s)) \quad = \quad \mathrm{tr}(\wp(L)(s')) \cdot L(p)(a)$$

Now we are done since by induction hypothsis

$$\mathrm{Tr}(L)(\mathrm{tr}(s')) \quad = \quad \mathrm{tr}(\wp(L)(s'))$$

and by Lem. 6.1 we have

$$\mathrm{tr}_D(s) = p$$

- Otherwise $s = s' \to (p.d, q)$ and we have

  $$\mathrm{Tr}(L)(\mathrm{tr}(s)) \quad = \quad \mathrm{Tr}(L)(\mathrm{tr}(s')) \cdot d$$

  and

  $$\mathrm{tr}(\wp(L)(s)) \quad = \quad \mathrm{tr}(\wp(L)(s')) \cdot d$$

  and we are done by induction hypothesis. ◀

▶ **Lemma 7.6.** *The following map is a bijection:*

$$\langle \mathrm{tr}, \wp(L) \rangle \quad : \quad \wp_\Sigma(\mathcal{A}, M \circ L) \quad \longrightarrow \quad \mathrm{Tr}_\Sigma \times_{\mathrm{Tr}_\Gamma} \wp_\Gamma(\mathcal{A}, M)$$

where $\mathrm{Tr}_\Sigma \times_{\mathrm{Tr}_\Gamma} \wp_\Gamma(\mathcal{A}, M)$ *is*

$$\{(w, s) \in \mathrm{Tr}_\Sigma \times \wp_\Gamma(\mathcal{A}, M) \mid \mathrm{Tr}(L)(w) = \mathrm{tr}(s)\} \quad \subseteq \quad \mathrm{Tr}_\Sigma \times \wp_\Gamma(\mathcal{A}, M)$$

▶ Remark (On the injectivity of $\langle \mathrm{tr}, \wp(L) \rangle$). Note that to get the injectivity of

$$\langle \mathrm{tr}, \wp(L) \rangle \quad : \quad \wp_\Sigma(\mathcal{A}, M \circ L) \quad \longrightarrow \quad \mathrm{Tr}_\Sigma \times \wp_\Gamma(\mathcal{A}, M)$$

the synchronization by traces (*i.e.* given by the first component tr of the pair $\langle \mathrm{tr}, \wp(L) \rangle$) is required, since the tree map $L \in \mathbf{Tree}[\Sigma, \Gamma]$ (hence the map $\wp(L)$) is not required to be injective.

Proof. We first show the injectivity of

$$\langle \mathrm{tr}, \wp(L) \rangle \quad : \quad \wp_\Sigma(\mathcal{A}, M \circ L) \quad \longrightarrow \quad \mathrm{Tr}_\Sigma \times \wp_\Gamma(\mathcal{A}, M)$$

that is, for all $s, t \in \wp_\Sigma(\mathcal{A}, M \circ L)$,

$$\langle \mathrm{tr}, \wp(L) \rangle(s) \quad = \quad \langle \mathrm{tr}, \wp(L) \rangle(t) \qquad \Longrightarrow \qquad s = t$$

First note that since $\wp(L)$ is length-preserving, we can w.l.o.g. assume $|s| = |t|$. We reason by induction on $n = |s| = |t|$. If $n = 0$, then $s = t = \varepsilon$ and we are done.

For the inductive step, note that since games in $\mathbf{SAG}_\Sigma$ and $\mathbf{SAG}_\Gamma$ are positive and alternating, the plays s and t must end with the same kind of move. There are two cases:

- Assume $s = s' \to (p, a, \gamma)$ and $t = t' \to (p', a', \gamma')$.
  Since $\wp(L)(s) = \wp(L)(t)$, we have

  $$\wp(L)(s') = \wp(L)(t') \quad \text{and} \quad p = p' \quad \text{and} \quad \gamma = \gamma'$$

  Moreover, since $\mathrm{tr}(s) = \mathrm{tr}(t)$, we get $\mathrm{tr}(s') = \mathrm{tr}(t')$ and $a = a'$ and we conclude by induction hypothesis.
- Otherwise $s = s' \to (p, q)$ and $t = t' \to (p', q')$.
  Since $\wp(L)(s) = \wp(L)(t)$, we have

  $$\wp(L)(s') = \wp(L)(t') \quad \text{and} \quad p = p' \quad \text{and} \quad q = q'$$

  Moreover, since $\mathrm{tr}(s) = \mathrm{tr}(t)$, we get $\mathrm{tr}(s') = \mathrm{tr}(t')$ and we conclude by induction hypothesis.

We now show the surjectivity of

$$\langle \mathrm{tr}, \wp(L) \rangle \quad : \quad \wp_\Sigma(\mathcal{A}, M \circ L) \quad \longrightarrow \quad \mathrm{Tr}_\Sigma \times_{\mathrm{Tr}_\Gamma} \wp_\Gamma(\mathcal{A}, M) \subseteq \mathrm{Tr}_\Sigma \times \wp_\Gamma(\mathcal{A}, M)$$

By induction on $s \in \wp_\Gamma(\mathcal{A}, M)$ we show that

- For all $w \in \mathrm{Tr}_\Sigma$ such that $\mathrm{Tr}(L)(w) = \mathrm{tr}(s)$, there is a play $t \in \wp_\Sigma(\mathcal{A}, M \circ L)$ such that $\mathrm{tr}(t) = w$, $\wp(L)(t) = s$ and moreover, the projections of s and t on $Q_\mathcal{A}^*$ and on $\mathcal{P}(Q_\mathcal{A} \times D)^*$ coincide.

In the base case $s = \varepsilon : *$, we must have $w = \varepsilon$ and we take $t = \varepsilon$. For the induction step, we consider two cases

- Assume $s = s' \to (p, b, \gamma)$.
  Given $w \in \mathrm{Tr}_\Sigma$ such that $\mathrm{Tr}(L)(w) = \mathrm{tr}(s)$, we must have $w = w' \cdot a$ and $\mathrm{Tr}(L)(w') = \mathrm{tr}(s')$ and $L(w'_D)(a) = b$.
  Let t' be obtained by applying the induction hypothesis on s' and w', so that $\mathrm{tr}(t') = s'$ and $\wp(L)(t') = s'$ and s' and t' have the same projection on $Q_\mathcal{A}^*$ and on $\mathcal{P}(Q_\mathcal{A} \times D)^*$. Take

  $$t \quad := \quad t' \to (p, a, \gamma)$$

  Note that s and t have the same projection on $Q_\mathcal{A}^*$ and $\mathcal{P}(Q_\mathcal{A} \times D)^*$. It remains to check that t is legal play on $\mathcal{G}(\mathcal{A}, M \circ L)$. Let q be the last state of t' and s'. First, note that $\gamma \in \delta(q, M(p)(b))$ by assumption. Since $\mathrm{tr}_D(s) = p$ by Lem. 6.1, and $w'_D = w_D = \mathrm{Tr}(L)(w)_D$ by Lem. 7.4, it follows from $\mathrm{Tr}(L)(w) = \mathrm{tr}(s)$ that $p = w'_D$. We thus get $\gamma \in \delta(q, (M \circ L)(p)(a))$.
- Otherwise, $s = s' \to (p.d, q)$.
  Given $w \in \mathrm{Tr}_\Sigma$ such that $\mathrm{Tr}(L)(w) = \mathrm{tr}(s)$, we must have $w = w' \cdot d$. and $\mathrm{Tr}(L)(w') = \mathrm{tr}(s')$.
  Let t' be obtained by applying the induction hypothesis on s' and w', so that $\mathrm{tr}(t') = s'$ and $\wp(L)(t') = s'$ and s' and t' have the same projection on $Q_\mathcal{A}^*$ and on $\mathcal{P}(Q_\mathcal{A} \times D)^*$. Take

  $$t \quad := \quad t' \to (p.d, q)$$

  which is legal since t' and s' have the same projection on $\mathcal{P}(Q_\mathcal{A} \times D)^*$. Moreover, s and t have the same projection on $Q_\mathcal{A}^*$ and on $\mathcal{P}(Q_\mathcal{A} \times D)^*$. ◀

▶ **Proposition 7.7.**

$$\begin{tikzcd}
\wp_\Sigma(\mathcal{A}, M \circ L) \arrow[r, "\wp(L)"] \arrow[d, "\mathrm{tr}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \wp_\Gamma(\mathcal{A}, M) \arrow[d, "\mathrm{tr}"] \\
\mathrm{Tr}_\Sigma \arrow[r, "\mathrm{Tr}(L)"] & \mathrm{Tr}_\Gamma
\end{tikzcd}$$

Proof. Commutation of the diagram is ensured by Lem. 7.5 and Lem. 7.6 gives the isomorphism in **Set**:

$$\wp_\Sigma(\mathcal{A}, M \circ L) \quad \simeq \quad \mathrm{Tr}_\Sigma \times_{\mathrm{Tr}_\Gamma} \wp_\Gamma(\mathcal{A}, M)$$

◀

▶ Remark (On the Definition of L^* and Tr_Σ). It is not clear wether it is interesting to extend $\mathrm{Tr}(L) : \mathrm{Tr}_\Sigma \to \mathrm{Tr}_\Gamma$ to

$$L^\star \quad : \quad D^* \times \Sigma^* \longrightarrow D^* \times \Gamma^*$$

so that the following is a pullback:

$$\begin{tikzcd}
\wp_\Sigma(\mathcal{A}, M \circ L) \arrow[r, "\wp(L)"] \arrow[d, "\mathrm{tr}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \wp_\Gamma(\mathcal{A}, M) \arrow[d, "\mathrm{tr}"] \\
D^* \times \Sigma^* \arrow[r, "L^*"] & D^* \times \Gamma^*
\end{tikzcd}$$

7.3.2 Synchronous Arrow.

We are now going to see that the pullback property of Prop. 7.7 extends to the synchronous arrow, in the sense that in **Set**:

$$\begin{tikzcd}
\wp^{\mathsf{P}}_\Sigma(\mathcal{G}(\mathcal{A}, M \circ L) \multimap\!\!\circledast\ \mathcal{G}(\mathcal{B}, N \circ L)) \arrow[r, "\wp(L)_{\multimap\circledast}"] \arrow[d, "\mathrm{tr}^{\multimap\circledast}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \wp^{\mathsf{P}}_\Gamma(\mathcal{G}(\mathcal{A}, M) \multimap\!\!\circledast\ \mathcal{G}(\mathcal{B}, N)) \arrow[d, "\mathrm{tr}^{\multimap\circledast}"] \\
\mathrm{Tr}_\Sigma \arrow[r, "\mathrm{Tr}(L)"] & \mathrm{Tr}_\Gamma
\end{tikzcd}$$

where the map

$$\wp^{\mathsf{P}}_\Gamma(\mathcal{G}(\mathcal{A}, M) \multimap\!\!\circledast\ \mathcal{G}(\mathcal{B}, N)) \quad \xrightarrow{\mathrm{tr}^{\multimap\circledast}} \quad \mathrm{Tr}_\Gamma$$

is defined in Def. 6.9.

This property (actually Lem. 7.9, see (13) below) will lead to Lem. 7.12 and Cor. 7.13 which are crucial for the functoriality of substitution (Prop. 8.4).

We will use the pullback lemma (see e.g. [12, Exercise 1.5.5, p. 30]) *via* the two following properties:

$$\begin{tikzcd}
\wp^{\mathsf{P}}_\Sigma(\mathcal{G}(\mathcal{A}, M \circ L) \multimap\!\!\circledast\ \mathcal{G}(\mathcal{B}, N \circ L)) \arrow[r, "\wp(L)_{\multimap\circledast}"] \arrow[d, "\mathrm{HS}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \wp^{\mathsf{P}}_\Gamma(\mathcal{G}(\mathcal{A}, M) \multimap\!\!\circledast\ \mathcal{G}(\mathcal{B}, N)) \arrow[d, "\mathrm{HS}"] \\
\wp_\Sigma(\mathcal{A}, M \circ L) \times_{\mathrm{Tr}_\Sigma} \wp_\Sigma(\mathcal{B}, N \circ L) \arrow[r, "\wp(L) \times \wp(L)"] & \wp_\Gamma(\mathcal{A}, M) \times_{\mathrm{Tr}_\Gamma} \wp_\Gamma(\mathcal{B}, N)
\end{tikzcd} \tag{12}$$

and

$$\begin{tikzcd}
\wp_\Sigma(\mathcal{A}, M \circ L) \times_{\mathrm{Tr}_\Sigma} \wp_\Sigma(\mathcal{B}, N \circ L) \arrow[r, "\wp(L) \times \wp(L)"] \arrow[d, "\mathrm{tr}^{\multimap\circledast}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \wp_\Gamma(\mathcal{A}, M) \times_{\mathrm{Tr}_\Gamma} \wp_\Gamma(\mathcal{B}, N) \arrow[d, "\mathrm{tr}^{\multimap\circledast}"] \\
\mathrm{Tr}_\Sigma \arrow[r, "\mathrm{Tr}(L)"] & \mathrm{Tr}_\Gamma
\end{tikzcd} \tag{13}$$

Property (12) will be shown in Lem. 7.10. As for (13), first note that

▶ **Lemma 7.8.** *In* **Set***,*

$$\begin{tikzcd}
\wp_\Sigma(\mathcal{A}, M \circ L) \times \wp_\Sigma(\mathcal{B}, N \circ L) \arrow[r, "\wp(L) \times \wp(L)"] \arrow[d, "\mathrm{tr} \times \mathrm{tr}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \wp_\Gamma(\mathcal{A}, M) \times \wp_\Gamma(\mathcal{B}, N) \arrow[d, "\mathrm{tr} \times \mathrm{tr}"] \\
\mathrm{Tr}_\Sigma \times \mathrm{Tr}_\Sigma \arrow[r, "\mathrm{Tr}(L) \times \mathrm{Tr}(L)"] & \mathrm{Tr}_\Gamma \times \mathrm{Tr}_\Gamma
\end{tikzcd}$$

Proof. First, by Prop. 7.7 we have

$$\begin{tikzcd} \wp_\Sigma(\mathcal{A}, M \circ L) \arrow[r, "\wp(L)"] \arrow[d, "\mathrm{tr}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \wp_\Gamma(\mathcal{A}, M) \arrow[d, "\mathrm{tr}"] \\ \mathrm{Tr}_\Sigma \arrow[r, "\mathrm{Tr}(L)"] & \mathrm{Tr}_\Gamma \end{tikzcd}$$

and

$$\begin{tikzcd} \wp_\Sigma(\mathcal{B}, N \circ L) \arrow[r, "\wp(L)"] \arrow[d, "\mathrm{tr}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \wp_\Gamma(\mathcal{B}, N) \arrow[d, "\mathrm{tr}"] \\ \mathrm{Tr}_\Sigma \arrow[r, "\mathrm{Tr}(L)"] & \mathrm{Tr}_\Gamma \end{tikzcd}$$

Since limits commute (see e.g. [13, Sect. IX.2 & IX.8]) we have

$$\begin{tikzcd} A \times A' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \times B' \arrow[d] \\ C \times C' \arrow[r] & D \times D' \end{tikzcd} \quad \text{whenever} \quad \begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B' \arrow[d] \\ C' \arrow[r] & D' \end{tikzcd}$$

and we get

$$\begin{tikzcd} \wp_\Sigma(\mathcal{A}, M \circ L) \times \wp_\Sigma(\mathcal{B}, N \circ L) \arrow[r, "\wp(L) \times \wp(L)"] \arrow[d, "\mathrm{tr} \times \mathrm{tr}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \wp_\Gamma(\mathcal{A}, M) \times \wp_\Gamma(\mathcal{B}, N) \arrow[d, "\mathrm{tr} \times \mathrm{tr}"] \\ \mathrm{Tr}_\Sigma \times \mathrm{Tr}_\Sigma \arrow[r, "\mathrm{Tr}(L) \times \mathrm{Tr}(L)"] & \mathrm{Tr}_\Gamma \times \mathrm{Tr}_\Gamma \end{tikzcd}$$

◀

▶ **Lemma 7.9.** *In* **Set***,*

$$\begin{tikzcd} \wp_\Sigma(\mathcal{A}, M \circ L) \times_{\mathrm{Tr}_\Sigma} \wp_\Sigma(\mathcal{B}, N \circ L) \arrow[r, "\wp(L) \times \wp(L)"] \arrow[d, "\mathrm{tr}^{-\circledast}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \wp_\Gamma(\mathcal{A}, M) \times_{\mathrm{Tr}_\Gamma} \wp_\Gamma(\mathcal{B}, N) \arrow[d, "\mathrm{tr}^{-\circledast}"] \\ \mathrm{Tr}_\Sigma \arrow[r, "\mathrm{Tr}(L)"] & \mathrm{Tr}_\Gamma \end{tikzcd}$$

Proof. We show

$$\begin{tikzcd} \wp_\Sigma(\mathcal{A}, M \circ L) \times_{\mathrm{Tr}_\Sigma} \wp_\Sigma(\mathcal{B}, N \circ L) \arrow[r, "\wp(L) \times \wp(L)"] \arrow[d, "\mathrm{tr} \times \mathrm{tr}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \wp_\Gamma(\mathcal{A}, M) \times_{\mathrm{Tr}_\Gamma} \wp_\Gamma(\mathcal{B}, N) \arrow[d, "\mathrm{tr} \times \mathrm{tr}"] \\ \mathrm{Tr}_\Sigma \times_{\mathrm{Tr}_\Sigma} \mathrm{Tr}_\Sigma \arrow[r, "\mathrm{Tr}(L) \times \mathrm{Tr}(L)"] & \mathrm{Tr}_\Gamma \times_{\mathrm{Tr}_\Gamma} \mathrm{Tr}_\Gamma \end{tikzcd}$$

and then conclude by definition of the maps

$$\begin{array}{rrcl} & \wp^{\mathsf{P}}_\Gamma(\mathcal{G}(\mathcal{A}, M) -\!\circledast\, \mathcal{G}(\mathcal{B}, N)) & \xrightarrow{\mathrm{tr}^{-\circledast}} & \mathrm{Tr}_\Gamma \\ \text{and} & \wp^{\mathsf{P}}_\Sigma(\mathcal{G}(\mathcal{A}, M \circ L) -\!\circledast\, \mathcal{G}(\mathcal{B}, N \circ L)) & \xrightarrow{\mathrm{tr}^{-\circledast}} & \mathrm{Tr}_\Sigma \end{array}$$

We first check the commutation of the diagram. Given

$(s, t) \in \wp_\Sigma(\mathcal{A}, M \circ L) \times_{\mathrm{Tr}_\Sigma} \wp_\Sigma(\mathcal{B}, N \circ L)$

since $\mathrm{tr}(s) = \mathrm{tr}(t)$ we have

$$\mathrm{Tr}(L) \circ \mathrm{tr}(s) \quad = \quad \mathrm{Tr}(L) \circ \mathrm{tr}(t)$$

and by Lem. 7.5 we have

$$\mathrm{tr} \circ \wp(L)(s) \quad = \quad \mathrm{tr} \circ \wp(L)(t)$$

hence

$$(\wp(L)(s),\ \wp(L)(t)) \in \wp_\Gamma(\mathcal{A}, M) \times_{\mathrm{Tr}_\Gamma} \wp_\Gamma(\mathcal{B}, N)$$

For the pullback property, we show that $\langle (\mathrm{tr} \times \mathrm{tr}), \wp(L) \times \wp(L) \rangle$ is a bijection from

$$\wp_\Sigma(\mathcal{A}, M \circ L) \times_{\mathrm{Tr}_\Sigma} \wp_\Sigma(\mathcal{B}, N \circ L)$$

to

$$(\mathrm{Tr}_\Sigma \times_{\mathrm{Tr}_\Sigma} \mathrm{Tr}_\Sigma) \times_{\mathrm{Tr}_\Gamma \times_{\mathrm{Tr}_\Gamma} \mathrm{Tr}_\Gamma} (\wp_\Gamma(\mathcal{A}, M) \times_{\mathrm{Tr}_\Gamma} \wp_\Gamma(\mathcal{B}, N))$$

Injectivity follows from the pullback property of Lem. 7.8. As for surjectivity, given

$$(w, w', s, t) \in (\mathrm{Tr}_\Sigma \times_{\mathrm{Tr}_\Sigma} \mathrm{Tr}_\Sigma) \times_{\mathrm{Tr}_\Gamma \times_{\mathrm{Tr}_\Gamma} \mathrm{Tr}_\Gamma} (\wp_\Gamma(\mathcal{A}, M) \times_{\mathrm{Tr}_\Gamma} \wp_\Gamma(\mathcal{B}, N))$$

by the pullback property of Lem. 7.8 there is some

$$(u, v) \in \wp_\Sigma(\mathcal{A}, M \circ L) \times \wp_\Sigma(\mathcal{B}, N \circ L)$$

such that

$$\mathrm{tr}(u) = w \quad \mathrm{tr}(v) = w' \quad \wp(L)(u) = s \quad \wp(L)(v) = t$$

But $w = w'$ since $(w, w') \in \mathrm{Tr}_\Sigma \times_{\mathrm{Tr}_\Sigma} \mathrm{Tr}_\Sigma$ and it follows that

$$(u, v) \in \wp_\Sigma(\mathcal{A}, M \circ L) \times_{\mathrm{Tr}_\Sigma} \wp_\Sigma(\mathcal{B}, N \circ L)$$

◀

We now turn to (12).

▶ **Lemma 7.10.**

$$\begin{tikzcd}
\wp^{\mathsf{P}}_\Sigma(\mathcal{G}(\mathcal{A}, M \circ L) \multimap\!\circledast\, \mathcal{G}(\mathcal{B}, N \circ L)) \arrow[r, "\wp(L)_{\multimap\circledast}"] \arrow[d, "\mathrm{HS}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \wp^{\mathsf{P}}_\Gamma(\mathcal{G}(\mathcal{A}, M) \multimap\!\circledast\, \mathcal{G}(\mathcal{B}, N)) \arrow[d, "\mathrm{HS}"] \\
\wp_\Sigma(\mathcal{A}, M \circ L) \times_{\mathrm{Tr}_\Sigma} \wp_\Sigma(\mathcal{B}, N \circ L) \arrow[r, "\wp(L) \times \wp(L)"] & \wp_\Gamma(\mathcal{A}, M) \times_{\mathrm{Tr}_\Gamma} \wp_\Gamma(\mathcal{B}, N)
\end{tikzcd}$$

Proof. Commutation of the diagram is ensured by Lem. 7.2 together with Prop. 6.4.

As for the pullback property, we show that the map $\langle \mathrm{HS}, \wp(L)_{\multimap\circledast} \rangle$ is a bijection from $\wp^{\mathsf{P}}_\Sigma(\mathcal{G}(\mathcal{A}, M \circ L) \multimap\!\circledast\, \mathcal{G}(\mathcal{B}, N \circ L))$ to

$$\wp_\Sigma(\mathcal{A}, M \circ L) \times_{\mathrm{Tr}_\Sigma} \wp_\Sigma(\mathcal{B}, N \circ L) \quad \times_{\wp_\Gamma(\mathcal{A}, M) \times_{\mathrm{Tr}_\Gamma} \wp_\Gamma(\mathcal{B}, N)} \quad \wp^{\mathsf{P}}_\Gamma(\mathcal{G}(\mathcal{A}, M) \multimap\!\circledast\, \mathcal{G}(\mathcal{B}, N))$$

The injectivity follows from the injectivity of HS (Lem. 6.6):

$$\mathrm{HS} \quad : \quad \wp_\Sigma(\mathcal{G}(\mathcal{A}, M \circ L) \multimap\!\circledast\, \mathcal{G}(\mathcal{B}, N \circ L)) \quad \longrightarrow \quad \wp_\Sigma(\mathcal{A}, M \circ L) \times \wp_\Sigma(\mathcal{B}, N \circ L)$$

As for surjectivity, consider

$$(s,t) \in \wp_\Sigma(\mathcal{A}, M \circ L) \times_{\mathrm{Tr}_\Sigma} \wp_\Sigma(\mathcal{B}, N \circ L) \qquad \text{and} \qquad u \in \wp^{\mathsf{P}}_\Gamma(\mathcal{G}(\mathcal{A}, M) \mathbin{-\!\circledast} \mathcal{G}(\mathcal{B}, N))$$

such that

$$\mathrm{HS}(u) \quad = \quad (\wp(L)(s),\ \wp(L)(t))$$

Since $(s,t) \in \wp_\Sigma(\mathcal{A}, M \circ L) \times_{\mathrm{Tr}_\Sigma} \wp_\Sigma(\mathcal{B}, N \circ L)$, by Cor. 6.7 there is some

$$v \in \wp^{\mathsf{P}}_\Sigma(\mathcal{G}(\mathcal{A}, M \circ L) \mathbin{-\!\circledast} \mathcal{G}(\mathcal{B}, N \circ L))$$

such that $\mathrm{HS}(v) = (s,t)$. Moreover, it follows from Lem 7.2 that

$$(\wp(L) \times \wp(L)) \circ \mathrm{HS}(v) \quad = \quad \mathrm{HS} \circ \wp(L)_{\mathbin{-\!\circledast}}(v)$$

hence

$$\mathrm{HS} \circ \wp(L)_{\mathbin{-\!\circledast}}(v) \quad = \quad \mathrm{HS}(u)$$

and it follows from Lem. 6.6 (injectivity of HS) that $\wp(L)_{\mathbin{-\!\circledast}}(v) = u$. ◀

▶ **Proposition 7.11.** *We have, in* **Set***,*

$$\begin{tikzcd}
\wp^{\mathsf{P}}_\Sigma(\mathcal{G}(\mathcal{A}, M \circ L) \mathbin{-\!\circledast} \mathcal{G}(\mathcal{B}, N \circ L)) \arrow[r, "\wp(L)_{\mathbin{-\!\circledast}}"] \arrow[d, "\mathrm{tr}^{\mathbin{-\!\circledast}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \wp^{\mathsf{P}}_\Gamma(\mathcal{G}(\mathcal{A}, M) \mathbin{-\!\circledast} \mathcal{G}(\mathcal{B}, N)) \arrow[d, "\mathrm{tr}^{\mathbin{-\!\circledast}}"] \\
\mathrm{Tr}_\Sigma \arrow[r, "\mathrm{Tr}(L)"] & \mathrm{Tr}_\Gamma
\end{tikzcd}$$

Proof. By the pullback lemma (see e.g. [12, Exercise 1.5.5, p. 30]), applied to Lem. 7.9 and Lem. 7.10. ◀

▶ Remark. Note that for O-plays we do *not* have

$$\begin{tikzcd}
\wp_\Sigma(\mathcal{G}(\mathcal{A}, M) \mathbin{-\!\circledast} \mathcal{G}(\mathcal{B}, N)) \arrow[r] \arrow[d] & \wp_\Sigma(\mathcal{B}, N) \arrow[d, "\mathrm{tr}"] \\
\wp_\Sigma(\mathcal{A}, M) \arrow[r, "\mathrm{tr}"'] & \mathrm{Tr}_\Sigma
\end{tikzcd}$$

and it follows that there is no sense to ask:

$$\begin{tikzcd}
\wp_\Sigma(\mathcal{G}(\mathcal{A}, M \circ L) \mathbin{-\!\circledast} \mathcal{G}(\mathcal{B}, N \circ L)) \arrow[r, "\wp(L)_{\mathbin{-\!\circledast}}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \wp_\Gamma(\mathcal{G}(\mathcal{A}, M) \mathbin{-\!\circledast} \mathcal{G}(\mathcal{B}, N)) \arrow[d] \\
\mathrm{Tr}_\Sigma \arrow[r, "\mathrm{Tr}(L)"] & \mathrm{Tr}_\Gamma
\end{tikzcd}$$

7.3.3 A Lifting Property on Plays.

The pullback properties Prop. 7.7 and Prop. 7.11 lead in particluar to the following lifting property, which is crucial for the functoriality of substitution (Prop. 8.4).

▶ **Lemma 7.12.** *Given* $(s,t) \in \wp_\Gamma(\mathcal{A}, M) \times_{\mathrm{Tr}_\Gamma} \wp_\Gamma(\mathcal{B}, N)$*, if* $s = \wp(L)(u)$ *for some* $u \in \wp_\Sigma(\mathcal{A}, M \circ L)$*, then there is* $v \in \wp_\Sigma(\mathcal{B}, N \circ L)$ *such that* $t = \wp(L)(v)$ *and*

$$(u,v) \in \wp_\Sigma(\mathcal{A}, M \circ L) \times_{\mathrm{Tr}_\Sigma} \wp_\Sigma(\mathcal{A}, N \circ L)$$

Proof. Let

$$w \quad := \quad \mathrm{tr}(u) \quad \in \quad \mathrm{Tr}_\Sigma$$

By Lem. 7.5, we have

$$\mathrm{tr}(t) \quad = \quad \mathrm{tr}(s) \quad = \quad \mathrm{tr} \circ \wp(L)(u) \quad = \quad \mathrm{Tr}(L) \circ \mathrm{tr}(u) \quad = \quad \mathrm{Tr}(L)(w)$$

Lem. 7.9 gives

$$(s', t') \in \wp_\Sigma(\mathcal{A}, M \circ L) \times_{\mathrm{Tr}_\Sigma} \wp_\Sigma(\mathcal{B}, N \circ L)$$

such that

$$\mathrm{tr}(s') \quad = \quad \mathrm{tr}(t') \quad = \quad w \qquad \text{and} \qquad \wp(L)(s') = s \qquad \text{and} \qquad \wp(L)(t') = t$$

In particular, we have

$$\mathrm{tr}(s') \quad = \quad \mathrm{tr}(u) \qquad \text{and} \qquad \wp(L)(s') \quad = \quad \wp(L)(u)$$

and it follows that

$$s' \quad = \quad u$$

from Prop. 7.7 (actually the injectivity of

$$\langle \mathrm{tr}, \wp(L) \rangle \quad : \quad \wp_\Sigma(\mathcal{A}, M \circ L) \quad \to \quad \mathrm{Tr}_\Sigma \times \wp_\Gamma(\mathcal{A}, M)$$

in Lem. 7.6). Hence we are done by taking

$$v \quad := \quad t'$$

◀

▶ **Corollary 7.13.** *Assume given* $(s, t) \in \wp_\Sigma(\mathcal{A}, M \circ L) \times_{\mathrm{Tr}_\Sigma} \wp_\Sigma(\mathcal{C}, P \circ L)$ *and* $u' \in \wp_\Gamma(\mathcal{B}, N)$ *such that*

$$(\wp(L)(s), u') \in \wp_\Gamma(\mathcal{A}, M) \times_{\mathrm{Tr}_\Gamma} \wp_\Gamma(\mathcal{B}, N) \quad \textit{and} \quad (u', \wp(L)(t)) \in \wp_\Gamma(\mathcal{B}, N) \times_{\mathrm{Tr}_\Gamma} \wp_\Gamma(\mathcal{C}, P)$$

There is $u \in \wp_\Sigma(\mathcal{A}, M \circ L)$ *such that* $\wp(L)(u) = u'$ *and*

$$(s, u) \in \wp_\Sigma(\mathcal{A}, M \circ L) \times_{\mathrm{Tr}_\Sigma} \wp_\Sigma(\mathcal{B}, N \circ L) \quad \textit{and} \quad (u, t) \in \wp_\Sigma(\mathcal{B}, N \circ L) \times_{\mathrm{Tr}_\Sigma} \wp_\Sigma(\mathcal{C}, P \circ L)$$

Proof. We apply Lem. 7.12 to

$$(\wp(L)(s), u') \in \wp_\Gamma(\mathcal{A}, M) \times_{\mathrm{Tr}_\Gamma} \wp_\Gamma(\mathcal{B}, N)$$

and get some $u \in \wp_\Sigma(\mathcal{B}, N \circ L)$ such that $\wp(L)(u) = u'$ and

$$(s, u) \in \wp_\Sigma(\mathcal{A}, M \circ L) \times_{\mathrm{Tr}_\Sigma} \wp_\Sigma(\mathcal{B}, N \circ L)$$

Since

$$\mathrm{tr}(u) \quad = \quad \mathrm{tr}(s) \quad = \quad \mathrm{tr}(t)$$

we obtain

$$(u, t) \in \wp_\Sigma(\mathcal{B}, N \circ L) \times_{\mathrm{Tr}_\Sigma} \wp_\Sigma(\mathcal{C}, P \circ L)$$

◀

7.4 Relation with Change-of-Base in Plays.

Recall that the change-of-base functor $L^\bullet : \mathbf{Set}/\mathrm{Tr}_\Gamma \to \mathbf{Set}/\mathrm{Tr}_\Sigma$ satisfies:

$$\begin{tikzcd} L^\bullet(\wp_\Gamma(\mathcal{A}, M)) \arrow[r] \arrow[d, "L^\bullet(\mathrm{tr})"'] \arrow[dr, phantom, "\lrcorner", very near start] & \wp_\Gamma(\mathcal{A}, M) \arrow[d, "\mathrm{tr}"] \\ \mathrm{Tr}_\Sigma \arrow[r, "\mathrm{Tr}(L)"] & \mathrm{Tr}_\Gamma \end{tikzcd}$$

where

$$L^\bullet(\wp_\Gamma(\mathcal{A}, M)) \quad = \quad \mathrm{Tr}_\Sigma \times_{\mathrm{Tr}_\Sigma} \wp_\Gamma(\mathcal{A}, M)$$

and $L^\bullet(\wp_\Gamma(\mathcal{A}, M)) \xrightarrow{L^\bullet(\mathrm{tr})} \mathrm{Tr}_\Gamma$ is

$$\pi_1 \quad : \quad \mathrm{Tr}_\Sigma \times_{\mathrm{Tr}_\Sigma} \wp_\Gamma(\mathcal{A}, M) \quad \longrightarrow \quad \mathrm{Tr}_\Sigma$$

and $L^\bullet(\wp_\Gamma(\mathcal{A}, M)) \longrightarrow \wp_\Gamma(\mathcal{A}, M)$ is

$$\pi_2 \quad : \quad \mathrm{Tr}_\Sigma \times_{\mathrm{Tr}_\Sigma} \wp_\Gamma(\mathcal{A}, M) \quad \longrightarrow \quad \wp_\Gamma(\mathcal{A}, M)$$

As an immediate consequence of Prop. 7.7 (pullbacks in **Set**), we get, in $\mathbf{Set}/\mathrm{Tr}_\Sigma$:

▶ **Corollary 7.14.** *In* $\mathbf{Set}/\mathrm{Tr}_\Sigma$*:*

$$L^\bullet(\wp_\Gamma(\mathcal{A}, M) \xrightarrow{\mathrm{tr}} \mathrm{Tr}_\Gamma) \quad \simeq \quad \wp_\Sigma(\mathcal{A}, M \circ L) \xrightarrow{\mathrm{tr}} \mathrm{Tr}_\Sigma$$

8 Substitution Functors

This Section is devoted to the definition of substitution functors

$$L^*{}_{\mathbf{SAG}} : \mathbf{SAG}_\Gamma \longrightarrow \mathbf{SAG}_\Sigma \qquad L^*{}_{\mathrm{W}} : \mathbf{SAG}^{\mathrm{W}}_\Gamma \longrightarrow \mathbf{SAG}^{\mathrm{W}}_\Sigma \qquad L^*{}_{\mathrm{R}} : \mathbf{SAG}^{\mathrm{R}}_\Gamma \longrightarrow \mathbf{SAG}^{\mathrm{R}}_\Sigma$$

obtained from (synchronous) tree morphisms $L \in \mathbf{Tree}[\Sigma, \Gamma]$.

When no problematic ambiguity arises, we drop the subscripts and simply write

$$L^* \quad : \quad \mathbf{SAG}^{(\mathrm{W/R})}_\Gamma \quad \longrightarrow \quad \mathbf{SAG}^{(\mathrm{W/R})}_\Sigma$$

8.1 Substitution Functors on Games

Consider $L \in \mathbf{Tree}[\Sigma, \Gamma]$.

The action of L^* on objects of $\mathbf{SAG}_\Gamma$ is given by change-of-base, as described in Sect. 7:

$$L^*(\Gamma \vdash \mathcal{G}(\mathcal{A}, M)) \quad := \quad \Sigma \vdash \mathcal{G}(\mathcal{A}, M \circ L))$$

where, according to Prop. 7.7:

$$\begin{tikzcd}
\wp_\Sigma(\mathcal{A}, M \circ L) \arrow[r, "\wp(L)"] \arrow[d, "\mathrm{tr}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \wp_\Gamma(\mathcal{A}, M) \arrow[d, "\mathrm{tr}"] \\
\mathrm{Tr}_\Sigma \arrow[r, "\mathrm{Tr}(L)"] & \mathrm{Tr}_\Gamma
\end{tikzcd}$$

8.2 Definition of the Substitution Functors on Strategies

We shall now proceed to the definition of the substitution functor $L^* : \mathbf{SAG}^{(\mathrm{W/R})}_\Gamma \to \mathbf{SAG}^{(\mathrm{W/R})}_\Sigma$ induced by L.

▶ **Definition 8.1.** Consider

$$\Gamma \vdash \sigma : \mathcal{G}(\mathcal{A}, M) \multimap_{\circledast} \mathcal{G}(\mathcal{B}, N)$$

Define $L^*(\sigma) \subseteq \wp_\Sigma(\mathcal{G}(\mathcal{A}, M \circ L) \multimap_{\circledast} \mathcal{G}(\mathcal{B}, N \circ L))$ as

$$L^*(\sigma) \quad := \quad \wp(L)^{-1}_{\multimap_{\circledast}}(\sigma)$$

where

$$\wp(L)_{\multimap_{\circledast}} \quad : \quad \wp_\Sigma(\mathcal{G}(\mathcal{A}, M \circ L) \multimap_{\circledast} \mathcal{G}(\mathcal{B}, N \circ L)) \quad \longrightarrow \quad \wp_\Gamma(\mathcal{G}(\mathcal{A}, M) \multimap_{\circledast} \mathcal{G}(\mathcal{B}, N))$$

▶ **Proposition 8.2** (Prop. 4.2). *(i) $L^*(\sigma)$ is a strategy on $\mathcal{G}(A, M \circ L) \multimap_{\circledast} \mathcal{G}(B, N \circ L)$.*
(ii) If moreover σ is a morphism of $\mathbf{SAG}^{\mathrm{W}}_\Gamma$ (resp. $\mathbf{SAG}^{\mathrm{R}}_\Gamma$) then $L^*(\sigma)$ is a morphism of $\mathbf{SAG}^{\mathrm{W}}_\Sigma$ (resp. $\mathbf{SAG}^{\mathrm{R}}_\Sigma$).

Proof. (i) First, $L^*(\sigma)$ is a set of P-plays on $\mathcal{G}(A, M \circ L) \multimap_{\circledast} \mathcal{G}(B, N \circ L)$ since $\wp(L)_{\multimap_{\circledast}}$ is length-preserving by definition.

A play t of $L^*(\sigma)$ must be of one of the two following forms:

$$\begin{array}{ccccc}
s & \xrightarrow{\mathsf{O}} & ((p, a, \gamma_{\mathcal{A}}), (p, q_{\mathcal{B}})) & \xrightarrow{\mathsf{P}} & ((p, a, \gamma_{\mathcal{A}}), (p, a, \gamma_{\mathcal{B}})) \\
s & \xrightarrow{\mathsf{O}} & ((p, a, \gamma_{\mathcal{A}}), (p.d, q_{\mathcal{B}})) & \xrightarrow{\mathsf{P}} & ((p.d, q_{\mathcal{A}}), (p.d, q_{\mathcal{B}}))
\end{array}$$

By definition of $L^*(\sigma)$, there is a play $u \in \sigma$ such that $\wp(L)_{\multimap\circledast}(t) = u$. It follows that u is of one of the two following forms

$$\begin{array}{lllll} v & \xrightarrow{\mathsf{O}} & ((p, L(p)(a), \gamma_\mathcal{A}), (p, q_\mathcal{B})) & \xrightarrow{\mathsf{P}} & ((p, L(p)(a), \gamma_\mathcal{A}), (p, L(p)(a), \gamma_\mathcal{B})) \\ v & \xrightarrow{\mathsf{O}} & ((p, L(p)(a), \gamma_\mathcal{A}), (p.d, q_\mathcal{B})) & \xrightarrow{\mathsf{P}} & ((p.d, q_\mathcal{A}), (p.d, q_\mathcal{B})) \end{array}$$

with $\wp(L)_{\multimap\circledast}(s) = v$. Since σ is closed under P-prefix, we have $v \in \sigma$, and it follows that $s \in L^*(\sigma)$.

As for P-determinism, consider a play t' of $L^*(\sigma)$, of one of the two following forms:

$$\begin{array}{lllll} s & \xrightarrow{\mathsf{O}} & ((p, a, \gamma_\mathcal{A}), (p, q_\mathcal{B})) & \xrightarrow{\mathsf{P}} & ((p, a, \gamma_\mathcal{A}), (p, a, \gamma'_\mathcal{B})) \\ s & \xrightarrow{\mathsf{O}} & ((p, a, \gamma_\mathcal{A}), (p.d, q_\mathcal{B})) & \xrightarrow{\mathsf{P}} & ((p.d, q'_\mathcal{A}), (p.d, q_\mathcal{B})) \end{array}$$

Reasoning as above, we get a play $u' \in \sigma$ of one of the two following forms

$$\begin{array}{lllll} v & \xrightarrow{\mathsf{O}} & ((p, L(p)(a), \gamma), (p, q_\mathcal{B})) & \xrightarrow{\mathsf{P}} & ((p, L(p)(a), \gamma_\mathcal{A}), (p, L(p)(a), \gamma'_\mathcal{B})) \\ v & \xrightarrow{\mathsf{O}} & ((p, L(p)(a), \gamma_\mathcal{A}), (p.d, q_\mathcal{B})) & \xrightarrow{\mathsf{P}} & ((p.d, q'_\mathcal{A}), (p.d, q_\mathcal{B})) \end{array}$$

and it follows that $\gamma'_\mathcal{B} = \gamma_\mathcal{B}$ and $q'_\mathcal{A} = q_\mathcal{A}$ by P-determinism of σ.

(ii) It follows from the proof of (i) above that to any play of $L(\sigma)$ corresponds a play of σ with the same projections on $Q_\mathcal{A}$ and $Q_\mathcal{B}$. So all infinite plays of $L^*(\sigma)$ are winning w.r.t. $\multimap\circledast$ (resp. $\circledast\!\multimap\!\circledast$) as soon as σ is winning w.r.t. $\multimap\circledast$ (resp. $\circledast\!\multimap\!\circledast$).

As for totality consider the following two situations, where in both cases we assume that the play is legal in $\mathcal{G}(\mathcal{A}, M \circ L) \multimap\circledast \mathcal{G}(\mathcal{B}, N \circ L)$ and that $s \in L^*(\sigma)$

$$\begin{array}{lll} s & \xrightarrow{\mathsf{O}} & ((p, a, \gamma_\mathcal{A}), (p, q_\mathcal{B})) \\ s & \xrightarrow{\mathsf{O}} & ((p, a, \gamma_\mathcal{A}), (p.d, q_\mathcal{B})) \end{array}$$

Then, reasoning as in the proof of (i) above, we obtain in both cases a play $v \in \sigma$ with $\wp(L)_{\multimap\circledast}(s) = v$.

In both cases, we have $\gamma_\mathcal{A} \in \delta_\mathcal{A}(q_\mathcal{A}, (M \circ L)(p)(a))$ and it follows that the two following plays are legal in $\mathcal{G}(\mathcal{A}, M) \multimap\circledast \mathcal{G}(\mathcal{B}, N)$:

$$\begin{array}{lll} v & \xrightarrow{\mathsf{O}} & ((p, L(p)(a), \gamma_\mathcal{A}), (p, q_\mathcal{B})) \\ v & \xrightarrow{\mathsf{O}} & ((p, L(p)(a), \gamma_\mathcal{A}), (p.d, q_\mathcal{B})) \end{array}$$

By totality of σ, we thus get in each case a play $u \in \sigma$ of one of the two following forms:

$$\begin{array}{lllll} v & \xrightarrow{\mathsf{O}} & ((p, L(p)(a), \gamma_\mathcal{A}), (p, q_\mathcal{B})) & \xrightarrow{\mathsf{P}} & ((p, L(p)(a), \gamma_\mathcal{A}), (p, L(p)(a), \gamma_\mathcal{B})) \\ v & \xrightarrow{\mathsf{O}} & ((p, L(p)(a), \gamma_\mathcal{A}), (p.d, q_\mathcal{B})) & \xrightarrow{\mathsf{P}} & ((p.d, q_\mathcal{A}), (p.d, q_\mathcal{B})) \end{array}$$

and it follows that in each case $L^*(\sigma)$ contains a play of one of the two following forms:

$$\begin{array}{lllll} s & \xrightarrow{\mathsf{O}} & ((p, a, \gamma_\mathcal{A}), (p, q_\mathcal{B})) & \xrightarrow{\mathsf{P}} & ((p, a, \gamma_\mathcal{A}), (p, a, \gamma_\mathcal{B})) \\ s & \xrightarrow{\mathsf{O}} & ((p, a, \gamma_\mathcal{A}), (p.d, q_\mathcal{B})) & \xrightarrow{\mathsf{P}} & ((p.d, q_\mathcal{A}), (p.d, q_\mathcal{B})) \end{array}$$

◀

Note that thanks to Lem. 7.2 we have

$$\begin{tikzcd} L^*(\sigma) \arrow[r, "\wp(L)_{\multimap\circledast}"] \arrow[d, "\mathrm{HS}"'] & \sigma \arrow[d, "\mathrm{HS}"] \\ \mathrm{HS}(L^*(\sigma)) \arrow[r, "\wp(L) \times \wp(L)"] & \mathrm{HS}(\sigma) \end{tikzcd} \tag{14}$$

▶ **Lemma 8.3.** *We have*

$$L^*(\sigma) \quad = \quad \mathrm{HS}^{-1}\left((\wp(L) \times \wp(L))^{-1}(\mathrm{HS}(\sigma))\right) \quad = \quad ((\wp(L) \times \wp(L)) \circ \mathrm{HS})^{-1}(\mathrm{HS}(\sigma))$$

where the HS*'s are maps:*

$$\begin{array}{lcrcl} \mathrm{HS} & : & \wp^{\mathsf{P}}_{\Gamma}(\mathcal{G}(\mathcal{A}, M) \multimap\!\circledast\, \mathcal{G}(\mathcal{B}, N)) & \longrightarrow & \wp_{\Gamma}(\mathcal{A}, M) \times \wp_{\Gamma}(\mathcal{B}, N) \\ \mathrm{HS} & : & \wp^{\mathsf{P}}_{\Sigma}(\mathcal{G}(\mathcal{A}, M \circ L) \multimap\!\circledast\, \mathcal{G}(\mathcal{B}, N \circ L)) & \longrightarrow & \wp_{\Sigma}(\mathcal{A}, M \circ L) \times \wp_{\Sigma}(\mathcal{B}, N \circ L) \end{array}$$

Proof. By definition of $L^*(\sigma)$, we have

$$\wp(L)_{\multimap\circledast}(L^*(\sigma)) \quad \subseteq \quad \sigma$$

hence

$$\mathrm{HS} \circ \wp(L)_{\multimap\circledast}(L^*(\sigma)) \quad \subseteq \quad \mathrm{HS}(\sigma)$$

and it follows from (14) that

$$(\wp(L) \times \wp(L)) \circ \mathrm{HS}(L^*(\sigma)) \quad \subseteq \quad \mathrm{HS}(\sigma)$$

hence

$$L^*(\sigma) \quad \subseteq \quad \mathrm{HS}^{-1}\left((\wp(L) \times \wp(L))^{-1}(\mathrm{HS}(\sigma))\right)$$

For the converse direction, let

$$t \in \mathrm{HS}^{-1}\left((\wp(L) \times \wp(L))^{-1}(\mathrm{HS}(\sigma))\right)$$

that is

$$t \quad \in \quad \wp^{\mathsf{P}}_{\Sigma}(\mathcal{G}(\mathcal{A}, M \circ L) \multimap\!\circledast\, \mathcal{G}(\mathcal{B}, N \circ L))$$

with

$$((\wp(L) \times \wp(L)) \circ \mathrm{HS})(t) \quad \in \quad \mathrm{HS}(\sigma)$$

We thus get by Lem. 7.2:

$$\mathrm{HS}(\wp(L)_{\multimap\circledast}(t)) \quad = \quad ((\wp(L) \times \wp(L)) \circ \mathrm{HS})(t) \quad \in \quad \mathrm{HS}(\sigma)$$

It follows that there is $u \in \sigma$ such that

$$\mathrm{HS}(u) \quad = \quad \mathrm{HS}(\wp(L)_{\multimap\circledast}(t))$$

By Lem. 6.6 (injectivity of HS on $\wp_{\Gamma}(\mathcal{G}(\mathcal{A}, M) \multimap\!\circledast\, \mathcal{G}(\mathcal{B}, N))$), we get $\wp(L)_{\multimap\circledast}(t) = u \in \sigma$ hence $t \in \wp(L)^{-1}_{\multimap\circledast}(\sigma)$.

We deduce that $t \in L^*(\sigma)$ since $t \in \wp^{\mathsf{P}}_{\Sigma}(\mathcal{G}(\mathcal{A}, M \circ L) \multimap\!\circledast\, \mathcal{G}(\mathcal{B}, N \circ L))$ by assumption. ◀

8.3 Functoriality of Substitution

We shall now see that L^* is functorial, *i.e.* $L^*(\mathrm{id}_{(\mathcal{A},M)}) = \mathrm{id}_{(\mathcal{A},M \circ L)}$, and moreover, given

$$\Gamma \vdash \sigma : \mathcal{G}(\mathcal{A}, M) \multimap\!\circledast\, \mathcal{G}(\mathcal{B}, N) \qquad \text{and} \qquad \Gamma \vdash \theta : \mathcal{G}(\mathcal{B}, N) \multimap\!\circledast\, \mathcal{G}(\mathcal{C}, P))$$

we have

$$L^*(\theta \circ \sigma) = L^*(\theta) \circ L^*(\sigma)$$

where

$$\Gamma \vdash L^*(\sigma) : \mathcal{G}(\mathcal{A}, M \circ L) \multimap\!\circledast\, \mathcal{G}(\mathcal{B}, N \circ L) \qquad \text{and} \qquad \Gamma \vdash L^*(\theta) : \mathcal{G}(\mathcal{B}, N \circ L) \multimap\!\circledast\, \mathcal{G}(\mathcal{C}, P \circ L)$$

▶ **Proposition 8.4** (Functoriality of Substitution – Prop. 4.3). *Given $L \in \mathbf{Tree}[\Sigma, \Gamma]$, L^* is a functor from $\mathbf{SAG}_\Gamma$ to $\mathbf{SAG}_\Sigma$:*

(i) $L^*(\mathrm{id}_{(\mathcal{A},M)}) = \mathrm{id}_{(\mathcal{A},M \circ L)}$.
(ii) $L^*(\theta \circ \sigma) = L^*(\theta) \circ L^*(\sigma)$ *whenever*

$$\Gamma \vdash \sigma : \mathcal{G}(\mathcal{A}, M) \multimap\!\circledast\, \mathcal{G}(\mathcal{B}, N) \qquad \textit{and} \qquad \Gamma \vdash \theta : \mathcal{G}(\mathcal{B}, N) \multimap\!\circledast\, \mathcal{G}(\mathcal{C}, P))$$

Proof. (i) Thanks to Prop. 4.11 is is sufficient to show that

$$\mathrm{HS}(L^*(\mathrm{id}_{(\mathcal{A},M)})) \quad = \quad \{(s, s) \mid s \in \wp_\Sigma(\mathcal{A}, M \circ L)\}$$

For the inclusion

$$\mathrm{HS}(L^*(\mathrm{id}_{(\mathcal{A},M)})) \quad \subseteq \quad \{(s, s) \mid s \in \wp_\Sigma(\mathcal{A}, M \circ L)\}$$

by Lem 8.3 we have

$$\mathrm{HS}(L^*(\mathrm{id}_{(\mathcal{A},M)})) \quad \subseteq \quad (\wp(L) \times \wp(L))^{-1} \circ \mathrm{HS}(\mathrm{id}_{(\mathcal{A},M)})$$

Let now $(s, t) \in \mathrm{HS}(L^*(\mathrm{id}_{(\mathcal{A},M)}))$. We thus get $\wp(L)(s) = \wp(L)(t)$ by Prop. 4.11 applied to $\mathrm{id}_{(\mathcal{A},M)}$. Since moreover $\mathrm{tr}(s) = \mathrm{tr}(t)$ we get

$$\mathrm{Tr}(L)(\mathrm{tr}(s)) \quad = \quad \mathrm{Tr}(L)(\mathrm{tr}(t)) \qquad \text{and} \qquad \mathrm{tr} \circ \wp(L)(s) \quad = \quad \mathrm{tr} \circ \wp(L)(t)$$

and it follows from Prop. 7.7 (actually Lem. 7.6) that $s = t$.
Conversely, given $s \in \wp_\Sigma(\mathcal{A}, M \circ L)$, by Prop. 4.11 we have

$$(\wp(L)(s), \wp(L)(s)) \in \mathrm{HS}(\mathrm{id}_{(\mathcal{A},M)})$$

hence

$$(s, s) \in (\wp(L) \times \wp(L))^{-1} \circ \mathrm{HS}(\mathrm{id}_{(\mathcal{A},M)})$$

On the other hand, $(s, s) \in \mathrm{id}_{\mathcal{A},M \circ L}$ by Prop. 4.11, and it follows that there is $w \in \wp^{\mathsf{P}}_\Sigma(\mathcal{G}(\mathcal{A}, M \circ L) \multimap\!\circledast\, \mathcal{G}(\mathcal{A}, M \circ L))$ such that $\mathrm{HS}(w) = (s, s)$. We thus get $(s, s) \in \mathrm{HS}(L^*(\mathrm{id}_{(\mathcal{A},M)}))$ by Lem. 8.3.

(ii) Thanks to Lem. 4.6.(ii) it is sufficient to show that

$$\mathrm{HS}(L^*(\theta \circ \sigma)) \quad = \quad \mathrm{HS}(L^*(\theta) \circ L^*(\sigma))$$

- For the inclusion

  $$\mathrm{HS}(L^*(\theta) \circ L^*(\sigma)) \quad \subseteq \quad \mathrm{HS}(L^*(\theta \circ \sigma))$$

  let $(s, t) = \mathrm{HS}(w)$ for some $w \in L^*(\theta) \circ L^*(\sigma)$. Since by Prop. 4.10

  $$\mathrm{HS}(L^*(\theta) \circ L^*(\sigma)) \quad = \quad \mathrm{HS}(L^*(\theta)) \circ \mathrm{HS}(L^*(\sigma)) ,$$

  we get some $u \in \wp_\Sigma(\mathcal{B}, N \circ L)$ such that

  $$(s, u) \in \mathrm{HS}(L^*(\theta)) \qquad \text{and} \qquad (u, t) \in \mathrm{HS}(L^*(\sigma))$$

  Since by Lem 8.3 we have

  $$(\wp(L) \times \wp(L)) \circ \mathrm{HS}(L^*(\theta)) \quad \subseteq \quad \mathrm{HS}(\theta)$$

and

$$(\wp(L) \times \wp(L)) \circ \mathrm{HS}(L^*(\sigma)) \quad \subseteq \quad \mathrm{HS}(\sigma)$$

using Prop. 4.10 again, we get

$$(\wp(L)(s), \wp(L)(t)) \in \mathrm{HS}(\theta) \circ \mathrm{HS}(\sigma) = \mathrm{HS}(\theta \circ \sigma)$$

hence

$$(s,t) \in (\wp(L) \times \wp(L))^{-1} \circ \mathrm{HS}(\theta \circ \sigma)$$

Since $\mathrm{HS}(w) = (s,t)$ with w a legal P-play by assumption, it follows from Lem. 8.3 that $(s,t) \in \mathrm{HS}(L^*(\theta \circ \sigma))$.

- For the converse inclusion, using Prop. 4.10 again, we show that

$$\mathrm{HS}(L^*(\theta \circ \sigma)) \quad \subseteq \quad \mathrm{HS}(L^*(\theta)) \circ \mathrm{HS}(L^*(\sigma))$$

Given $w \in L^*(\theta \circ \sigma)$ with $\mathrm{HS}(w) = (s,t)$, one has to exhibit some play $u \in \wp_\Sigma(\mathcal{B}, N \circ L)$ such that

$$(s,u) \in \mathrm{HS}(L^*(\theta)) \quad \text{and} \quad (u,t) \in \mathrm{HS}(L^*(\sigma))$$

By Lem 8.3, from $w \in L^*(\theta \circ \sigma)$ with $\mathrm{HS}(w) = (s,t)$ for some $s \in \wp_\Sigma(\mathcal{A}, M \circ L)$ and $t \in \wp_\Sigma(\mathcal{C}, P \circ L)$, we have

$$(\wp(L)(s), \wp(L)(t)) \in \mathrm{HS}(\theta \circ \sigma)$$

Hence there is some $u' \in \wp_\Gamma(\mathcal{B}, N)$ such that

$$(\wp(L)(s), u') \in \mathrm{HS}(\theta) \quad \text{and} \quad (u', \wp(L)(t)) \in \mathrm{HS}(\sigma)$$

It follows from Cor. 6.7 that $(\wp(L)(s), u') \in \wp_\Gamma(\mathcal{A}, M) \times_{\mathrm{Tr}_\Gamma} \wp_\Gamma(\mathcal{B}, N)$. Hence by Cor. 7.13 there is $u \in \wp_\Sigma(\mathcal{B}, N \circ L)$ such that $\wp(L)(u) = u'$ and

$$(s,u) \in \wp_\Sigma(\mathcal{A}, M \circ L) \times_{\mathrm{Tr}_\Sigma} \wp_\Sigma(\mathcal{A}, N \circ L) \quad \text{and} \quad (u,t) \in \wp_\Sigma(\mathcal{B}, N \circ L) \times_{\mathrm{Tr}_\Sigma} \wp_\Sigma(\mathcal{C}, P \circ L)$$

We thus get

$$(\wp(L)(s), \wp(L)(u)) \in \mathrm{HS}(\theta) \quad \text{and} \quad (\wp(L)(u), \wp(L)(t)) \in \mathrm{HS}(\sigma)$$

so that

$$(s,u) \in (\wp(L) \times \wp(L))^{-1}(\mathrm{HS}(\theta)) \quad \text{and} \quad (u,t) \in (\wp(L) \times \wp(L))^{-1}(\mathrm{HS}(\sigma))$$

On the other hand, since

$$(s,u) \in \wp_\Sigma(\mathcal{A}, M \circ L) \times_{\mathrm{Tr}_\Sigma} \wp_\Sigma(\mathcal{B}, N \circ L) \quad \text{and} \quad (u,t) \in \wp_\Sigma(\mathcal{B}, N \circ L) \times_{\mathrm{Tr}_\Sigma} \wp_\Sigma(\mathcal{C}, P \circ L)$$

by Cor. 6.7 and by definition of $\_ \multimap\!\!\circledast \_$, there are

$$a \in \wp_\Sigma^{\mathsf{P}}(\mathcal{G}(\mathcal{A}, M \circ L) \multimap\!\!\circledast \mathcal{G}(\mathcal{B}, N \circ L)) \quad \text{and} \quad b \in \wp_\Sigma^{\mathsf{P}}(\mathcal{G}(\mathcal{B}, N \circ L) \multimap\!\!\circledast \mathcal{G}(\mathcal{C}, P \circ L))$$

such that

$$\mathrm{HS}(a) = (s,u) \qquad \text{and} \qquad \mathrm{HS}(b) = (u,t)$$

Now we are done since it follows from Lem 8.3 that

$$(s,u) \in \mathrm{HS}(L^*(\theta)) \quad \text{and} \quad (u,t) \in \mathrm{HS}(L^*(\sigma))$$

◀

▶ **Corollary 8.5.** *L^* is a functor from $\mathbf{SAG}_\Gamma^{\mathrm{W/R}}$ to $\mathbf{SAG}_\Sigma^{\mathrm{W/R}}$.*

8.4 A Universal Property of Substitution

Consider $L \in \mathbf{Tree}[\Sigma, \Gamma]$. Note that by definition of $L^*(\sigma)$ as $\wp(L)^{-1}_{-\circledast}(\sigma)$, we have

$$\begin{tikzcd}
L^*(\sigma) \arrow[r, "\wp(L)_{-\circledast}"] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \sigma \arrow[d, hook] \\
\wp^{\mathsf{P}}_{\Sigma}(\mathcal{G}(\mathcal{A}, M \circ L) \multimap\!\!\circledast\, \mathcal{G}(\mathcal{B}, N \circ L)) \arrow[r, "\wp(L)_{-\circledast}"] & \wp^{\mathsf{P}}_{\Gamma}(\mathcal{G}(\mathcal{A}, M) \multimap\!\!\circledast\, \mathcal{G}(\mathcal{B}, N))
\end{tikzcd}$$

It thus follows from Prop. 7.11

$$\begin{tikzcd}
\wp^{\mathsf{P}}_{\Sigma}(\mathcal{G}(\mathcal{A}, M \circ L) \multimap\!\!\circledast\, \mathcal{G}(\mathcal{B}, N \circ L)) \arrow[r, "\wp(L)_{-\circledast}"] \arrow[d, "\mathrm{tr}^{-\circledast}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \wp^{\mathsf{P}}_{\Gamma}(\mathcal{G}(\mathcal{A}, M) \multimap\!\!\circledast\, \mathcal{G}(\mathcal{B}, N)) \arrow[d, "\mathrm{tr}^{-\circledast}"] \\
\mathrm{Tr}_{\Sigma} \arrow[r, "\mathrm{Tr}(L)"] & \mathrm{Tr}_{\Gamma}
\end{tikzcd}$$

and the pullback lemma (see e.g. [12, Exercise 1.5.5, p. 30]) that we have, in **Set**:

$$\begin{tikzcd}
L^*(\sigma) \arrow[r, "\wp(L)_{-\circledast}"] \arrow[d, "\mathrm{tr}^{-\circledast}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \sigma \arrow[d, "\mathrm{tr}^{-\circledast}"] \\
\mathrm{Tr}_{\Sigma} \arrow[r, "\mathrm{Tr}(L)"] & \mathrm{Tr}_{\Gamma}
\end{tikzcd} \tag{15}$$

9 Fibrations of Acceptance Games and Automata

In this Section, we briefely present the split fibrations *of acceptance games*

$$\mathsf{game} : \mathbf{SAG} \to \mathbf{Tree} \qquad \mathsf{game}^{\mathrm{W}} : \mathbf{SAG}^{\mathrm{W}} \to \mathbf{Tree} \qquad \mathsf{game}^{\mathrm{R}} : \mathbf{SAG}^{\mathrm{R}} \to \mathbf{Tree}$$

defined by Grothendieck completion of the split indexed categories

$$(-)^*{}_{\mathbf{SAG}} : \mathbf{Tree}^{\mathrm{op}} \to \mathbf{Cat} \qquad (-)^*{}_{\mathrm{W}} : \mathbf{Tree}^{\mathrm{op}} \to \mathbf{Cat} \qquad (-)^*{}_{\mathrm{R}} : \mathbf{Tree}^{\mathrm{op}} \to \mathbf{Cat}$$

issued from substitution.

9.1 The Split Indexed Category of Substitution

We show that substitution leads to a split indexed category

$$(-)^* : \mathbf{Tree}^{\mathrm{op}} \to \mathbf{Cat}$$

(in the sense of [12, Def. 1.4.4, pp. 50–51]):

▶ **Proposition 9.1** (Prop. 4.4). *(i) With $\mathrm{Id}_\Sigma \in \mathbf{Tree}[\Sigma, \Sigma]$, the functors*

$$\mathrm{Id}_\Sigma^* \quad : \quad \mathbf{SAG}_\Sigma^{(\mathrm{W/R})} \quad \longrightarrow \quad \mathbf{SAG}_\Sigma^{(\mathrm{W/R})}$$

is the identity functor

$$\mathbf{SAG}_\Sigma^{(\mathrm{W/R})} \quad \longrightarrow \quad \mathbf{SAG}_\Sigma^{(\mathrm{W/R})}$$

(ii) Given $L \in \mathbf{Tree}[\Sigma, \Gamma]$ and $K \in \mathbf{Tree}[\Gamma, \Delta]$, we have

$$(K \circ L)^* \quad = \quad L^* \circ K^*$$

where $K \circ L \in \mathbf{Tree}[\Sigma, \Delta]$ and

$$(K \circ L)^*, L^* \circ K^* \quad : \quad \mathbf{SAG}_\Delta^{(\mathrm{W/R})} \quad \longrightarrow \quad \mathbf{SAG}_\Sigma^{(\mathrm{W/R})}$$

Proof. (i) Since $\wp_{\multimap\circledast}(\mathrm{Id}_\Sigma)$ is the identity on

$$\wp_\Sigma(\mathcal{G}(\mathcal{A}, M \circ L) \multimap\circledast \mathcal{G}(\mathcal{A}, M \circ L)) \quad = \quad \wp_\Sigma(\mathcal{G}(\mathcal{A}, M) \multimap\circledast \mathcal{G}(\mathcal{A}, M))$$

we have

$$\mathrm{Id}_\Sigma^*(\sigma) \quad = \quad \wp_{\multimap\circledast}(\mathrm{Id}_\Sigma)^{-1}(\sigma) \quad = \quad \sigma$$

(ii) Since $\wp(K \circ L)_{\multimap\circledast} = \wp(K)_{\multimap\circledast} \circ \wp(L)_{\multimap\circledast}$ we have

$$(K \circ L)^*(\sigma) \quad = \quad \wp(K \circ L)^{-1}_{\multimap\circledast}(\sigma) \quad = \quad (\wp(L)^{-1}_{\multimap\circledast} \circ \wp(K)^{-1}_{\multimap\circledast})(\sigma) \quad = \quad L^*(K^*(\sigma))$$

◀

By combining Prop. 9.1 with Prop. 8.4, we thus obtain that

▶ **Corollary 9.2.** *Substitution induces a functors*

$$(-)^*{}_{\mathbf{SAG}} : \mathbf{Tree}^{\mathrm{op}} \to \mathbf{Cat} \qquad (-)^*{}_{\mathrm{W}} : \mathbf{Tree}^{\mathrm{op}} \to \mathbf{Cat} \qquad (-)^*{}_{\mathrm{R}} : \mathbf{Tree}^{\mathrm{op}} \to \mathbf{Cat}$$

9.2 The Fibration of Acceptance Game

We now define the split fibrations

$$\mathsf{game}^{(\mathrm{W/R})} : \mathbf{SAG}^{(\mathrm{W/R})} \to \mathbf{Tree}$$

by Grothendieck completion (see e.g. [12, §1.10]) of the corresponding split indexed category

$$(-)^* : \mathbf{Tree}^{\mathrm{op}} \to \mathbf{Cat}$$

9.2.1 The Total Categories $\mathbf{SAG}^{(\mathrm{W/R})}$:

Objects are substituted acceptance games $\Sigma \vdash \mathcal{G}(\mathcal{A}, M)$ where $\Gamma \vdash \mathcal{A}$ and $\Sigma \vdash M : \Gamma$.

Morphisms from $\Sigma \vdash \mathcal{G}(\mathcal{A}, M)$ to $\Gamma \vdash \mathcal{G}(\mathcal{B}, N)$ are given by pairs (L, σ) of a tree $L \in \mathbf{Tree}[\Sigma, \Gamma]$ and a strategy

$$\Sigma \vdash \sigma : \mathcal{G}(\mathcal{A}, M) \multimap_{\circledast} \mathcal{G}(\mathcal{B}, N \circ L)$$

such that, for $\mathbf{SAG}^{\mathrm{W}}$ we additionally require:

$$\Sigma \vdash \sigma \Vdash \mathcal{G}(\mathcal{A}, M) \multimap_{\circledast} \mathcal{G}(\mathcal{B}, N \circ L)$$

and for $\mathbf{SAG}^{\mathrm{R}}$ we additionally require:

$$\Sigma \vdash \sigma \Vdash \mathcal{G}(\mathcal{A}, M) \circledast\!\!\multimap_{\circledast} \mathcal{G}(\mathcal{B}, N \circ L)$$

The total categories $\mathbf{SAG}^{(\mathrm{W/R})}$ is thus the Grothendieck completions of the corresponding $(-)^* : \mathbf{Tree}^{\mathrm{op}} \to \mathbf{Cat}$. Thanks to the functoriality of substituion (Prop 8.4), composition is defined componentwise: if

$$\Sigma \vdash \mathcal{G}(\mathcal{A}, M) \xrightarrow{(L,\sigma)} \Delta \vdash \mathcal{G}(\mathcal{B}, N) \xrightarrow{(K,\theta)} \Gamma \vdash \mathcal{G}(\mathcal{C}, P)$$

then

$$\begin{array}{rcccccc} & \Sigma \vdash & \sigma & \Vdash & \mathcal{G}(\mathcal{A}, M) & \multimap_{\circledast} & \mathcal{G}(\mathcal{B}, N \circ L) \\ \text{and} & \Delta \vdash & \theta & \Vdash & \mathcal{G}(\mathcal{B}, N) & \multimap_{\circledast} & \mathcal{G}(\mathcal{C}, P \circ K) \\ \text{hence} & \Sigma \vdash & L^*(\theta) & \Vdash & \mathcal{G}(\mathcal{B}, N \circ L) & \multimap_{\circledast} & \mathcal{G}(\mathcal{C}, P \circ K \circ L) \end{array}$$

We let

$$(K, \theta) \circ (L, \sigma) := (K \circ L, L^*(\theta) \circ \sigma)$$

9.2.2 The Functors $\mathsf{game}^{(\mathrm{W/R})} : \mathbf{SAG}^{(\mathrm{W/R})} \to \mathbf{Tree}$

maps a game $\Sigma \vdash \mathcal{G}(\mathcal{A}, M)$ to the alphabet Σ, and a morphism

$$\Sigma \vdash \mathcal{G}(\mathcal{A}, M) \xrightarrow{(L,\sigma)} \Gamma \vdash \mathcal{G}(\mathcal{B}, N)$$

to $L \in \mathbf{Tree}[\Sigma, \Gamma]$.

The fibre categories $\mathsf{game}^{-1}(\Sigma)$ are isomorphic to the categories $\mathbf{SAG}^{(\mathrm{W/R})}_{\Sigma}$: they have games $\Sigma \vdash \mathcal{G}(\mathcal{A}, M)$ as objects and morphisms are of the form

$$\Sigma \vdash \mathcal{G}(\mathcal{A}, M) \xrightarrow{(\mathrm{Id}_\Sigma,\sigma)} \Sigma \vdash \mathcal{G}(\mathcal{B}, N)$$

9.2.3 Cartesian Liftings.

Given $L \in \mathbf{Tree}[\Gamma, \Sigma]$, we let $\overline{L} := (L, \mathrm{id}_\Gamma)$ where

$$\Gamma \vdash \mathrm{id}_\Gamma \Vdash \mathcal{G}(\mathcal{A}, M \circ L) \multimap\!\!\circledast\, \mathcal{G}(\mathcal{A}, M \circ L)$$

is the identity strategy. We thus have, in $\mathbf{SAG}^{(\mathrm{W/R})}$,

$$L^*(\Sigma \vdash \mathcal{G}(\mathcal{A}, M)) \xrightarrow{\ \overline{L}\ } \Sigma \vdash \mathcal{G}(\mathcal{A}, M)$$

In other words, given $\Sigma \vdash \mathcal{G}(\mathcal{A}, M)$ and

$$L \quad : \quad \Gamma \quad \longrightarrow_{\mathbf{Tree}} \quad \mathsf{game}^{(\mathrm{W/R})}(\Sigma \vdash \mathcal{G}(\mathcal{A}, M)) \ (= \Sigma)$$

we have

$$\overline{L} \quad : \quad L^*(\Sigma \vdash \mathcal{G}(\mathcal{A}, M)) \quad \longrightarrow_{\mathbf{SAG}^{(\mathrm{W/R})}} \quad \Sigma \vdash \mathcal{G}(\mathcal{A}, M)$$

with $\mathsf{game}(\overline{L}) = L$ and moreover for every

$$(K, \theta) \quad : \quad \Delta \vdash \mathcal{G}(\mathcal{B}, N) \longrightarrow \Sigma \vdash \mathcal{G}(\mathcal{A}, M)$$

with

$$\mathsf{game}(K, \theta) = K = L \circ P$$

there is a unique τ such that:

$$\begin{tikzcd}
\Delta \vdash \mathcal{G}(\mathcal{B}, N) \arrow[dr, dashed, "(P,\tau)"'] \arrow[drr, bend left, "(K,\theta)"] & & \\
& L^*(\Sigma \vdash \mathcal{G}(\mathcal{A}, M)) \arrow[r, "\overline{L}"'] & \Sigma \vdash \mathcal{G}(\mathcal{A}, M)
\end{tikzcd}$$

$$\begin{tikzcd}
\Delta \arrow[dr, "P"'] \arrow[drr, bend left, "K"] & & \\
& \Gamma \arrow[r, "L"'] & \Sigma
\end{tikzcd}$$

The unique τ claimed above is easily obtainable by unfolding the definitions: Since (by Prop. 8.4.(i)) we have

$$\overline{L} \circ (P, \tau) \quad = \quad (L \circ P, L^*(\mathrm{id}_\Gamma) \circ \tau) \quad = \quad (L \circ P, \tau)$$

the equality

$$(K, \theta) \quad = \quad (L \circ P, \theta) \quad = \quad \overline{L} \circ (P, \tau)$$

imposes $\tau := \theta$.

9.2.4 The Fibration of Acceptance Game.

Thanks to the functoriality of substitution $(-)^* : \mathbf{Tree}^{\mathrm{op}} \to \mathbf{Cat}$ (Cor. 9.2), we thus get by [12, Prop. 1.10.2(i)]:

▶ **Proposition 9.3.** *The functors* $\mathsf{game}^{(\mathrm{W/R})} : \mathbf{SAG}^{(\mathrm{W/R})} \to \mathbf{Tree}$*, together with* $L \mapsto L^*, \overline{L}$ *as above, forms a split fibration.*

9.3 Fibration of Automata

We write $\Sigma \vdash \sigma : \mathcal{A} \multimap\!\circledast\, \mathcal{B}$ for

$$\Sigma \vdash \sigma : \mathcal{G}(\mathcal{A}, \mathrm{Id}_\Sigma) \multimap\!\circledast\, \mathcal{G}(\mathcal{B}, \mathrm{Id}_\Sigma)$$

and $\Sigma \vdash \sigma \Vdash \mathcal{A} \multimap\!\circledast\, \mathcal{B}$ for

$$\Sigma \vdash \sigma \Vdash \mathcal{G}(\mathcal{A}, \mathrm{Id}_\Sigma) \multimap\!\circledast\, \mathcal{G}(\mathcal{B}, \mathrm{Id}_\Sigma)$$

and finally, $\Sigma \vdash \sigma \Vdash \mathcal{A} \circledast\!\!-\!\!\circledast\, \mathcal{B}$ for

$$\Sigma \vdash \sigma \Vdash \mathcal{G}(\mathcal{A}, \mathrm{Id}_\Sigma) \circledast\!\!-\!\!\circledast\, \mathcal{G}(\mathcal{B}, \mathrm{Id}_\Sigma)$$

9.3.1 The Categories $\mathbf{Aut}_\Sigma^{(\mathrm{W/R})}$:

Objects are automata $\Sigma \vdash \mathcal{A}$
Morphisms from $\Sigma \vdash \mathcal{A}$ to $\Sigma \vdash \mathcal{B}$ are strategies

$$\Sigma \vdash \sigma : \mathcal{A} \multimap\!\circledast\, \mathcal{B}$$

such that, for $\mathbf{Aut}^{\mathrm{W}}$ we additionally require:

$$\Sigma \vdash \sigma \Vdash \mathcal{A} \multimap\!\circledast\, \mathcal{B}$$

and for $\mathbf{Aut}^{\mathrm{R}}$ we additionally require:

$$\Sigma \vdash \sigma \Vdash \mathcal{A} \circledast\!\!-\!\!\circledast\, \mathcal{B}$$

9.3.2 The Faithful Functor $\mathrm{Emb} : \mathbf{Alph} \to \mathbf{Tree}$

is the identity on objects and maps $\beta \in \mathbf{Alph}[\Sigma, \Gamma]$ to the constant tree $\mathrm{Emb}(\beta) \in \mathbf{Tree}[\Sigma, \Gamma]$ with $\mathrm{Emb}(\beta)(p) := \beta$ for all $p \in D^*$.

We will often simply write $\beta \in \mathbf{Tree}[\Sigma, \Gamma]$ for the morphism $\mathrm{Emb}(\beta)$ induced by $\beta \in \mathbf{Alph}[\Sigma, \Gamma]$.

According to Cor. 9.2, the functor $\mathrm{Emb} : \mathbf{Alph} \to \mathbf{Tree}$ induces a split indexed category

$$(-)^* : \mathbf{Alph}^{\mathrm{op}} \to \mathbf{Cat}$$

where, for $\beta \in \mathbf{Alph}[\Sigma, \Gamma]$, we have

$$\beta^* \quad : \quad \mathbf{SAG}_\Gamma^{(\mathrm{W/R})} \quad \longrightarrow \quad \mathbf{SAG}_\Sigma^{(\mathrm{W/R})}$$

9.3.3 Substitutions induced by Alphabet Morphisms.

Substitutions issued from alphabet morphisms can be internalized in automata. Given $\Gamma \vdash \mathcal{A}$ with $\mathcal{A} = (Q, q^\imath, \delta, \Omega)$ and $\beta \in \mathbf{Alph}[\Sigma, \Gamma]$, define the automaton $\Sigma \vdash \mathcal{A}[\beta]$ as

$$\mathcal{A}[\beta] \quad := \quad (Q, q^\imath, \delta_\beta, \Omega)$$

where

$$\delta_\beta(q, b) \quad := \quad \delta(q, \beta(b))$$

▶ **Proposition 9.4.** *Given $\Gamma \vdash \mathcal{A}$ and $\beta \in \mathbf{Tree}[\Sigma, \Gamma]$, we have*

$$\Sigma \vdash \mathcal{G}(\mathcal{A}[\beta], \mathrm{Id}_\Sigma) \quad = \quad \Sigma \vdash \mathcal{G}(\mathcal{A}, \beta)$$

Proof. We have

$$(p, q) \quad \xrightarrow{\mathsf{P}} \quad (p, a, \gamma) \qquad \text{in } \Sigma \vdash \mathcal{G}(\mathcal{A}[\beta], \mathrm{Id}_\Sigma)$$

iff

$$\gamma \quad \in \quad \delta_\beta(q, \mathrm{Id}_\Sigma(a)) \quad = \quad \delta_\beta(q, a) \quad = \quad \delta(q, \beta(a))$$

iff

$$(p, q) \quad \xrightarrow{\mathsf{P}} \quad (p, a, \gamma) \qquad \text{in } \Sigma \vdash \mathcal{G}(\mathcal{A}, \beta)$$

Moreover,

$$(p, a, \gamma) \quad \xrightarrow{\mathsf{O}} \quad (p.d, q) \qquad \text{in } \Sigma \vdash \mathcal{G}(\mathcal{A}[\beta], \mathrm{Id}_\Sigma)$$

iff $(q, d) \in \gamma$ iff

$$(p, a, \gamma) \quad \xrightarrow{\mathsf{O}} \quad (p.d, q) \qquad \text{in } \Sigma \vdash \mathcal{G}(\mathcal{A}, \beta)$$

◀

▶ **Corollary 9.5.** *Substitution along $\beta \in \mathbf{Alph}[\Sigma, \Gamma]$ induce functors*

$$\beta^* \quad : \quad \mathbf{Aut}_\Gamma^{(\mathrm{W/R})} \quad \longrightarrow \quad \mathbf{Aut}_\Sigma^{(\mathrm{W/R})}$$

9.3.4 The Total Category $\mathbf{Aut}^{(\mathrm{W/R})}$:

Objects are automata $\Sigma \vdash A$.
Morphisms from $\Sigma \vdash \mathcal{A}$ to $\Gamma \vdash \mathcal{B}$ are pairs (β, σ) where $\beta \in \mathbf{Alph}[\Sigma, \Gamma]$ and $\Sigma \vdash \sigma \Vdash \mathcal{A} \multimap\!\!\circledast\, \mathcal{B}[\beta]$.

10 A Synchronous Symmetric Monoidal Product

In this section, we define a synchronous monoidal product $\_ \circledast \_$ (and its variant $\_ \circledast_{\equiv} \_$ for reduction games) and show its basic properties. This product will be equipped in Sect. 11 with a symmetric monoidal structure, leading to symmetric monoidal fibrations. The symmetric monoidal structre actually assumes automata to be *complete*. Moreover, the symmetric monoidal structure on acceptance games is easier to describe starting from a first partial version of $\_\circledast\_$, which only allows products of games $\Sigma \vdash \mathcal{G}(\mathcal{A}, M)$ and $\Sigma \vdash \mathcal{G}(\mathcal{B}, M)$ with the *same* substituted tree morphism M. This variant of $\circledast$ is called *uniform*.

10.1 The Relational Tensorial Product in $\mathrm{Set}^{\to}$

10.1.1 Symmetric Monoidal Categories.

Following [16, 13], a *symmetric monoidal category* is a category $\mathbb{C}$ equipped with a bifunctor $\_ \otimes \_$ and an object $\mathbf{I}$ together with natural isomorphisms:

$$
\begin{array}{rcrcl}
\alpha_{A,B,C} & : & (A \otimes B) \otimes C & \longrightarrow & A \otimes (B \otimes C) \\
\lambda_A & : & \mathbf{I} \otimes A & \longrightarrow & A \\
\rho_A & : & A \otimes \mathbf{I} & \longrightarrow & A \\
\gamma_{A,B} & : & A \otimes B & \longrightarrow & B \otimes A
\end{array}
$$

satisfying $\gamma_{A,B} = \gamma_{B,A}^{-1}$ and usual coherence diagrams (see e.g. [16, 13]).

10.1.2 The Monoidal Category Rel of Sets and Relations.

In **Rel**, the monoidal product $\_ \otimes_{\mathbf{Rel}} \_$ is given by:

On Objects: $A \otimes_{\mathbf{Rel}} B := A \times B$.

On Morphisms: given $R : A \to\!\!\!\!\!\!+ C$ and $P : B \to\!\!\!\!\!\!+ D$, we define $R \otimes_{\mathbf{Rel}} P : A \otimes_{\mathbf{Rel}} B \to\!\!\!\!\!\!+ C \otimes_{\mathbf{Rel}} D$ as

$$
R \otimes_{\mathbf{Rel}} P \quad := \quad \{((a,b),(c,d)) \mid (a,c) \in R \text{ and } (b,d) \in P\}
$$

The unit $\mathbf{I}$ is the singletton set $\mathbf{I} := \{\bullet\}(= \mathbf{1})$, and the natural isomorphisms are given by:

$$
\begin{array}{rcl}
\mathring{\alpha}_{A,B,C} & := & \{(((a,b),c)\ ,\ (a,(b,c))) \mid a \in A \text{ and } b \in B \text{ and } c \in C\} \\
\mathring{\lambda}_A & := & \{((\bullet, a)\ ,\ a) \mid a \in A\} \\
\mathring{\rho}_A & := & \{((a, \bullet)\ ,\ a) \mid a \in A\} \\
\mathring{\gamma}_{A,B} & := & \{((a,b)\ ,\ (b,a)) \mid a \in A \text{ and } b \in B\}
\end{array}
$$

10.1.3 Monoidal Structure in $\mathrm{Rel}(\mathrm{Set}/J)$.

Define, in $\mathbf{Rel}(\mathbf{Set}/J)$ the operation $\_ \otimes_{\mathbf{Rel}(\mathbf{Set}/J)} \_$ (simply denoted $\_ \otimes \_$ when no confusion arises):

On Objects: for (A, g) and (B, h) objects in $\mathbf{Rel}(\mathbf{Set}/J)$ the tensor product $A \otimes B$ is $A \times_J B$ with the corresponding map, that is

$$
A \otimes B \quad := \quad \{(a,b) \in A \times B \mid g(a) = h(b)\} \quad \xrightarrow{g \circ \pi_1 = h \circ \pi_2} \quad J
$$

On Morphisms: given $R \in \mathbf{Rel}(\mathbf{Set}/J)[A, C]$ and $P \in \mathbf{Rel}(\mathbf{Set}/J)[B, D]$, we define $R \otimes P \in \mathbf{Rel}(\mathbf{Set}/J)[A \otimes B, C \otimes D]$ as

$$R \otimes P \quad := \quad \{((a,b),(c,d)) \in (A \otimes B) \times (C \otimes D) \mid (a,c) \in R \text{ and } (b,d) \in P\}$$

Note that given $((a,b),(c,d)) \in R \otimes P$, writing $C \xrightarrow{k} J$ and $D \xrightarrow{l} J$, we have $((a,b),(c,d)) \in (A \otimes B) \times_J (C \otimes D)$, since $(a,c) \in R$ implies $g(a) = k(c)$ and since $(b,d) \in P$ implies $h(b) = l(d)$.

The same holds for $((a,b),(c,d)) \in R \otimes_{\mathbf{Rel}} P$, but $((a,b),(c,d)) \in R \otimes_{\mathbf{Rel}} P$ does *not* imply $((a,b),(c,d)) \in R \otimes P$ since we may *not* have $(a,b) \in A \otimes B$ nor $(c,d) \in C \otimes D$.

▶ **Proposition 10.1.** *The product* $\_ \otimes \_$ *is a bifunctor on* $\mathbf{Rel}(\mathbf{Set}/J)$*:*

(i) $1_A \otimes 1_B = 1_{A \otimes B}$

(ii) Given

$$A \xrightarrow{R_0}\!\!\!\!\!\!\!\!\!+\;\,_{/J} B \xrightarrow{R_1}\!\!\!\!\!\!\!\!\!+\;\,_{/J} C \quad \textit{and} \quad A' \xrightarrow{R'_0}\!\!\!\!\!\!\!\!\!+\;\,_{/J} B' \xrightarrow{R'_1}\!\!\!\!\!\!\!\!\!+\;\,_{/J} C'$$

we have

$$(R_1 \circ R_0) \otimes (R'_1 \circ R'_0) \quad = \quad (R_1 \otimes R'_1) \circ (R_0 \otimes R'_0)$$

Proof. (i) Write $A \xrightarrow{g} J$ and $B \xrightarrow{h} J$.

We have

$$\begin{array}{rcl} 1_A \otimes 1_B & = & \{((a,b),(a,b)) \in (A \otimes B) \times (A \otimes B) \mid a \in A \;\&\; b \in B\} \\ & = & \{((a,b),(a,b)) \in (A \otimes B) \times (A \otimes B) \mid (a,b) \in A \otimes B\} \\ & = & 1_{A \otimes B} \end{array}$$

(ii) Consider

$$((a,a'),(c,c')) \in (A \otimes A') \times (C \otimes C')$$

Then we have

$$((a,a'),(c,c')) \in (R_1 \circ R_0) \otimes (R'_1 \circ R'_0)$$

if and only if

$$(a,c) \in R_1 \circ R_0 \quad \text{and} \quad (a',c') \in R'_1 \circ R'_0$$

if and only if there are $b \in B$ and $b' \in B'$ such that

$$(a,b) \in R_0 \quad (b,c) \in R_1 \quad (a',b') \in R'_0 \quad (b',c') \in R'_1$$

if and only if (since $((a,a'),(b,b')) \in (A \otimes A') \times (B \otimes B')$, and similarly for $((b,b'),(c,c'))$)

$$((a,a'),(b,b')) \in R_0 \otimes R'_0 \quad \text{and} \quad ((b,b'),(c,c')) \in R_1 \otimes R'_1$$

if and only if

$$((a,a'),(c,c')) \in (R_1 \otimes R'_1) \circ (R_0 \otimes R'_0)$$

◀

For the unit, we *choose* some

$$\mathbf{I} \quad = \quad (I \quad \xrightarrow{\jmath} \quad J)$$

such that

$$\jmath \quad : \quad I \xrightarrow{\simeq} J$$

The natural isomorphisms are given by:

$$\begin{array}{rcl} \mathring{\alpha}_{A,B,C} & := & \{(((a,b),c)\ ,\ (a,(b,c))) \mid g_A(a) = g_B(b) = g_C(c)\} \\ \mathring{\lambda}_A & := & \{((e,a)\ ,\ a) \mid \jmath(e) = g_A(a)\} \\ \mathring{\rho}_A & := & \{((a,e)\ ,\ a) \mid g_A(a) = \jmath(e)\} \\ \mathring{\gamma}_{A,B} & := & \{((a,b)\ ,\ (b,a)) \mid g_A(a) = g_B(b)\} \end{array}$$

We easily get:

▶ **Lemma 10.2.** *We have isomorphisms*

$$\begin{array}{rcrcl} \mathring{\alpha}_{A,B,C} & : & (A \otimes B) \otimes C & \twoheadrightarrow_{/J} & A \otimes (B \otimes C) \\ \mathring{\lambda}_A & : & \mathbf{I} \otimes A & \twoheadrightarrow_{/J} & A \\ \mathring{\rho}_A & : & A \otimes \mathbf{I} & \twoheadrightarrow_{/J} & A \\ \mathring{\gamma}_{A,B} & : & A \otimes B & \twoheadrightarrow_{/J} & B \otimes A \end{array}$$

Proof. For $\mathring{\alpha}_{A,B,C}$, if $((a,b),c) \in (A \otimes B) \otimes C$ then we have $g_A(a) = g_B(b) = g_C(c)$, hence

$$((a,b),c)\ \mathring{\alpha}_{A,B,C}\ (a,(b,c)) \quad \text{and} \quad (a,(b,c))\ \mathring{\alpha}^{-1}_{A,B,C}\ ((a,b),c)$$

and thus

$$((a,b),c)\ (\mathring{\alpha}^{-1}_{A,B,C} \circ \mathring{\alpha}_{A,B,C})\ ((a,b),c)$$

It follows that

$$1_{(A\otimes B)\otimes C} \quad \subseteq \quad \mathring{\alpha}^{-1}_{A,B,C} \circ \mathring{\alpha}_{A,B,C}$$

Conversely, consider

$$((a,b),c)\ (\mathring{\alpha}^{-1}_{A,B,C} \circ \mathring{\alpha}_{A,B,C})\ ((a',b'),c')$$

Then there are a'', b'' and c'' such that

$$((a,b),c)\ \mathring{\alpha}_{A,B,C}\ (a'',(b'',c'')) \quad \text{and} \quad (a'',(b'',c''))\ \mathring{\alpha}^{-1}_{A,B,C}\ ((a',b'),c')$$

But this implies $a = a'' = a'$, $b = b'' = b'$, and $c = c'' = c'$, as well as $g_A(a) = g_B(b) = g_C(c)$. We thus have

$$\mathring{\alpha}^{-1}_{A,B,C} \circ \mathring{\alpha}_{A,B,C} \quad \subseteq \quad 1_{(A\otimes B)\otimes C}$$

We similarly get

$$\mathring{\alpha}_{A,B,C} \circ \mathring{\alpha}^{-1}_{A,B,C} \quad = \quad 1_{A\otimes(B\otimes C)}$$

The case of $\mathring{\gamma}_{A,B}$ is dealt-with similarly.

For $\mathring{\lambda}_A$, if $(e, a) \in \mathbf{I} \otimes A$ then we have $((e, a), a) \in \mathring{\lambda}_A$ and $(a, (e, a)) \in \mathring{\lambda}_A^{-1}$, and it follows that

$$1_{\mathbf{I}\otimes A} \subseteq \mathring{\lambda}_A^{-1} \circ \mathring{\lambda}_A$$

On the other hand, since $\jmath : I \to J$ is surjective, for all $a \in A$ there is some $e \in I$ such that $g_A(a) = \jmath(e)$, hence $(e, a) \in \mathbf{I} \otimes A$, from which it follows that $((e, a), a) \in \mathring{\lambda}_A$, and $(a, (e, a)) \in \mathring{\lambda}_A^{-1}$ hence

$$1_A \subseteq \mathring{\lambda}_A \circ \mathring{\lambda}_A^{-1}$$

For the converse inclusions, consider

$$(e, a)\ (\mathring{\lambda}_A^{-1} \circ \mathring{\lambda}_A)\ (e', a')$$

hence there is $a'' \in A$ such that

$$(e, a)\ \mathring{\lambda}_A\ a'' \quad \text{and} \quad a''\ \mathring{\lambda}_A^{-1}\ (e', a')$$

But this implies $a = a'' = a'$, hence $e = e'$ since $\jmath(e) = \jmath(e')$ and $\jmath$ is injective, and we get

$$\mathring{\lambda}_A^{-1} \circ \mathring{\lambda}_A \subseteq 1_{\mathbf{I}\otimes A}$$

Similarly, if

$$a\ (\mathring{\lambda}_A \circ \mathring{\lambda}_A^{-1})\ a'$$

then there are $(e, a'') \in \mathbf{I} \otimes A$ such that

$$a\ \mathring{\lambda}_A^{-1}\ (e, a'') \quad \text{and} \quad (e, a'')\ \mathring{\lambda}_A\ a'$$

which implies $a = a'' = a'$, and we get

$$\mathring{\lambda}_A \circ \mathring{\lambda}_A^{-1} \subseteq 1_A$$

The case of $\mathring{\rho}_A$ is dealt-with similarly. ◀

▶ **Lemma 10.3.** *We have natural transformations:*

$$\begin{array}{rcrcl}
\mathring{\alpha}_{A,B,C} & : & (A \otimes B) \otimes C & \mathrel{\to\!\!\!\!\!+}_{/J} & A \otimes (B \otimes C) \\
\mathring{\lambda}_A & : & \mathbf{I} \otimes A & \mathrel{\to\!\!\!\!\!+}_{/J} & A \\
\mathring{\rho}_A & : & A \otimes \mathbf{I} & \mathrel{\to\!\!\!\!\!+}_{/J} & A \\
\mathring{\gamma}_{A,B} & : & A \otimes B & \mathrel{\to\!\!\!\!\!+}_{/J} & B \otimes A
\end{array}$$

Proof. Assume given objects (A, g_A), $(A', g'_{A'})$ etc

(i) Given

$$P : A \mathrel{\to\!\!\!\!\!+}_{/J} A' \qquad Q : B \mathrel{\to\!\!\!\!\!+}_{/J} B' \qquad R : C \mathrel{\to\!\!\!\!\!+}_{/J} C'$$

we have to check

$$\mathring{\alpha}_{A',B',C'} \circ ((P \otimes Q) \otimes R) \quad = \quad (P \otimes (Q \otimes R)) \circ \mathring{\alpha}_{A,B,C}$$

Proof. Consider $((a,b),c) \in (A \otimes B) \otimes C$ and $(a',(b',c')) \in A' \otimes (B' \otimes C')$ such that

$$((a,b),c) \quad \mathring{\alpha}_{A',B',C'} \circ ((P \otimes Q) \otimes R) \quad (a',(b',c'))$$

Note that

$$((a,b),c) \quad ((P \otimes Q) \otimes R) \quad ((a',b'),c')$$

Since

$$g_A(a) \quad = \quad g_{A'}(a') \quad = \quad g_B(b) \quad = \quad g_{B'}(b') \quad = \quad g_C(c) \quad = \quad g_{C'}(c')$$

we have

$$(a,(b,c)) \quad \mathring{\alpha}_{A,B,C} \quad ((a,b),c)$$

and

$$((a,b),c) \quad (P \otimes (Q \otimes R)) \circ \mathring{\alpha}_{A,B,C} \quad (a',(b',c'))$$

We thus have

$$\mathring{\alpha}_{A',B',C'} \circ ((P \otimes Q) \otimes R) \quad \subseteq \quad (P \otimes (Q \otimes R)) \circ \mathring{\alpha}_{A,B,C}$$

For the other direction, consider $((a,b),c) \in (A \otimes B) \otimes C$ and $(a',(b',c') \in A' \otimes (B' \otimes C')$ such that

$$((a,b),c) \quad (P \otimes (Q \otimes R)) \circ \mathring{\alpha}_{A,B,C} \quad (a',(b',c'))$$

We have

$$((a,b),c) \quad \mathring{\alpha}_{A,B,C} \quad (a,(b,c)) \qquad \text{and} \qquad (a,(b,c)) \quad (P \otimes (Q \otimes R)) \quad (a',(b',c'))$$

Now since

$$g_A(a) \quad = \quad g_{A'}(a') \quad = \quad g_B(b) \quad = \quad g_{B'}(b') \quad = \quad g_C(c) \quad = \quad g_{C'}(c')$$

we have

$$((a,b),c) \quad ((P \otimes Q) \otimes R) \quad ((a',b'),c')$$

and

$$((a',b'),c') \quad \mathring{\alpha}_{A',B',C'} \quad (a',(b',c'))$$

It follows that

$$(P \otimes (Q \otimes R)) \circ \mathring{\alpha}_{A,B,C} \quad \subseteq \quad \mathring{\alpha}_{A',B',C'} \circ ((P \otimes Q) \otimes R)$$

◀

(ii) For the naturality of $\mathring{\gamma}_{A,B}$, given

$$P : A \mathrel{+\!\!\!\rightarrow}_{/J} A' \qquad Q : B \mathrel{+\!\!\!\rightarrow}_{/J} B'$$

we have to check

$$\mathring{\gamma}_{A',B'} \circ P \otimes Q \quad = \quad Q \otimes R \circ \mathring{\gamma}_{A,B}$$

and this can be done similarly as for $\mathring{\alpha}_{A,B,C}$ above.

(iii) As for $\mathring{\lambda}$, given

$$P : A \mathrel{\rightarrow\!\!\!\!+}_{/J} A'$$

we have to check

$$\mathring{\lambda}_{A'} \circ (1_{\mathbf{I}} \otimes P) \quad = \quad P \circ \mathring{\lambda}_A$$

Proof. Consider

$$((e, a), a') \in (\mathbf{I} \otimes A) \times_J A'$$

Then we have

$$((e, a), a') \in P \circ \mathring{\lambda}_A$$

if and only if

$$(a, a') \in P$$

But $(a, a') \in P$ implies $((e, a), (e, a')) \in 1_{\mathbf{I}} \otimes P$ since $g_A(a) = g_{A'}(a')$. We moreover get $((e, a'), a') \in \mathring{\lambda}_{A'}$ and it follows that

$$P \circ \mathring{\lambda}_A \quad \subseteq \quad \mathring{\lambda}_{A'} \circ (1_{\mathbf{I}} \otimes P)$$

Conversely, if

$$((e, a), a') \in \mathring{\lambda}_{A'} \circ (1_{\mathbf{I}} \otimes P)$$

then for some $(e', a'') \in \mathbf{I} \otimes A'$ we get

$$((e, a), (e', a'')) \in 1_{\mathbf{I}} \otimes P \quad \text{and} \quad ((e', a''), a') \in \mathring{\lambda}_{A'}$$

But we then get $a'' = a'$ and $\jmath(e') = g_{A'}(a')$, and since $g_A(a) = g_{A'}(a'')$ it follows that $\jmath(e) = \jmath(e')$, hence $e = e'$ since $\jmath$ is injective. Since moreover $(a, a') \in P$, we conclude that

$$((e, a), a') \in P \circ \mathring{\lambda}_A$$

◀

(iv) For $\mathring{\rho}_A$ given

$$P : A \mathrel{\rightarrow\!\!\!\!+}_{/J} A'$$

we have to check

$$\mathring{\rho}_{A'} \circ (P \otimes 1_{\mathbf{I}}) \quad = \quad P \circ \mathring{\rho}_A$$

and this can be done similarly as for $\mathring{\lambda}_A$. ◀

▶ **Proposition 10.4.** *The category* $\mathbf{Rel}(\mathbf{Set}/J)$*, equipped with the above data, is symmetric monoidal.*

Proof. According to Prop. 10.1, Lem. 10.2 and Lem. 10.3, it remains to check the coherence diagrams of symmetric monoidal categories. We follow [16][4].

Assume given objects (A, g_A), (B, g_B), (C, g_C) and (D, g_D) of $\mathbf{Set}/J$. We have to show:

(i) $\mathring{\alpha}_{A,B,(C\otimes D)} \circ \mathring{\alpha}_{A\otimes B,C,D} = (1_A \otimes \mathring{\alpha}_{B,C,D}) \circ \mathring{\alpha}_{A,(B\otimes C),D} \circ (\mathring{\alpha}_{A,B,C} \otimes 1_D)$ in

$$(((A \otimes B) \otimes C) \otimes D) \mathrel{\longrightarrow\!\!\!\!\!\!+}_{/J} A \otimes (B \otimes (C \otimes D))$$

Proof. Consider

$$a \in A \qquad b \in B \qquad c \in C \qquad d \in D$$

with $g_A(a) = g_B(b) = g_C(c) = g_D(d)$.
We then have

$$(((a, b), c), d) \quad \mathring{\alpha}_{A,B,(C\otimes D)} \circ \mathring{\alpha}_{A\otimes B,C,D} \quad (a, (b, (c, d)))$$

and

$$(((a, b), c), d) \quad (1_A \otimes \mathring{\alpha}_{B,C,D}) \circ \mathring{\alpha}_{A,(B\otimes C),D} \circ (\mathring{\alpha}_{A,B,C} \otimes 1_D) \quad (a, (b, (c, d)))$$

Now we are done since both

$$(((a, b), c), d) \quad \mathring{\alpha}_{A,B,(C\otimes D)} \circ \mathring{\alpha}_{A\otimes B,C,D} \quad (a', (b', (c', d')))$$

and

$$(((a, b), c), d) \quad (1_A \otimes \mathring{\alpha}_{B,C,D}) \circ \mathring{\alpha}_{A,(B\otimes C),D} \circ (\mathring{\alpha}_{A,B,C} \otimes 1_D) \quad (a', (b', (c', d')))$$

separately imply

$$a = a' \qquad b = b' \qquad c = c' \qquad d = d'$$

◀

(ii) $(1_A \otimes \mathring{\lambda}_A) \circ \mathring{\alpha}_{A,\mathbf{I},B} = \mathring{\rho}_A \otimes 1_B$ in

$$(A \otimes \mathbf{I}) \otimes B \quad \mathrel{\longrightarrow\!\!\!\!\!\!+}_{/J} \quad A \otimes B$$

Proof. Consider a, b and e such that $((a, e), b) \in (A \otimes \mathbf{I}) \otimes B$.
We have $(a, b) \in A \otimes B$ and

$$((a, e), b) \quad \mathring{\rho}_A \otimes 1_B \quad (a, b)$$

On the other hand, we have

$$((a, e), b) \quad \mathring{\alpha}_{A,\mathbf{I},B} \quad (a, (e, b))$$

and since $g_A(a) = \boldsymbol{\jmath}(e) = g_B(b)$ we get

$$(a, (e, b)) \quad (1_A \otimes \mathring{\lambda}_A) \quad (a, b)$$

[4] By [16, Prop. 3], the diagram [13, VII.7.(2)] relating the braiding $\mathring{\gamma}$ with $\mathring{\lambda}$ and $\mathring{\rho}$ is unnecessary.

Now we are done since both

$$((a, e), b) \quad \mathring{\rho}_A \otimes 1_B \quad (a', b')$$

and

$$((a, e), b) \quad (1_A \otimes \mathring{\lambda}_A) \circ \mathring{\alpha}_{A,\mathbf{I},B} \quad (a', b')$$

separately imply

$$a = a' \qquad b = b'$$

◀

(iii) $\mathring{\alpha}_{B,C,A} \circ (\mathring{\gamma}_{A,(B \otimes C)}) \circ \mathring{\alpha}_{A,B,C} = (1_B \otimes \mathring{\gamma}_{A,C}) \circ \mathring{\alpha}_{B,A,C} \circ (\mathring{\gamma}_{A,B} \otimes 1_C)$ in

$$(A \otimes B) \otimes C \quad \twoheadrightarrow\!\!\!\!\!\!\!\!\!\;+\;_{/J} \quad B \otimes (C \otimes A)$$

Proof. Can be check similarly as (i) above. ◀

(iv) $\mathring{\gamma}_{B,A} = (\mathring{\gamma}_{A,B})^{-1}$ in

$$B \otimes A \quad \twoheadrightarrow\!\!\!\!\!\!\!\!\!\;+\;_{/J} \quad A \otimes B$$

Proof. Recall that

$$\mathring{\gamma}_{A,B} \quad = \quad \{((a, b)\ ,\ (b, a)) \mid a \in A \text{ and } b \in B \text{ and } g_A(a) = g_B(b)\}$$

We thus have

$$\begin{array}{rcl} (\mathring{\gamma}_{A,B})^{-1} & = & \{((b, a)\ ,\ (a, b)) \mid a \in A \text{ and } b \in B \text{ and } g_A(a) = g_B(b)\} \\ & = & \mathring{\gamma}_{B,A} \end{array}$$

◀

◀

▶ Remark ($\mathbf{Rel}$(cod) : $\mathbf{Rel}(\mathbf{Set}^{\to}) \to \mathbf{Set}$ as a monoidal fibration). Because of the definition of $(-)^{\bullet}$ in $\mathbf{Set}^{\to}$ by pullbacks, it seems that the monoidal product $\_ \otimes_{\mathbf{Rel}(\mathbf{Set}/(-))} \_$ is not preserved on the noise by substitution.

On the other hand, [19, Ex. 5.8] mentions, for $\mathbb{C}$ a *regular category*, the $\mathbb{C}$-indexed monoidal category $A \mapsto \mathrm{Sub}(A)$, which need not be strict, since relations over $\mathbb{C}$ form a bicategory rather than a category.

10.1.4 Some Usefull Facts on Set and $\mathrm{Rel}(\mathrm{Set}/J)$

We now state two usefull easy facts concerning some interactions of **Set** with $\mathbf{Rel}(\mathbf{Set}/J)$.

▶ **Lemma 10.5.** *Consider, in* $\mathbf{Rel}(\mathbf{Set}/J)$*, maps*

$$P : A \twoheadrightarrow\!\!\!\!\!\!\!\!\!\;+\;_{/J} C \qquad R : B \twoheadrightarrow\!\!\!\!\!\!\!\!\!\;+\;_{/J} D$$

and consider, in $\mathbf{Set}/J$*, functions*

$$f_A : A \to_{/J} A' \quad f_B : B \to_{/J} B' \quad f_C : C \to_{/J} C' \quad f_D : D \to_{/J} D'$$

Then we have

$$(f_A \times f_C)(P) \otimes (f_B \times f_D)(R) \quad = \quad ((f_A \times f_B) \times (f_C \times f_D))(P \otimes R)$$

where

$$(f_A \times f_C)(P) \quad = \quad \{(f_A(a), f_C(c)) \mid (a,c) \in P\}$$

and similarly for $(f_B \times f_D)(R)$*, and*

$$((f_A{\times}f_B){\times}(f_C{\times}f_D))(P{\otimes}R) \quad = \{((f_A(a), f_B(b)), (f_C(c), f_D(d))) \mid ((a,b),(c,d)) \in P{\otimes}R\}$$

Proof. We first check that

$$\begin{array}{ll} f_A \times f_B : A \otimes B \to_{/J} A' \otimes B' & f_C \times f_D : C \otimes D \to_{/J} C' \otimes D' \\ f_A \times f_C : A \otimes C \to_{/J} A' \otimes C' & f_B \times f_D : B \otimes D \to_{/J} B' \otimes D' \end{array}$$

It is sufficient to look at

$$f_A \times f_B : A \otimes B \to_{/J} A' \otimes B'$$

Recall that by definition $A \otimes B = A \times_J B$ and similarly for $A' \otimes B'$. Now, given $a \in A$ and $b \in B$ with $g_A(a) = g_B(b)$, by assumption on f_A and f_B we get $g_A(a) = g_{A'}(f_A(a))$ and similarly for f_B, from which it follows that

$$g_{A'}(f_A(a)) = g_{B'}(f_B(b))$$

Now we have

$$((a', b'), (c', d')) \in (f_A \times f_C)(P) \otimes (f_B \times f_D)(R)$$

if and only if there are

$$((a,b),(c,d)) \in (A \otimes B) \times (C \otimes D)$$

such that

$$f_A(a) = a' \quad f_B(b) = b' \quad f_C(c) = c' \quad f_D(d) = d'$$

and

$$(a,c) \in P \qquad (b,d) \in R$$

that is

$$((a,b),(c,d)) \in P \otimes R$$

But this is equivalent to

$$((a', b'), (c', d')) \in ((f_A \times f_B) \times (f_C \times f_D))(P \otimes R)$$

◀

▶ **Lemma 10.6.** *Consider composable relations*

$$p : A \nrightarrow_{/J} B \qquad R : B \nrightarrow_{/J} C$$

and maps

$$f_A : A \longrightarrow_{/J} A' \qquad f_B : B \longrightarrow_{/J} B' \qquad f_C : C \longrightarrow_{/J} C'$$

such that f_B is a bijection.
Then we have

$$(f_A \times f_C)(R \circ P) \quad = \quad [(f_B \times f_C)(R)] \circ [(f_A \times f_B)(P)]$$

Proof. Given

$$(f_A(a), f_C(c)) \in (f_A \times f_C)(R \circ P)$$

we have

$$(a, c) \in R \circ P$$

Hence there is some b such that

$$(a, b) \in P \qquad (b, c) \in R$$

It follows that

$$(f_A(a), f_B(b)) \in (f_A \times f_B)(P) \qquad (f_B(b), f_C(c)) \in (f_B \times f_C)(R)$$

Hence

$$(f_A(a), f_C(c)) \in [(f_B \times f_C)(R)] \circ [(f_A \times f_B)(P)]$$

Conversely, given

$$(f_A(a), f_C(c)) \in [(f_B \times f_C)(R)] \circ [(f_A \times f_B)(P)]$$

there is $b' \in f_B(B)$ such that

$$(f_A(a), b') \in (f_A \times f_B)(P) \qquad (b', f_C(c)) \in (f_B \times f_C)(R)$$

Now since f_B is a bijection, there ia unique $b \in B$ such that $b' = f_B(b)$, and it follows that

$$(a, b) \in P \qquad (b, c) \in R$$

Hence

$$(f_A(a), f_C(c)) \in (f_A \times f_C)(R \circ P)$$

◀

10.2 Complete Automata

Recall from Sect.2 that an automaton $\mathcal{A}$ is *complete* if for every $(q, a) \in Q \times \Sigma$, the set $\delta(q, a)$ is not empty and moreover for every $\gamma \in \delta(q, a)$ and every direction $d \in D$, we have $(q', d) \in \gamma$ for some $q' \in Q$.

Given an automaton $\mathcal{A}$ with

$$\mathcal{A} \quad = \quad (Q, q^\imath, \delta, \Omega)$$

its *completion* is the automaton

$$\widehat{\mathcal{A}} \quad := \quad (\widehat{Q}, q^\imath, \widehat{\delta}, \widehat{\Omega})$$

where

- $\widehat{Q} := Q + \{\mathsf{true}, \mathsf{false}\}$,
- the transition function $\widehat{\delta}$ is defined as follows:

$$\begin{array}{rcll} \widehat{\delta}(\mathsf{true}, q) & := & \{\{(\mathsf{true}, d) \mid d \in D\}\} & \\ \widehat{\delta}(\mathsf{false}, q) & := & \{\{(\mathsf{false}, d) \mid d \in D\}\} & \\ \widehat{\delta}(q, a) & := & \{\{(\mathsf{false}, d) \mid d \in D\}\} & \text{if } q \in Q \text{ and } \delta(q, a) = \emptyset \\ \widehat{\delta}(q, a) & := & \{\widehat{\gamma} \mid \gamma \in \delta(q, a)\} & \text{otherwise} \end{array}$$

  where, given $\gamma \in \delta(q, a)$ we let

$$\widehat{\gamma} \quad := \quad \gamma \cup \{(\mathsf{true}, d) \mid \text{there is no } q \in Q \text{ s.t. } (q, d) \in \gamma\}$$

- $\widehat{\Omega} := \Omega + Q^* \cdot \mathsf{true} \cdot \widehat{Q}^\omega$.

▶ **Proposition 10.7.** $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\widehat{\mathcal{A}})$.

Full Subcategories and Fibrations. Restricting to complete automata gives rise to full subcategories $\widehat{\mathbf{SAG}}_\Sigma^{(\mathrm{W})}$ and $\widehat{\mathbf{Aut}}_\Sigma^{(\mathrm{W})}$ or resp. $\mathbf{SAG}_\Sigma$ and $\mathbf{Aut}^{(\mathrm{W})}$, and thus induces fibrations

$$\widehat{\mathsf{game}} \;:\; \widehat{\mathbf{SAG}}^{(\mathrm{W})} \longrightarrow \mathbf{Tree} \qquad \widehat{\mathsf{aut}} \;:\; \widehat{\mathbf{Aut}}^{(\mathrm{W})} \longrightarrow \mathbf{Tree}$$

10.3 Synchronous Monoidal Product on Automata

Assume given complete automata $\Sigma \vdash \mathcal{A}$ and $\Sigma \vdash \mathcal{B}$. Define $\Sigma \vdash \mathcal{A} \circledast \mathcal{B}$ as follows:

$$\mathcal{A} \circledast \mathcal{B} \quad := \quad (Q_\mathcal{A} \times Q_\mathcal{B}, (q^\imath_\mathcal{A}, q^\imath_\mathcal{B}), \delta_{\mathcal{A} \circledast \mathcal{B}}, \Omega_{\mathcal{A} \circledast \mathcal{B}})$$

where

$$\delta_{\mathcal{A} \circledast \mathcal{B}}((q_\mathcal{A}, q_\mathcal{B}), a) \quad := \quad \bigcup_{\substack{\gamma_\mathcal{A} \in \delta_\mathcal{A}(q_\mathcal{A}, a) \\ \gamma_\mathcal{B} \in \delta_\mathcal{B}(q_\mathcal{B}, a)}} \gamma_\mathcal{A} \circledast \gamma_\mathcal{B}$$

and

$$\gamma_\mathcal{A} \circledast \gamma_\mathcal{B} \quad := \quad \{((q'_\mathcal{A}, q'_\mathcal{B}), d) \mid d \in D \text{ and } (q'_\mathcal{A}, d) \in \gamma_\mathcal{A} \text{ and } (q'_\mathcal{B}, d) \in \gamma_\mathcal{B}\}$$

and moreover

$$(q^n_\mathcal{A}, q^n_\mathcal{B})_{n \in \mathbb{N}} \in \Omega_{\mathcal{A} \circledast \mathcal{B}} \qquad \text{iff} \qquad ((q^n_\mathcal{A})_{n \in \mathbb{N}} \in \Omega_\mathcal{A} \quad \text{and} \quad (q^n_\mathcal{B})_{n \in \mathbb{N}} \in \Omega_\mathcal{B})$$

For synchronous reductions (in categories $\widehat{\mathbf{SAG}}^{\mathrm{R}}_{\Sigma}$ and $\widehat{\mathbf{Aut}}^{\mathrm{R}}_{\Sigma}$), we will rather use the product $\Sigma \vdash \mathcal{A} \circledast_{\equiv} \mathcal{B}$ defined as

$$\mathcal{A} \circledast_{\equiv} \mathcal{B} \quad := \quad (Q_{\mathcal{A}} \times Q_{\mathcal{B}}, (q^{\imath}_{\mathcal{A}}, q^{\imath}_{\mathcal{B}}), \delta_{\mathcal{A} \circledast \mathcal{B}}, \Omega_{\mathcal{A} \circledast_{\equiv} \mathcal{B}})$$

where

$$(q^n_{\mathcal{A}}, q^n_{\mathcal{B}})_{n \in \mathbb{N}} \in \Omega_{\mathcal{A} \circledast_{\equiv} \mathcal{B}} \qquad \text{iff} \qquad ((q^n_{\mathcal{A}})_{n \in \mathbb{N}} \in \Omega_{\mathcal{A}} \quad \Longleftrightarrow \quad (q^n_{\mathcal{B}})_{n \in \mathbb{N}} \in \Omega_{\mathcal{B}})$$

10.4 Action on Acceptance Games of the Synchronous Monoidal Product

Given *complete* $\Sigma \vdash \mathcal{A}$ and $\Sigma \vdash \mathcal{B}$ as above, we define projections

$$\varpi_1 : \wp_{\Sigma}(\mathcal{A} \circledast \mathcal{B}, M) \quad \to \quad \wp_{\Sigma}(\mathcal{A}, M) \qquad \text{and} \qquad \varpi_2 : \wp_{\Sigma}(\mathcal{A} \circledast \mathcal{B}, M) \quad \to \quad \wp_{\Sigma}(\mathcal{B}, M)$$

Note that since $\mathcal{A}$ and $\mathcal{B}$ are complete:

▶ **Lemma 10.8.** $\gamma_{\mathcal{A}} \circledast \gamma_{\mathcal{B}} = \gamma'_{\mathcal{A}} \circledast \gamma'_{\mathcal{B}}$ *implies* $\gamma_{\mathcal{A}} = \gamma'_{\mathcal{A}}$ *and* $\gamma_{\mathcal{B}} = \gamma'_{\mathcal{B}}$.

Proof. Assume that (say) $\gamma_{\mathcal{A}} \neq \gamma'_{\mathcal{A}}$ so that (say) we have

$$(q_{\mathcal{A}}, d) \in \gamma_{\mathcal{A}} \setminus \gamma'_{\mathcal{A}}$$

Then, since $\mathcal{B}$ is complete, there is $q_{\mathcal{B}}$ such that $(q_{\mathcal{B}}, q) \in \gamma_{\mathcal{B}}$. It follows that

$$((q_{\mathcal{A}}, q_{\mathcal{B}}), d) \in \gamma_{\mathcal{A}} \circledast \gamma_{\mathcal{B}} \setminus \gamma'_{\mathcal{A}} \circledast \gamma'_{\mathcal{B}}$$

◀

Using Lem. 10.8, we now define the projections

$$\varpi_1 : \wp_{\Sigma}(\mathcal{A} \circledast \mathcal{B}, M) \quad \to \quad \wp_{\Sigma}(\mathcal{A}, M) \qquad \text{and} \qquad \varpi_2 : \wp_{\Sigma}(\mathcal{A} \circledast \mathcal{B}, M) \quad \to \quad \wp_{\Sigma}(\mathcal{B}, M)$$

The projection $\varpi_1 : \wp_{\Sigma}(\mathcal{A} \circledast \mathcal{B}, M) \to \wp_{\Sigma}(\mathcal{A}, M)$ is defined by induction as follows:

$$\begin{array}{rcl}
\varpi_1(\varepsilon : (\varepsilon, (q^{\imath}_{\mathcal{A}}, q^{\imath}_{\mathcal{B}}))) & := & \varepsilon : (\varepsilon, q^{\imath}_{\mathcal{A}}) \\
\varpi_1(s \to (p, a, \gamma_{\mathcal{A}} \circledast \gamma_{\mathcal{B}})) & := & \varpi_1(s) \to (p, a, \gamma_{\mathcal{A}}) \\
\varpi_1(s \to (p \cdot d, (q_{\mathcal{A}}, q_{\mathcal{B}}))) & := & \varpi_1(s) \to (p \cdot d, q_{\mathcal{A}})
\end{array}$$

Note that in the second case above, ϖ_1 is well defined thanks to Lem. 10.8. The other projection $\varpi_2 : \wp_{\Sigma}(\mathcal{A} \circledast \mathcal{B}, M) \to \wp_{\Sigma}(\mathcal{B}, M)$ is defined similarly.

Note that it readily follows that the following diagram commutes:

$$\begin{tikzcd}
\wp_{\Sigma}(\mathcal{A} \circledast \mathcal{B}, M) \arrow[r, "\varpi_2"] \arrow[d, "\varpi_1"'] & \wp_{\Sigma}(\mathcal{B}, M) \arrow[d, "\mathrm{tr}"] \\
\wp_{\Sigma}(\mathcal{A}, M) \arrow[r, "\mathrm{tr}"'] & \mathrm{Tr}_{\Sigma}
\end{tikzcd} \tag{16}$$

We write

$$\mathrm{SP} \quad := \quad \langle \varpi_1, \varpi_2 \rangle \quad : \quad \wp_{\Sigma}(\mathcal{A} \circledast \mathcal{B}, M) \quad \longrightarrow \quad \wp_{\Sigma}(\mathcal{A}, M) \times_{\mathrm{Tr}_{\Sigma}} \wp_{\Sigma}(\mathcal{B}, M)$$

▶ **Proposition 10.9.** *We have, in* **Set***:*

$$
\begin{tikzcd}
\wp_\Sigma(\mathcal{A} \circledast \mathcal{B}, M) \arrow[r, "\varpi_2"] \arrow[d, "\varpi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \wp_\Sigma(\mathcal{B}, M) \arrow[d, "\mathrm{tr}"] \\
\wp_\Sigma(\mathcal{A}, M) \arrow[r, "\mathrm{tr}"'] & \mathrm{Tr}_\Sigma
\end{tikzcd}
$$

Proof. We show that we have a bijection

$$\mathrm{SP} : \wp_\Sigma(\mathcal{A} \circledast \mathcal{B}, M) \quad \xrightarrow{\simeq} \quad \wp_\Sigma(\mathcal{A}, M) \times_{\mathrm{Tr}_\Sigma} \wp_\Sigma(\mathcal{B}, M)$$

For the injectivity, consider $s, t \in \wp_\Sigma(\mathcal{A} \circledast \mathcal{B}, M)$ such that $\mathrm{SP}(s) = \mathrm{SP}(t)$. Note that since ϖ_1 and ϖ_2 are length-preserving, we must have $|s| = |t|$. We thus reason by induction on $|s| = |t|$. In the base case, we must have $s = t = \varepsilon : (\varepsilon, (q^\imath_\mathcal{A}, q^\imath_\mathcal{B}))$ and we are done. For the induction step, there are two cases:

- If

$$s \quad = \quad s' \to (p, a, \gamma_\mathcal{A} \circledast \gamma_\mathcal{B}) \qquad \text{and} \qquad t \quad = \quad t' \to (p', a', \gamma'_\mathcal{A} \circledast \gamma'_\mathcal{B})$$

  then we must have $p = p'$ and $a = a'$ and $\gamma_\mathcal{A} = \gamma'_\mathcal{A}$ and $\gamma_\mathcal{B} = \gamma'_\mathcal{B}$, and we are done by induction hypothesis.
- Otherwise, we must have

$$s \quad = \quad s' \to (p \cdot d, (q_\mathcal{A}, q_\mathcal{B})) \qquad \text{and} \qquad t \quad = \quad t' \to (p' \cdot d', a', (q'_\mathcal{A}, q'_\mathcal{B}))$$

  and again, we are done thanks to the induction hypothesis.

  For surjectivity, given

$$(s, t) \in \wp_\Sigma(\mathcal{A}, M) \times_{\mathrm{Tr}_\Sigma} \wp_\Sigma(\mathcal{B}, M)$$

we must build

$$u \in \wp_\Sigma(\mathcal{A} \circledast \mathcal{B}, M)$$

such that

$$\varpi_1(u) \quad = \quad s \qquad \text{and} \qquad \varpi_2(u) \quad = \quad t$$

Note that $\mathrm{tr}(s) = \mathrm{tr}(t)$ implies $|s| = |t|$. We thus reason by induction on $|s| = |t|$. In the case case $s = t = \varepsilon$ and we take

$$u \quad := \quad \varepsilon : (\varepsilon, (q^\imath_\mathcal{A}, q^\imath_\mathcal{B}))$$

For the induction step, we consider two cases:

- If

$$s \quad = \quad s' \to (p, a, \gamma_\mathcal{A}) \qquad \text{and} \qquad t \quad = \quad t' \to (p', a', \gamma_\mathcal{B})$$

  then $\mathrm{tr}(s) = \mathrm{tr}(t)$ implies $p = p'$ and $a = a'$. Moreover, we have $\mathrm{tr}(s') = \mathrm{tr}(t')$ and thus, by induction hypothesis, there is some u' such that $\mathrm{SP}(u') = (s', t')$. It follows that s' and t' are of the form:

$$s' \quad = \quad *_\mathcal{A} \to^* (p, q_\mathcal{A}) \qquad \text{and} \qquad t' \quad = \quad *_\mathcal{B} \to^* (p, q_\mathcal{B})$$

  with $\gamma_\mathcal{A} \in \delta_\mathcal{A}(q_\mathcal{A}, a)$ and $\gamma_\mathcal{B} \in \delta_\mathcal{B}(q_\mathcal{B}, a)$. It follows that we extend u' as follows:

$$u' \to (p, a, \gamma_\mathcal{A} \circledast \gamma_\mathcal{B})$$

- Otherwise, we must have

$$s \quad = \quad s' \to (p \cdot d, q_{\mathcal{A}}) \qquad \text{and} \qquad t \quad = \quad t' \to (p' \cdot d', q_{\mathcal{B}})$$

then we must have $p = p'$ and $d = d'$ since $\mathrm{tr}(s') = \mathrm{tr}(t')$. Moreover $\mathrm{tr}(s') = \mathrm{tr}(t')$ and by induction hypothesis, we have $(s', t') = \mathrm{SP}(u')$ for some u'. On the other hand, since $\mathrm{tr}(s') = \mathrm{tr}(t')$, we must have, for some $a \in \Sigma$,

$$s' \quad = \quad *_{\mathcal{A}} \to^* (p, a, \gamma_{\mathcal{A}}) \qquad \text{and} \qquad t' \quad = \quad *_{\mathcal{B}} \to^* (p, a, \gamma_{\mathcal{B}})$$

with $(q_{\mathcal{A}}, d) \in \gamma_{\mathcal{A}}$ and $(q_{\mathcal{B}}, d) \in \gamma_{\mathcal{B}}$. Note that we must have

$$u' \quad = \quad *_{\mathcal{A} \circledast \mathcal{B}} \to^* (p, a, \gamma_{\mathcal{A}} \circledast \gamma_{\mathcal{B}})$$

But we have $((q_{\mathcal{A}}, q_{\mathcal{B}}), d) \in \gamma_{\mathcal{A}} \circledast \gamma_{\mathcal{B}}$ and we are done by taking

$$u \quad := \quad u' \to (p \cdot d, (q_{\mathcal{A}}, q_{\mathcal{B}}))$$

◀

10.5 Action on the Synchronous Arrow of the Synchronous Monoidal Product

We now extend the projections ϖ_1 and ϖ_2 of Sect. 10.4 to the plays of the synchronous arrow.

In the whole Section, we assume given *complete* automata $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$. Using Lem. 10.8, define

$$\varpi_1 \quad : \quad \wp_\Sigma \left(\mathcal{G}(\mathcal{A} \circledast \mathcal{B}, M) \multimap_\circledast \mathcal{G}(\mathcal{C} \circledast \mathcal{D}, N)\right) \quad \to \quad \wp_\Sigma \left(\mathcal{G}(\mathcal{A}, M) \multimap_\circledast \mathcal{G}(\mathcal{C}, N)\right)$$

as follows:

$$\begin{array}{rcl}
\varpi_1 \left((\varepsilon, (q^\imath_{\mathcal{A}}, q^\imath_{\mathcal{B}}))\ ,\ (\varepsilon, (q^\imath_{\mathcal{C}}, q^\imath_{\mathcal{D}}))\right) & := & \left((\varepsilon, q^\imath_{\mathcal{A}})\ ,\ (\varepsilon, q^\imath_{\mathcal{C}})\right) \\
\varpi_1 \left(s \to ((p, a, \gamma_{\mathcal{A}} \circledast \gamma_{\mathcal{B}})\ ,\ (p, (q_{\mathcal{C}}, q_{\mathcal{D}})))\right) & := & \varpi_1(s) \to ((p, a, \gamma_{\mathcal{A}})\ ,\ (p, q_{\mathcal{C}})) \\
\varpi_1 \left(s \to ((p, a, \gamma_{\mathcal{A}} \circledast \gamma_{\mathcal{B}})\ ,\ (p, a, \gamma_{\mathcal{C}} \circledast \gamma_{\mathcal{D}}))\right) & := & \varpi_1(s) \to ((p, a, \gamma_{\mathcal{A}})\ ,\ (p, a, \gamma_{\mathcal{C}})) \\
\varpi_1 \left(s \to ((p, a, \gamma_{\mathcal{A}} \circledast \gamma_{\mathcal{B}})\ ,\ (p \cdot d, (q_{\mathcal{C}}, q_{\mathcal{D}})))\right) & := & \varpi_1(s) \to ((p, a, \gamma_{\mathcal{A}})\ ,\ (p \cdot d, q_{\mathcal{C}})) \\
\varpi_1 \left(s \to ((p \cdot d, (q_{\mathcal{A}}, q_{\mathcal{B}}))\ ,\ (p \cdot d, (q_{\mathcal{C}}, q_{\mathcal{D}})))\right) & := & \varpi_1(s) \to ((p \cdot d, q_{\mathcal{A}})\ ,\ (p \cdot d, q_{\mathcal{C}}))
\end{array}$$

The second projection

$$\varpi_2 \quad : \quad \wp_\Sigma \left(\mathcal{G}(\mathcal{A} \circledast \mathcal{B}, M) \multimap_\circledast \mathcal{G}(\mathcal{C} \circledast \mathcal{D}, N)\right) \quad \to \quad \wp_\Sigma \left(\mathcal{G}(\mathcal{B}, M) \multimap_\circledast \mathcal{G}(\mathcal{D}, N)\right)$$

is defined similarly.

▶ **Lemma 10.10.** *We have, in* **Set**,

$$\begin{tikzcd}
\wp^{\mathsf{P}}_\Sigma \left(\mathcal{G}(\mathcal{A} \circledast \mathcal{B}, M) \multimap_\circledast \mathcal{G}(\mathcal{C} \circledast \mathcal{D}, N)\right) \arrow[r, "\varpi_2"] \arrow[d, "\varpi_1"'] & \wp^{\mathsf{P}}_\Sigma \left(\mathcal{G}(\mathcal{B}, M) \multimap_\circledast \mathcal{G}(\mathcal{D}, N)\right) \arrow[d, "\mathrm{tr}^{-\circledast}"] \\
\wp^{\mathsf{P}}_\Sigma \left(\mathcal{G}(\mathcal{A}, M) \multimap_\circledast \mathcal{G}(\mathcal{C}, N)\right) \arrow[r, "\mathrm{tr}^{-\circledast}"'] & \mathrm{Tr}_\Sigma
\end{tikzcd} \tag{17}$$

Proof. By induction on

$$s \quad \in \quad \wp^{\mathsf{P}}_{\Sigma}\left(\mathcal{G}(\mathcal{A} \circledast \mathcal{B}, M) \multimap\!\!\circledast\, \mathcal{G}(\mathcal{C} \circledast \mathcal{D}, N)\right)$$

we show that

$$\mathrm{tr}^{-\circledast} \circ \varpi_1(s) \quad = \quad \mathrm{tr}^{-\circledast} \circ \varpi_2(s)$$

In the base case, we have

$$s \quad = \quad ((\varepsilon, (q^{\imath}_{\mathcal{A}}, q^{\imath}_{\mathcal{B}}))\ ,\ (\varepsilon, (q^{\imath}_{\mathcal{C}}, q^{\imath}_{\mathcal{D}})))$$

with

$$\varpi_1(s) \quad = \quad ((\varepsilon, q^{\imath}_{\mathcal{A}})\ ,\ (\varepsilon, q^{\imath}_{\mathcal{C}})) \qquad \text{and} \qquad \varpi_2(s) \quad = \quad ((\varepsilon, q^{\imath}_{\mathcal{B}})\ ,\ (\varepsilon, q^{\imath}_{\mathcal{D}}))$$

and we are done since

$$\mathrm{tr}^{-\circledast} \circ \varpi_1(s) \quad = \quad \varepsilon \quad = \quad \mathrm{tr}^{-\circledast} \circ \varpi_2(s)$$

For the induction, there are two cases:

Case of $s = t \xrightarrow{\mathsf{O}}\xrightarrow{\mathsf{P}} ((p, a, \gamma_{\mathcal{A}} \circledast \gamma_{\mathcal{B}})\ ,\ (p, a, \gamma_{\mathcal{C}} \circledast \gamma_{\mathcal{D}}))$.
In this case,

$$\begin{array}{lrcl} & \varpi_1(s) & = & \varpi_1(t) \xrightarrow{\mathsf{O}}\xrightarrow{\mathsf{P}} ((p, a, \gamma_{\mathcal{A}})\ ,\ (p, a, \gamma_{\mathcal{C}})) \\ \text{and} & \varpi_2(s) & = & \varpi_2(t) \xrightarrow{\mathsf{O}}\xrightarrow{\mathsf{P}} ((p, a, \gamma_{\mathcal{B}})\ ,\ (p, a, \gamma_{\mathcal{D}})) \end{array}$$

Hence

$$\begin{array}{lrcl} & \mathrm{tr}^{-\circledast} \circ \varpi_1(s) & = & \mathrm{tr}^{-\circledast} \circ \varpi_1(t) \cdot a \\ \text{and} & \mathrm{tr}^{-\circledast} \circ \varpi_2(s) & = & \mathrm{tr}^{-\circledast} \circ \varpi_2(t) \cdot a \end{array}$$

and we are done since $\mathrm{tr}^{-\circledast} \circ \varpi_1(t) = \mathrm{tr}^{-\circledast} \circ \varpi_2(t)$ by induction hypothesis.

Case of $s = t \xrightarrow{\mathsf{O}}\xrightarrow{\mathsf{P}} ((p \cdot d, (q_{\mathcal{A}}, q_{\mathcal{B}}))\ ,\ (p \cdot d, (q_{\mathcal{C}}, q_{\mathcal{D}})))$.
In this case,

$$\begin{array}{lrcl} & \varpi_1(s) & = & \varpi_1(t) \xrightarrow{\mathsf{O}}\xrightarrow{\mathsf{P}} ((p \cdot d, q_{\mathcal{A}})\ ,\ (p \cdot d, q_{\mathcal{C}})) \\ \text{and} & \varpi_2(s) & = & \varpi_2(t) \xrightarrow{\mathsf{O}}\xrightarrow{\mathsf{P}} ((p \cdot d, q_{\mathcal{B}})\ ,\ (p \cdot d, q_{\mathcal{D}})) \end{array}$$

Hence

$$\begin{array}{lrcl} & \mathrm{tr}^{-\circledast} \circ \varpi_1(s) & = & \mathrm{tr}^{-\circledast} \circ \varpi_1(t) \cdot d \\ \text{and} & \mathrm{tr}^{-\circledast} \circ \varpi_2(s) & = & \mathrm{tr}^{-\circledast} \circ \varpi_2(t) \cdot d \end{array}$$

and we are done since $\mathrm{tr}^{-\circledast} \circ \varpi_1(t) = \mathrm{tr}^{-\circledast} \circ \varpi_2(t)$ by induction hypothesis.

◀

▶ Remark. Note that Lem. 10.10 can not be extended to all plays of $(\mathcal{G}(\mathcal{A} \circledast \mathcal{B}, M) \multimap\!\!\circledast\, \mathcal{G}(\mathcal{C} \circledast \mathcal{D}, N))$ since $\mathrm{tr}^{-\circledast}$ is only defined on P-plays (see Rem 6.2.3).

Consider now the structure isomorphism in **Set**:

$$m \quad : \quad (A \times B) \times (C \times D) \quad \xrightarrow{\simeq} \quad (A \times C) \times (B \times D)$$

Note that m restricts to a bijection in $\mathbf{Set}/J$: Given objects (A, g), (B, h), (C, k) and (D, l) of $\mathbf{Set}/J$, we have, in $\mathbf{Set}/J$

$$m \quad : \quad (A \times_J B) \times_J (C \times_J D) \quad \xrightarrow{\simeq} \quad (A \times_J C) \times_J (B \times_J D)$$

▶ **Lemma 10.11.** *In* **Set***, we have*

$$(\mathrm{HS} \times \mathrm{HS}) \circ \mathrm{SP}_{-\circledast} \quad = \quad m \circ (\mathrm{SP} \times \mathrm{SP}) \circ \mathrm{HS}$$

in

$$\wp_\Sigma\left(\mathcal{G}(\mathcal{A} \circledast \mathcal{B}, M) \multimap\!\!\circledast\, \mathcal{G}(\mathcal{C} \circledast \mathcal{D}, N)\right) \quad \longrightarrow \quad (\wp_\Sigma(\mathcal{A}, M) \times \wp_\Sigma(\mathcal{C}, N)) \times (\wp_\Sigma(\mathcal{B}, M) \times \wp_\Sigma(\mathcal{D}, N))$$

In diagram:

$$\begin{tikzcd}
\wp_\Sigma\left(\mathcal{G}(\mathcal{A} \circledast \mathcal{B}, M) \multimap\!\!\circledast\, \mathcal{G}(\mathcal{C} \circledast \mathcal{D}, N)\right) \arrow[r, "\mathrm{HS}"] \arrow[d, "\mathrm{SP}_{-\circledast}"'] & \wp_\Sigma(\mathcal{A} \circledast \mathcal{B}, M) \times \wp_\Sigma(\mathcal{C} \circledast \mathcal{D}, N) \arrow[d, "\mathrm{SP}\times\mathrm{SP}"] \\
\wp_\Sigma\left(\mathcal{G}(\mathcal{A}, M) \multimap\!\!\circledast\, \mathcal{G}(\mathcal{C}, N)\right) \times \wp_\Sigma\left(\mathcal{G}(\mathcal{B}, M) \multimap\!\!\circledast\, \mathcal{G}(\mathcal{D}, N)\right) \arrow[d, "\mathrm{HS}\times\mathrm{HS}"'] & (\wp_\Sigma(\mathcal{A}, M) \times \wp_\Sigma(\mathcal{B}, M)) \times (\wp_\Sigma(\mathcal{C}, N) \times \wp_\Sigma(\mathcal{D}, N)) \arrow[dl, "m"] \\
(\wp_\Sigma(\mathcal{A}, M) \times \wp_\Sigma(\mathcal{C}, N)) \times (\wp_\Sigma(\mathcal{B}, M) \times \wp_\Sigma(\mathcal{D}, N)) &
\end{tikzcd}$$

Proof. By induction on

$$t \in \wp_\Sigma\left(\mathcal{G}(\mathcal{A} \circledast \mathcal{B}, M) \multimap\!\!\circledast\, \mathcal{G}(\mathcal{C} \circledast \mathcal{D}, N)\right)$$

In the base case, we have

$$t \quad = \quad ((\varepsilon, (q^\imath_{\mathcal{A}}, q^\imath_{\mathcal{B}}))\ ,\ (\varepsilon, (q^\imath_{\mathcal{C}}, q^\imath_{\mathcal{D}})))$$

and we have

$$(\mathrm{HS}\times\mathrm{HS})\circ\mathrm{SP}_{-\circledast}(t) \quad = \quad (((\varepsilon, q^\imath_{\mathcal{A}}), (\varepsilon, q^\imath_{\mathcal{C}})), ((\varepsilon, q^\imath_{\mathcal{B}}), (\varepsilon, q^\imath_{\mathcal{D}}))) \quad = \quad m\circ(\mathrm{SP}\times\mathrm{SP})\circ\mathrm{HS}(t)$$

For the induction step, there are four cases.

Case of $t = s \to ((p, a, \gamma_{\mathcal{A}} \circledast \gamma_{\mathcal{B}})\ ,\ (p, (q_{\mathcal{C}}, q_{\mathcal{D}})))$.

We have

$$\mathrm{SP}_{-\circledast}(t) \quad = \quad (\varpi_1(t) \to ((p, a, \gamma_{\mathcal{A}}), (p, q_{\mathcal{C}}))\ ,\ \varpi_2(t) \to ((p, a, \gamma_{\mathcal{B}}), (p, q_{\mathcal{D}})))$$

Hence $(\mathrm{HS} \times \mathrm{HS}) \circ \mathrm{SP}_{-\circledast}(t)$ is

$$\begin{array}{llll} (& (\varpi_1(s){\upharpoonright}\mathcal{G}(\mathcal{A}, M) \to (p, a, \gamma_{\mathcal{A}}) & , \quad \varpi_1(s){\upharpoonright}\mathcal{G}(\mathcal{C}, N) \to (p, q_{\mathcal{C}})) & , \\ & (\varpi_2(s){\upharpoonright}\mathcal{G}(\mathcal{B}, M) \to (p, a, \gamma_{\mathcal{B}}) & , \quad \varpi_2(s){\upharpoonright}\mathcal{G}(\mathcal{D}, N) \to (p, q_{\mathcal{D}})) &) \end{array}$$

and $(\mathrm{HS} \times \mathrm{HS}) \circ \mathrm{SP}_{-\circledast}(s)$ is

$$\begin{array}{llll} (& (\varpi_1(s){\upharpoonright}\mathcal{G}(\mathcal{A}, M) & , \quad \varpi_1(s){\upharpoonright}\mathcal{G}(\mathcal{C}, N)) & , \\ & (\varpi_2(s){\upharpoonright}\mathcal{G}(\mathcal{B}, M) & , \quad \varpi_2(s){\upharpoonright}\mathcal{G}(\mathcal{D}, N)) &) \end{array}$$

On the other hand,

$$\mathrm{HS}(t) \quad = \quad (s{\upharpoonright}\mathcal{G}(\mathcal{A} \circledast \mathcal{B}, M) \to (p, a, \gamma_{\mathcal{A}} \circledast \gamma_{\mathcal{B}})\ ,\ s{\upharpoonright}\mathcal{G}(\mathcal{C} \circledast \mathcal{D}, N) \to (p, (q_{\mathcal{C}}, q_{\mathcal{D}})))$$

Hence, $(\mathrm{SP} \times \mathrm{SP}) \circ \mathrm{HS}(t)$ is

$$\begin{array}{llll} (& (\varpi_1(s{\upharpoonright}\mathcal{G}(\mathcal{A} \circledast \mathcal{B}, M)) \to (p, a, \gamma_{\mathcal{A}}) & , \quad \varpi_2(s{\upharpoonright}\mathcal{G}(\mathcal{A} \circledast \mathcal{B}, M)) \to (p, a, \gamma_{\mathcal{B}})) & \\ & (\varpi_1(s{\upharpoonright}\mathcal{G}(\mathcal{C} \circledast \mathcal{D}, N)) \to (p, q_{\mathcal{C}}) & , \quad \varpi_2(s{\upharpoonright}\mathcal{G}(\mathcal{C} \circledast \mathcal{D}, N)) \to (p, q_{\mathcal{D}})) &) \end{array}$$

and $(\mathrm{SP}\times\mathrm{SP})\circ\mathrm{HS}(s)$ is

$$\begin{array}{llll} (& (\varpi_1(s{\restriction}\mathcal{G}(\mathcal{A}\circledast\mathcal{B},M)) & , & \varpi_2(s{\restriction}\mathcal{G}(\mathcal{A}\circledast\mathcal{B},M))) \\ & (\varpi_1(s{\restriction}\mathcal{G}(\mathcal{C}\circledast\mathcal{D},N)) & , & \varpi_2(s{\restriction}\mathcal{G}(\mathcal{C}\circledast\mathcal{D},N))) \quad) \end{array}$$

Now we are done since by induction hypothesis

$$(\mathrm{HS}\times\mathrm{HS})\circ\mathrm{SP}_{-\circledast}(s) \quad = \quad m\circ(\mathrm{SP}\times\mathrm{SP})\circ\mathrm{HS}(s)$$

Case of $t = s \to ((p,a,\gamma_{\mathcal{A}}\circledast\gamma_{\mathcal{B}})\ ,\ (p,a,\gamma_{\mathcal{C}}\circledast\gamma_{\mathcal{D}}))$.
We have

$$\mathrm{SP}_{-\circledast}(t) \quad = \quad (\varpi_1(t)\to((p,a,\gamma_{\mathcal{A}}),(p,a,\gamma_{\mathcal{C}}))\ ,\ \varpi_2(t)\to((p,a,\gamma_{\mathcal{B}}),(p,a,\gamma_{\mathcal{D}})))$$

Hence $(\mathrm{HS}\times\mathrm{HS})\circ\mathrm{SP}_{-\circledast}(t)$ is

$$\begin{array}{llll} (& (\varpi_1(s){\restriction}\mathcal{G}(\mathcal{A},M)\to(p,a,\gamma_{\mathcal{A}}) & , & \varpi_1(s){\restriction}\mathcal{G}(\mathcal{C},N)\to(p,a,\gamma_{\mathcal{C}})) \quad , \\ & (\varpi_2(s){\restriction}\mathcal{G}(\mathcal{B},M)\to(p,a,\gamma_{\mathcal{B}}) & , & \varpi_2(s){\restriction}\mathcal{G}(\mathcal{D},N)\to(p,a,\gamma_{\mathcal{D}})) \quad) \end{array}$$

and $(\mathrm{HS}\times\mathrm{HS})\circ\mathrm{SP}_{-\circledast}(s)$ is

$$\begin{array}{llll} (& (\varpi_1(s){\restriction}\mathcal{G}(\mathcal{A},M) & , & \varpi_1(s){\restriction}\mathcal{G}(\mathcal{C},N)) \quad , \\ & (\varpi_2(s){\restriction}\mathcal{G}(\mathcal{B},M) & , & \varpi_2(s){\restriction}\mathcal{G}(\mathcal{D},N)) \quad) \end{array}$$

On the other hand,

$$\mathrm{HS}(t) \quad = \quad (s{\restriction}\mathcal{G}(\mathcal{A}\circledast\mathcal{B},M)\to(p,a,\gamma_{\mathcal{A}}\circledast\gamma_{\mathcal{B}})\ ,\ s{\restriction}\mathcal{G}(\mathcal{C}\circledast\mathcal{D},N)\to(p,a,\gamma_{\mathcal{C}}\circledast\gamma_{\mathcal{D}}))$$

Hence, $(\mathrm{SP}\times\mathrm{SP})\circ\mathrm{HS}(t)$ is

$$\begin{array}{llll} (& (\varpi_1(s{\restriction}\mathcal{G}(\mathcal{A}\circledast\mathcal{B},M))\to(p,a,\gamma_{\mathcal{A}}) & , & \varpi_2(s{\restriction}\mathcal{G}(\mathcal{A}\circledast\mathcal{B},M))\to(p,a,\gamma_{\mathcal{B}})) \\ & (\varpi_1(s{\restriction}\mathcal{G}(\mathcal{C}\circledast\mathcal{D},N))\to(p,a,\gamma_{\mathcal{C}}) & , & \varpi_2(s{\restriction}\mathcal{G}(\mathcal{C}\circledast\mathcal{D},N))\to(p,a,\gamma_{\mathcal{D}})) \quad) \end{array}$$

and $(\mathrm{SP}\times\mathrm{SP})\circ\mathrm{HS}(s)$ is

$$\begin{array}{llll} (& (\varpi_1(s{\restriction}\mathcal{G}(\mathcal{A}\circledast\mathcal{B},M)) & , & \varpi_2(s{\restriction}\mathcal{G}(\mathcal{A}\circledast\mathcal{B},M))) \\ & (\varpi_1(s{\restriction}\mathcal{G}(\mathcal{C}\circledast\mathcal{D},N)) & , & \varpi_2(s{\restriction}\mathcal{G}(\mathcal{C}\circledast\mathcal{D},N))) \quad) \end{array}$$

Now we are done since by induction hypothesis

$$(\mathrm{HS}\times\mathrm{HS})\circ\mathrm{SP}_{-\circledast}(s) \quad = \quad m\circ(\mathrm{SP}\times\mathrm{SP})\circ\mathrm{HS}(s)$$

Case of $t = s \to ((p,a,\gamma_{\mathcal{A}}\circledast\gamma_{\mathcal{B}})\ ,\ (p\cdot d,(q_{\mathcal{C}},q_{\mathcal{D}})))$.
We have

$$\mathrm{SP}_{-\circledast}(t) \quad = \quad (\varpi_1(t)\to((p,a,\gamma_{\mathcal{A}}),(p\cdot d,q_{\mathcal{C}}))\ ,\ \varpi_2(t)\to((p,a,\gamma_{\mathcal{B}}),(p\cdot d,q_{\mathcal{D}})))$$

Hence $(\mathrm{HS}\times\mathrm{HS})\circ\mathrm{SP}_{-\circledast}(t)$ is

$$\begin{array}{llll} (& (\varpi_1(s){\restriction}\mathcal{G}(\mathcal{A},M)\to(p,a,\gamma_{\mathcal{A}}) & , & \varpi_1(s){\restriction}\mathcal{G}(\mathcal{C},N)\to(p\cdot d,q_{\mathcal{C}})) \quad , \\ & (\varpi_2(s){\restriction}\mathcal{G}(\mathcal{B},M)\to(p,a,\gamma_{\mathcal{B}}) & , & \varpi_2(s){\restriction}\mathcal{G}(\mathcal{D},N)\to(p\cdot d,q_{\mathcal{D}})) \quad) \end{array}$$

and $(\mathrm{HS}\times\mathrm{HS})\circ\mathrm{SP}_{-\circledast}(s)$ is

$$\begin{array}{llll} (& (\varpi_1(s){\restriction}\mathcal{G}(\mathcal{A},M) & , & \varpi_1(s){\restriction}\mathcal{G}(\mathcal{C},N)) \quad , \\ & (\varpi_2(s){\restriction}\mathcal{G}(\mathcal{B},M) & , & \varpi_2(s){\restriction}\mathcal{G}(\mathcal{D},N)) \quad) \end{array}$$

On the other hand,

$$\mathrm{HS}(t) \quad = \quad (s{\upharpoonright}\mathcal{G}(\mathcal{A} \circledast \mathcal{B}, M) \to (p, a, \gamma_{\mathcal{A}} \circledast \gamma_{\mathcal{B}})\ ,\ s{\upharpoonright}\mathcal{G}(\mathcal{C} \circledast \mathcal{D}, N) \to (p \cdot d, (q_{\mathcal{C}}, q_{\mathcal{D}})))$$

Hence, $(\mathrm{SP} \times \mathrm{SP}) \circ \mathrm{HS}(t)$ is

$$\begin{array}{lllll} (& (\varpi_1(s{\upharpoonright}\mathcal{G}(\mathcal{A} \circledast \mathcal{B}, M)) \to (p, a, \gamma_{\mathcal{A}}) & , & \varpi_2(s{\upharpoonright}\mathcal{G}(\mathcal{A} \circledast \mathcal{B}, M)) \to (p, a, \gamma_{\mathcal{B}})) & \\ & (\varpi_1(s{\upharpoonright}\mathcal{G}(\mathcal{C} \circledast \mathcal{D}, N)) \to (p \cdot d, q_{\mathcal{C}}) & , & \varpi_2(s{\upharpoonright}\mathcal{G}(\mathcal{C} \circledast \mathcal{D}, N)) \to (p \cdot d, q_{\mathcal{D}})) &) \end{array}$$

and $(\mathrm{SP} \times \mathrm{SP}) \circ \mathrm{HS}(s)$ is

$$\begin{array}{lllll} (& (\varpi_1(s{\upharpoonright}\mathcal{G}(\mathcal{A} \circledast \mathcal{B}, M)) & , & \varpi_2(s{\upharpoonright}\mathcal{G}(\mathcal{A} \circledast \mathcal{B}, M))) & \\ & (\varpi_1(s{\upharpoonright}\mathcal{G}(\mathcal{C} \circledast \mathcal{D}, N)) & , & \varpi_2(s{\upharpoonright}\mathcal{G}(\mathcal{C} \circledast \mathcal{D}, N))) &) \end{array}$$

Now we are done since by induction hypothesis

$$(\mathrm{HS} \times \mathrm{HS}) \circ \mathrm{SP}_{-\circledast}(s) \quad = \quad m \circ (\mathrm{SP} \times \mathrm{SP}) \circ \mathrm{HS}(s)$$

Case of $t = s \to ((p \cdot d, (q_{\mathcal{A}}, q_{\mathcal{B}}))\ ,\ (p \cdot d, (q_{\mathcal{C}}, q_{\mathcal{D}})))$.
We have

$$\mathrm{SP}_{-\circledast}(t) \quad = \quad (\varpi_1(t) \to ((p \cdot d, q_{\mathcal{A}}), (p \cdot d, q_{\mathcal{C}}))\ ,\ \varpi_2(t) \to ((p \cdot d, q_{\mathcal{B}}), (p \cdot d, q_{\mathcal{D}})))$$

Hence $(\mathrm{HS} \times \mathrm{HS}) \circ \mathrm{SP}_{-\circledast}(t)$ is

$$\begin{array}{llllll} (& (\varpi_1(s){\upharpoonright}\mathcal{G}(\mathcal{A}, M) \to (p \cdot d, q_{\mathcal{A}}) & , & \varpi_1(s){\upharpoonright}\mathcal{G}(\mathcal{C}, N) \to (p \cdot d, q_{\mathcal{C}})) & , & \\ & (\varpi_2(s){\upharpoonright}\mathcal{G}(\mathcal{B}, M) \to (p \cdot d, q_{\mathcal{B}}) & , & \varpi_2(s){\upharpoonright}\mathcal{G}(\mathcal{D}, N) \to (p \cdot d, q_{\mathcal{D}})) &) & \end{array}$$

and $(\mathrm{HS} \times \mathrm{HS}) \circ \mathrm{SP}_{-\circledast}(s)$ is

$$\begin{array}{llllll} (& (\varpi_1(s){\upharpoonright}\mathcal{G}(\mathcal{A}, M) & , & \varpi_1(s){\upharpoonright}\mathcal{G}(\mathcal{C}, N)) & , & \\ & (\varpi_2(s){\upharpoonright}\mathcal{G}(\mathcal{B}, M) & , & \varpi_2(s){\upharpoonright}\mathcal{G}(\mathcal{D}, N)) &) & \end{array}$$

On the other hand,

$$\mathrm{HS}(t) \quad = \quad (s{\upharpoonright}\mathcal{G}(\mathcal{A} \circledast \mathcal{B}, M) \to (p \cdot d, (q_{\mathcal{A}}, q_{\mathcal{B}}))\ ,\ s{\upharpoonright}\mathcal{G}(\mathcal{C} \circledast \mathcal{D}, N) \to (p \cdot d, (q_{\mathcal{C}}, q_{\mathcal{D}})))$$

Hence, $(\mathrm{SP} \times \mathrm{SP}) \circ \mathrm{HS}(t)$ is

$$\begin{array}{lllll} (& (\varpi_1(s{\upharpoonright}\mathcal{G}(\mathcal{A} \circledast \mathcal{B}, M)) \to (p \cdot d, q_{\mathcal{A}}) & , & \varpi_2(s{\upharpoonright}\mathcal{G}(\mathcal{A} \circledast \mathcal{B}, M)) \to (p \cdot d, q_{\mathcal{B}})) & \\ & (\varpi_1(s{\upharpoonright}\mathcal{G}(\mathcal{C} \circledast \mathcal{D}, N)) \to (p \cdot d, q_{\mathcal{C}}) & , & \varpi_2(s{\upharpoonright}\mathcal{G}(\mathcal{C} \circledast \mathcal{D}, N)) \to (p \cdot d, q_{\mathcal{D}})) &) \end{array}$$

and $(\mathrm{SP} \times \mathrm{SP}) \circ \mathrm{HS}(s)$ is

$$\begin{array}{lllll} (& (\varpi_1(s{\upharpoonright}\mathcal{G}(\mathcal{A} \circledast \mathcal{B}, M)) & , & \varpi_2(s{\upharpoonright}\mathcal{G}(\mathcal{A} \circledast \mathcal{B}, M))) & \\ & (\varpi_1(s{\upharpoonright}\mathcal{G}(\mathcal{C} \circledast \mathcal{D}, N)) & , & \varpi_2(s{\upharpoonright}\mathcal{G}(\mathcal{C} \circledast \mathcal{D}, N))) &) \end{array}$$

Now we are done since by induction hypothesis

$$(\mathrm{HS} \times \mathrm{HS}) \circ \mathrm{SP}_{-\circledast}(s) \quad = \quad m \circ (\mathrm{SP} \times \mathrm{SP}) \circ \mathrm{HS}(s)$$

◀

10.6 Action on Strategies of the Synchronous Monoidal Product

We now define the action of $\_\circledast\_$ on strategies. As above, in the whole Section, we assume given *complete* automata $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$. Consider

$$\Sigma \vdash \sigma : \mathcal{G}(\mathcal{A}, M) \multimap\!\circledast\, \mathcal{G}(\mathcal{C}, N) \quad \text{and} \quad \Sigma \vdash \theta : \mathcal{G}(\mathcal{B}, M) \multimap\!\circledast\, \mathcal{G}(\mathcal{D}, N)$$

By Lem. 10.10, we have

$$\mathrm{SP}_{\multimap\circledast}\left(\wp^{\mathsf{P}}_{\Sigma}\left[\mathcal{G}(\mathcal{A}\circledast\mathcal{B}, M) \multimap\!\circledast\, \mathcal{G}(\mathcal{C}\circledast\mathcal{D}, N)\right]\right) \quad \subseteq \quad \wp^{\mathsf{P}}_{\Sigma}\left(\mathcal{G}(\mathcal{A}, M) \multimap\!\circledast\, \mathcal{G}(\mathcal{C}, N)\right) \times_{\mathrm{Tr}_\Sigma} \wp^{\mathsf{P}}_{\Sigma}\left(\mathcal{G}(\mathcal{B}, M) \multimap\!\circledast\, \mathcal{G}(\mathcal{D}, N)\right)$$

Now, since

$$\sigma \quad \subseteq \quad \wp^{\mathsf{P}}_{\Sigma}(\mathcal{G}(\mathcal{A}, M) \multimap\!\circledast\, \mathcal{G}(\mathcal{C}, N)) \qquad \text{and} \qquad \theta \quad \subseteq \quad \wp^{\mathsf{P}}_{\Sigma}(\mathcal{G}(\mathcal{B}, M) \multimap\!\circledast\, \mathcal{G}(\mathcal{D}, N))$$

we have

$$\mathrm{SP}^{-1}_{\multimap\circledast}(\sigma, \tau) \quad \subseteq \quad \wp^{\mathsf{P}}_{\Sigma}\left(\mathcal{G}(\mathcal{A}\circledast\mathcal{B}, M) \multimap\!\circledast\, \mathcal{G}(\mathcal{C}\circledast\mathcal{D}, N)\right)$$

▶ **Definition 10.12.** Given

$$\Sigma \vdash \sigma : \mathcal{G}(\mathcal{A}, M) \multimap\!\circledast\, \mathcal{G}(\mathcal{C}, N) \quad \text{and} \quad \Sigma \vdash \theta : \mathcal{G}(\mathcal{B}, M) \multimap\!\circledast\, \mathcal{G}(\mathcal{D}, N)$$

define

$$\Sigma \vdash \sigma \circledast \theta : \mathcal{G}(\mathcal{A}\circledast\mathcal{B}, M) \multimap\!\circledast\, \mathcal{G}(\mathcal{C}\circledast\mathcal{D}, N)$$

as

$$\sigma \circledast \theta \quad := \quad \mathrm{SP}^{-1}_{\multimap\circledast}(\sigma, \theta)$$

▶ **Proposition 10.13.** *Consider*

$$\Sigma \vdash \sigma : \mathcal{G}(\mathcal{A}, M) \multimap\!\circledast\, \mathcal{G}(\mathcal{C}, N) \qquad \textit{and} \qquad \Sigma \vdash \theta : \mathcal{G}(\mathcal{B}, M) \multimap\!\circledast\, \mathcal{G}(\mathcal{D}, N)$$

(i) $\Sigma \vdash \sigma \circledast \theta : \mathcal{G}(\mathcal{A}\circledast\mathcal{B}, M) \multimap\!\circledast\, \mathcal{G}(\mathcal{C}\circledast\mathcal{D}, N)$

(ii) If σ and τ are both total then $\sigma \circledast \tau$ is total.

(iii) If σ and τ are both morphisms of $\widehat{\mathbf{SAG}}^{\mathrm{W}}_{\Sigma}$ (resp. $\widehat{\mathbf{SAG}}^{R}_{\Sigma}$), then $\sigma \circledast \tau$ is a morphism of $\widehat{\mathbf{SAG}}^{\mathrm{W}}_{\Sigma}$ (resp. $\widehat{\mathbf{SAG}}^{R}_{\Sigma}$).

Proof. We show (i) that $\sigma \circledast \tau$ is a synchronous strategy, (ii) that $\sigma \circledast \tau$ is total when σ and τ are both total, and (iii) that $\sigma \circledast \tau$ is winning w.r.t. $\multimap\!\circledast$ as soon as σ and τ are both winning w.r.t. $\multimap\!\circledast$ (resp. $\circledast\!\!\multimap\!\circledast$).

(i) We have to show that $\sigma \circledast \tau$ is a P-deterministic P-prefix-closed set of negative P-plays. The last point is ensured (*via* Lem. 10.10) by construction of $\sigma \circledast \tau$, and P-prefix-closure follows from the fact that ϖ_1 and ϖ_2 are length-preserving.
It remains to check that $\sigma \circledast \tau$ is P-deterministic. There are two cases to consider.
   - Assume that $\sigma \circledast \tau$ contains the two following plays

$$\begin{array}{l} s \xrightarrow{\mathsf{O}} ((p, a, \gamma_{\mathcal{A}} \circledast \gamma_{\mathcal{B}})\ ,\ (p, (q_{\mathcal{C}}, q_{\mathcal{D}}))) \xrightarrow{\mathsf{P}} ((p, a, \gamma_{\mathcal{A}} \circledast \gamma_{\mathcal{B}})\ ,\ (p, a, \gamma_{\mathcal{C}} \circledast \gamma_{\mathcal{D}})) \\ s \xrightarrow{\mathsf{O}} ((p, a, \gamma_{\mathcal{A}} \circledast \gamma_{\mathcal{B}})\ ,\ (p, (q_{\mathcal{C}}, q_{\mathcal{D}}))) \xrightarrow{\mathsf{P}} ((p, a, \gamma_{\mathcal{A}} \circledast \gamma_{\mathcal{B}})\ ,\ (p, a, \gamma'_{\mathcal{C}} \circledast \gamma'_{\mathcal{D}})) \end{array}$$

   By construction, the two following plays belong to σ

$$\begin{array}{l} \varpi_1(s) \xrightarrow{\mathsf{O}} ((p, a, \gamma_{\mathcal{A}})\ ,\ (p, q_{\mathcal{C}})) \xrightarrow{\mathsf{P}} ((p, a, \gamma_{\mathcal{A}})\ ,\ (p, a, \gamma_{\mathcal{C}})) \\ \varpi_1(s) \xrightarrow{\mathsf{O}} ((p, a, \gamma_{\mathcal{A}})\ ,\ (p, q_{\mathcal{C}})) \xrightarrow{\mathsf{P}} ((p, a, \gamma_{\mathcal{A}})\ ,\ (p, a, \gamma'_{\mathcal{C}})) \end{array}$$

   hence $\gamma_{\mathcal{C}} = \gamma'_{\mathcal{C}}$ by P-determinism of σ.
   We similarly get $\gamma_{\mathcal{D}} = \gamma'_{\mathcal{D}}$ (using τ instead of σ).

- Assume that $\sigma \circledast \tau$ contains the two following plays

$$
\begin{array}{l}
s \xrightarrow{\mathsf{O}} ((p, a, \gamma_{\mathcal{A}} \circledast \gamma_{\mathcal{B}})\ ,\ (p \cdot d, (q_{\mathcal{C}}, q_{\mathcal{D}}))) \xrightarrow{\mathsf{P}} ((p \cdot d, (q_{\mathcal{A}}, q_{\mathcal{B}}))\ ,\ (p \cdot d, (q_{\mathcal{C}}, q_{\mathcal{D}}))) \\
s \xrightarrow{\mathsf{O}} ((p, a, \gamma_{\mathcal{A}} \circledast \gamma_{\mathcal{B}})\ ,\ (p \cdot d, (q_{\mathcal{C}}, q_{\mathcal{D}}))) \xrightarrow{\mathsf{P}} ((p \cdot d, (q'_{\mathcal{A}}, q'_{\mathcal{B}}))\ ,\ (p \cdot d, (q_{\mathcal{C}}, q_{\mathcal{D}})))
\end{array}
$$

  By construction, the two following plays belong to σ

$$
\begin{array}{l}
\varpi_1(s) \xrightarrow{\mathsf{O}} ((p, a, \gamma_{\mathcal{A}})\ ,\ (p \cdot d, q_{\mathcal{C}})) \xrightarrow{\mathsf{P}} ((p \cdot d, q_{\mathcal{A}})\ ,\ (p \cdot d, q_{\mathcal{C}})) \\
\varpi_1(s) \xrightarrow{\mathsf{O}} ((p, a, \gamma_{\mathcal{A}})\ ,\ (p \cdot d, q_{\mathcal{C}})) \xrightarrow{\mathsf{P}} ((p \cdot d, q'_{\mathcal{A}})\ ,\ (p \cdot d, q_{\mathcal{C}}))
\end{array}
$$

  hence $q_{\mathcal{A}} = q'_{\mathcal{A}}$ by P-determinism of σ.
  We similarly get $q_{\mathcal{B}} = q'_{\mathcal{B}}$ (using τ instead of σ).

(ii) We check that $\sigma \circledast \tau$ is total as soon as σ and τ are total.
So consider a play $s \in \sigma \circledast \tau$ extended by some O-move, say:

$$
s \quad \xrightarrow{\mathsf{O}} \quad ((p, a, \gamma_{\mathcal{A}} \circledast \gamma_{\mathcal{B}})\ ,\ (p, (q_{\mathcal{C}}, q_{\mathcal{D}})))
$$

By construction, we have

$$
\begin{array}{llll}
\varpi_1(s) & \xrightarrow{\mathsf{O}} & ((p, a, \gamma_{\mathcal{A}})\ ,\ (p, q_{\mathcal{C}})) & \in \sigma \\
\varpi_2(s) & \xrightarrow{\mathsf{O}} & ((p, a, \gamma_{\mathcal{B}})\ ,\ (p, q_{\mathcal{D}})) & \in \tau
\end{array}
$$

By totality of σ and τ, for some P-moves, we have

$$
\begin{array}{llllllll}
\varpi_1(s) & \xrightarrow{\mathsf{O}} & ((p, a, \gamma_{\mathcal{A}})\ ,\ (p, q_{\mathcal{C}})) & \xrightarrow{\mathsf{P}} & ((p, a, \gamma_{\mathcal{A}})\ ,\ (p, a, \gamma_{\mathcal{C}})) & \in & \sigma \\
\varpi_2(s) & \xrightarrow{\mathsf{O}} & ((p, a, \gamma_{\mathcal{B}})\ ,\ (p, q_{\mathcal{D}})) & \xrightarrow{\mathsf{P}} & ((p, a, \gamma_{\mathcal{B}})\ ,\ (p, a, \gamma_{\mathcal{D}})) & \in & \tau
\end{array}
$$

By construction of $\sigma \circledast \tau$, we conclude:

$$
s \quad \xrightarrow{\mathsf{O}} \quad ((p, a, \gamma_{\mathcal{A}} \circledast \gamma_{\mathcal{B}})\ ,\ (p, (q_{\mathcal{C}}, q_{\mathcal{D}}))) \quad \xrightarrow{\mathsf{P}} \quad ((p, a, \gamma_{\mathcal{A}} \circledast \gamma_{\mathcal{B}})\ ,\ (p, a, \gamma_{\mathcal{C}} \circledast \gamma_{\mathcal{D}})) \quad \in \sigma \circledast \tau
$$

The other possiblity is that

$$
s \quad \xrightarrow{\mathsf{O}} \quad ((p, a, \gamma_{\mathcal{A}} \circledast \gamma_{\mathcal{B}})\ ,\ (p \cdot d, (q_{\mathcal{C}}, q_{\mathcal{D}}))
$$

By construction, we have

$$
\begin{array}{llll}
\varpi_1(s) & \xrightarrow{\mathsf{O}} & ((p, a, \gamma_{\mathcal{A}})\ ,\ (p \cdot d, q_{\mathcal{C}})) & \in \sigma \\
\varpi_2(s) & \xrightarrow{\mathsf{O}} & ((p, a, \gamma_{\mathcal{B}})\ ,\ (p \cdot d, q_{\mathcal{D}})) & \in \tau
\end{array}
$$

By totality of σ and τ, for some P-moves, we have

$$
\begin{array}{llllllll}
\varpi_1(s) & \xrightarrow{\mathsf{O}} & ((p, a, \gamma_{\mathcal{A}})\ ,\ (p \cdot d, q_{\mathcal{C}})) & \xrightarrow{\mathsf{P}} & ((p \cdot d, q_{\mathcal{A}})\ ,\ (p \cdot d, q_{\mathcal{C}})) & \in & \sigma \\
\varpi_2(s) & \xrightarrow{\mathsf{O}} & ((p, a, \gamma_{\mathcal{B}})\ ,\ (p \cdot d, q_{\mathcal{D}})) & \xrightarrow{\mathsf{P}} & ((p \cdot d, q_{\mathcal{B}})\ ,\ (p \cdot d, q_{\mathcal{D}})) & \in & \tau
\end{array}
$$

By construction of $\sigma \circledast \tau$, we conclude:

$$
s \quad \xrightarrow{\mathsf{O}} \quad ((p, a, \gamma_{\mathcal{A}} \circledast \gamma_{\mathcal{B}})\ ,\ (p \cdot d, (q_{\mathcal{C}}, q_{\mathcal{D}}))) \quad \xrightarrow{\mathsf{P}} \quad ((p \cdot d, (q_{\mathcal{A}}, q_{\mathcal{B}}))\ ,\ (p \cdot d, (q_{\mathcal{C}}, q_{\mathcal{D}}))) \quad \in \sigma \circledast \tau
$$

(iii) Consider now an infinite play π of $\sigma \circledast \tau$. By construction of $\multimap\!\circledast$ and $\circledast\!\multimap\!\circledast$, the projections of π on $\wp(\mathcal{A} \circledast \mathcal{B}, M)$ and $\wp(\mathcal{C} \circledast \mathcal{D}, N)$ must be both infinite. Let $(q^n_{\mathcal{A}}, q^n_{\mathcal{B}})_{n \in \mathbb{N}}$ be the projection of π on the states of $\mathcal{A} \circledast \mathcal{B}$ (resp. $\mathcal{A} \circledast_{\equiv} \mathcal{B}$) and $(q^n_{\mathcal{C}}, q^n_{\mathcal{D}})_{n \in \mathbb{N}}$ be its projection on the states of $\mathcal{C} \circledast \mathcal{D}$ (resp. $\mathcal{C} \circledast_{\equiv} \mathcal{D}$).

Consider first the case of σ and τ morphisms of $\widehat{\mathbf{SAG}}^{\mathrm{W}}_{\Sigma}$ (*i.e.* both winning w.r.t. $\mathrel{-\!\!\circledast}$). If $(q^n_{\mathcal{A}}, q^n_{\mathcal{B}})_{n\in\mathbb{N}} \in \Omega_{\mathcal{A}\circledast\mathcal{B}}$, then we have both $(q^n_{\mathcal{A}})_{n\in\mathbb{N}} \in \Omega_{\mathcal{A}}$ and $(q^n_{\mathcal{B}})_{n\in\mathbb{N}} \in \Omega_{\mathcal{B}}$. By assumption on σ and τ, this implies $(q^n_{\mathcal{C}})_{n\in\mathbb{N}} \in \Omega_{\mathcal{C}}$ and $(q^n_{\mathcal{D}})_{n\in\mathbb{N}} \in \Omega_{\mathcal{D}}$, hence $(q^n_{\mathcal{C}}, q^n_{\mathcal{D}})_{n\in\mathbb{N}} \in \Omega_{\mathcal{C}\circledast\mathcal{D}}$,

Consider now the case of reduction games, that is of σ and τ both morphisms of $\widehat{\mathbf{SAG}}^{\mathrm{R}}_{\Sigma}$. We have to show

$$(q^n_{\mathcal{A}}, q^n_{\mathcal{B}})_{n\in\mathbb{N}} \in \Omega_{\mathcal{A}\circledast_{\equiv}\mathcal{B}} \quad\iff\quad (q^n_{\mathcal{C}}, q^n_{\mathcal{D}})_{n\in\mathbb{N}} \in \Omega_{\mathcal{C}\circledast_{\equiv}\mathcal{D}}$$

that is

$$[(q^n_{\mathcal{A}})_{n\in\mathbb{N}} \in \Omega_{\mathcal{A}} \iff (q^n_{\mathcal{B}})_{n\in\mathbb{N}} \in \Omega_{\mathcal{B}}] \iff [(q^n_{\mathcal{C}})_{n\in\mathbb{N}} \in \Omega_{\mathcal{C}} \iff (q^n_{\mathcal{D}})_{n\in\mathbb{N}} \in \Omega_{\mathcal{D}}]$$

under the assumptions

$$[(q^n_{\mathcal{A}})_{n\in\mathbb{N}} \in \Omega_{\mathcal{A}} \iff (q^n_{\mathcal{C}})_{n\in\mathbb{N}} \in \Omega_{\mathcal{C}}] \quad\text{and}\quad [(q^n_{\mathcal{B}})_{n\in\mathbb{N}} \in \Omega_{\mathcal{B}} \iff (q^n_{\mathcal{D}})_{n\in\mathbb{N}} \in \Omega_{\mathcal{D}}]$$

But this is a propositional tautology. ◀

10.7 Universal Properties

As above, in the whole Section, we assume given *complete* automata $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$.

Similarly to what we have done with Prop. 7.11 in Sect. 7.3, we are now going to show that diagram (17) of Lem. 10.10 is actually a pullback diagram. That is:

$$\begin{tikzcd}
\wp^{\mathsf{P}}_{\Sigma}\left(\mathcal{G}(\mathcal{A}\circledast\mathcal{B}, M) \mathrel{-\!\!\circledast} \mathcal{G}(\mathcal{C}\circledast\mathcal{D}, N)\right) \arrow[r, "\varpi_2"] \arrow[d, "\varpi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \wp^{\mathsf{P}}_{\Sigma}\left(\mathcal{G}(\mathcal{B}, M) \mathrel{-\!\!\circledast} \mathcal{G}(\mathcal{D}, N)\right) \arrow[d, "\mathrm{tr}^{-\circledast}"] \\
\wp^{\mathsf{P}}_{\Sigma}\left(\mathcal{G}(\mathcal{A}, M) \mathrel{-\!\!\circledast} \mathcal{G}(\mathcal{C}, N)\right) \arrow[r, "\mathrm{tr}^{-\circledast}"'] & \mathrm{Tr}_{\Sigma}
\end{tikzcd} \tag{18}$$

Simlarly as for Prop. 7.11, we will use the pullback lemma (see e.g. [12, Exercise 1.5.5, p. 30]).

▶ **Lemma 10.14.** *If*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d] \\
B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F \arrow[r, equal] \arrow[d, equal] & F \\
D \arrow[r] & F &
\end{tikzcd}
\quad \textit{and} \quad
\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[r] & E \arrow[d] \\
B \arrow[d] & F \arrow[r, equal] \arrow[d, equal] & F \\
D \arrow[r] & F &
\end{tikzcd}$$

then

$$\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d] \\
B \arrow[r] & F
\end{tikzcd}$$

Proof. By applying the pullback lemma to

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[r] & E \arrow[d] \\ B \arrow[d] & & F \arrow[d, equal] \\ D \arrow[r] & F \arrow[r, equal] & F \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} B \arrow[rr] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & & F \arrow[d, equal] \\ D \arrow[r] & F \arrow[r, equal] & F \end{tikzcd}$$

we get

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[r] & E \arrow[d] \\ B \arrow[r] & F \arrow[r, equal] & F \end{tikzcd}$$

and we conclude by a second application of the pullback lemma to

$$\begin{tikzcd} C \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d] \\ F \arrow[r, equal] & F \end{tikzcd}$$

◀

In order to obtain (18) we will apply Lem. 10.14, with

$$\begin{tikzcd} \wp^{\mathsf{P}}_{\Sigma}\left(\mathcal{G}(\mathcal{A},M) \mathbin{-\!\circledast} \mathcal{G}(\mathcal{C},N)\right) \arrow[r, "\mathrm{tr}^{-\circledast}"] \arrow[d, "\mathrm{HS}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Tr}_{\Sigma} \arrow[d, equal] \\ \wp(\mathcal{A},M) \times_{\mathrm{Tr}_{\Sigma}} \wp(\mathcal{C},N) \arrow[r] & \mathrm{Tr}_{\Sigma} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \wp^{\mathsf{P}}_{\Sigma}\left(\mathcal{G}(\mathcal{B},M) \mathbin{-\!\circledast} \mathcal{G}(\mathcal{D},N)\right) \arrow[r, "\mathrm{HS}"] \arrow[d, "\mathrm{tr}^{-\circledast}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \wp(\mathcal{B},M) \times_{\mathrm{Tr}_{\Sigma}} \wp(\mathcal{D},N) \arrow[d] \\ \mathrm{Tr}_{\Sigma} \arrow[r, equal] & \mathrm{Tr}_{\Sigma} \end{tikzcd}$$

for respectively

$$\begin{tikzcd} B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F \arrow[d, equal] \\ D \arrow[r] & F \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} C \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d] \\ F \arrow[r, equal] & F \end{tikzcd}$$

It remains to show

$$\begin{tikzcd} \wp^{\mathsf{P}}_{\Sigma}\left(\mathcal{G}(\mathcal{A} \circledast \mathcal{B},M) \mathbin{-\!\circledast} \mathcal{G}(\mathcal{C} \circledast \mathcal{D},N)\right) \arrow[r, "\mathrm{HS}\circ\varpi_2"] \arrow[d, "\mathrm{HS}\circ\varpi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \wp(\mathcal{B},M) \times_{\mathrm{Tr}_{\Sigma}} \wp(\mathcal{D},N) \arrow[d] \\ \wp(\mathcal{A},M) \times_{\mathrm{Tr}_{\Sigma}} \wp(\mathcal{C},N) \arrow[r] & \mathrm{Tr}_{\Sigma} \end{tikzcd} \tag{19}$$

Thanks to Lem. 10.11, property (19) will follow from

▶ **Lemma 10.15.** $\mathrm{SP}^2 \circ \mathrm{HS}$ *is a bijection from*

$$\wp^{\mathsf{P}}_{\Sigma}\left(\mathcal{G}(\mathcal{A} \circledast \mathcal{B},M) \mathbin{-\!\circledast} \mathcal{G}(\mathcal{C} \circledast \mathcal{D},N)\right)$$

to

$$\left(\wp(\mathcal{A},M) \times_{\mathrm{Tr}_{\Sigma}} \wp(\mathcal{B},M)\right) \times_{\mathrm{Tr}_{\Sigma}} \left(\wp(\mathcal{C},N) \times_{\mathrm{Tr}_{\Sigma}} \wp(\mathcal{D},N)\right)$$

Proof. First, it follows from Cor. 6.7 that HS is a bijection from

$$\wp^{\mathsf{P}}_{\Sigma}\left(\mathcal{G}(\mathcal{A} \circledast \mathcal{B}, M) \multimap_{\circledast} \mathcal{G}(\mathcal{C} \circledast \mathcal{D}, N)\right)$$

to

$$\wp(\mathcal{A} \circledast \mathcal{B}, M) \times_{\mathrm{Tr}_\Sigma} \wp(\mathcal{C} \circledast \mathcal{D}, N)$$

Then we are done since by Prop. 10.9 the following maps are bijections:

$$\begin{array}{lclcl}
\mathrm{SP} & : & \wp(\mathcal{A} \circledast \mathcal{B}, M) & \longrightarrow & \wp(\mathcal{A}, M) \times_{\mathrm{Tr}_\Sigma} \wp(\mathcal{B}, M) \\
\mathrm{SP} & : & \wp(\mathcal{C} \circledast \mathcal{D}, N) & \longrightarrow & \wp(\mathcal{C}, N) \times_{\mathrm{Tr}_\Sigma} \wp(\mathcal{D}, N)
\end{array}$$

◀

▶ **Lemma 10.16.** *In* **Set***,*

$$\begin{tikzcd}
\wp^{\mathsf{P}}_{\Sigma}\left(\mathcal{G}(\mathcal{A} \circledast \mathcal{B}, M) \multimap_{\circledast} \mathcal{G}(\mathcal{C} \circledast \mathcal{D}, N)\right) \arrow[r, "\mathrm{HS} \circ \varpi_2"] \arrow[d, "\mathrm{HS} \circ \varpi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \wp(\mathcal{B}, M) \times_{\mathrm{Tr}_\Sigma} \wp(\mathcal{D}, N) \arrow[d] \\
\wp(\mathcal{A}, M) \times_{\mathrm{Tr}_\Sigma} \wp(\mathcal{C}, N) \arrow[r] & \mathrm{Tr}_\Sigma
\end{tikzcd}$$

Proof. Commutation the diagram follows from Lem. 10.10 and Prop. 6.4.
We have to show that $\mathrm{HS}^2 \circ \mathrm{SP}$ is a bijection from

$$\wp^{\mathsf{P}}_{\Sigma}\left(\mathcal{G}(\mathcal{A} \circledast \mathcal{B}, M) \multimap_{\circledast} \mathcal{G}(\mathcal{C} \circledast \mathcal{D}, N)\right)$$

to

$$\left(\wp(\mathcal{A}, M) \times_{\mathrm{Tr}_\Sigma} \wp(\mathcal{C}, N)\right) \times_{\mathrm{Tr}_\Sigma} \left(\wp(\mathcal{B}, M) \times_{\mathrm{Tr}_\Sigma} \wp(\mathcal{D}, N)\right)$$

By Lem. 10.15 the map $\mathrm{SP}^2 \circ \mathrm{HS}$ is a bijection from

$$\wp^{\mathsf{P}}_{\Sigma}\left(\mathcal{G}(\mathcal{A} \circledast \mathcal{B}, M) \multimap_{\circledast} \mathcal{G}(\mathcal{C} \circledast \mathcal{D}, N)\right)$$

to

$$\left(\wp(\mathcal{A}, M) \times_{\mathrm{Tr}_\Sigma} \wp(\mathcal{B}, M)\right) \times_{\mathrm{Tr}_\Sigma} \left(\wp(\mathcal{C}, N) \times_{\mathrm{Tr}_\Sigma} \wp(\mathcal{D}, N)\right)$$

Now we are done since the structure map m restricts to a bijection in $\mathbf{Set}/\mathrm{Tr}_\Sigma$:

$$\begin{aligned}
\left(\wp(\mathcal{A}, M) \times_{\mathrm{Tr}_\Sigma} \wp(\mathcal{B}, M)\right) \times_{\mathrm{Tr}_\Sigma} \left(\wp(\mathcal{C}, N) \times_{\mathrm{Tr}_\Sigma} \wp(\mathcal{D}, N)\right) \quad \longrightarrow& \\
\left(\wp(\mathcal{A}, M) \times_{\mathrm{Tr}_\Sigma} \wp(\mathcal{C}, N)\right) \times_{\mathrm{Tr}_\Sigma} \left(\wp(\mathcal{B}, M) \times_{\mathrm{Tr}_\Sigma} \wp(\mathcal{D}, N)\right)&
\end{aligned}$$

◀

We thus have shown

▶ **Proposition 10.17.** *In* **Set***,*

$$\begin{tikzcd}
\wp^{\mathsf{P}}_{\Sigma}\left(\mathcal{G}(\mathcal{A} \circledast \mathcal{B}, M) \multimap_{\circledast} \mathcal{G}(\mathcal{C} \circledast \mathcal{D}, N)\right) \arrow[r, "\varpi_2"] \arrow[d, "\varpi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \wp^{\mathsf{P}}_{\Sigma}\left(\mathcal{G}(\mathcal{B}, M) \multimap_{\circledast} \mathcal{G}(\mathcal{D}, N)\right) \arrow[d, "\mathrm{tr}^{-\circledast}"] \\
\wp^{\mathsf{P}}_{\Sigma}\left(\mathcal{G}(\mathcal{A}, M) \multimap_{\circledast} \mathcal{G}(\mathcal{C}, N)\right) \arrow[r, "\mathrm{tr}^{-\circledast}"'] & \mathrm{Tr}_\Sigma
\end{tikzcd}$$

10.8 Characterization of the action of $\_\circledast\_$ on Strategies

As above, in the whole Section, we assume given *complete* automata $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$.

Thanks to the HS functor (see Prop. 6.3)

$$\mathrm{HS} \quad : \quad \mathbf{SAG}_\Sigma \quad \longrightarrow \quad \mathbf{Rel}(\mathbf{Set}/\mathrm{Tr}_\Sigma)$$

we relate the action of $\_\circledast\_$ on strategies to the tensorial product $\_\otimes\_$ of $\mathbf{Rel}(\mathbf{Set}/\mathrm{Tr}_\Sigma)$.

Consider

$$\Sigma \vdash \sigma : \mathcal{G}(\mathcal{A}, M) \multimap_{\circledast} \mathcal{G}(\mathcal{C}, N) \quad \text{and} \quad \Sigma \vdash \theta : \mathcal{G}(\mathcal{B}, M) \multimap_{\circledast} \mathcal{G}(\mathcal{D}, N)$$

Note that

$$\mathrm{HS}(\sigma \circledast \tau) \quad \subseteq \quad \wp_\Sigma(\mathcal{A} \circledast \mathcal{B}, M) \times_{\mathrm{Tr}_\Sigma} \wp_\Sigma(\mathcal{C} \circledast \mathcal{D}, N)$$

Recall that by Prop. 10.9 we have a bijection

$$\mathrm{SP} \quad : \quad \wp_\Sigma(\mathcal{A} \circledast \mathcal{B}, M) \quad \xrightarrow{\simeq} \quad \wp_\Sigma(\mathcal{A}, M) \times_{\mathrm{Tr}_\Sigma} \wp_\Sigma(\mathcal{B}, M)$$

We thus have, in **Set**,

$$(\mathrm{SP} \times \mathrm{SP}) \circ \mathrm{HS}(\sigma \circledast \tau) \quad \subseteq \quad [\wp_\Sigma(\mathcal{A}, M) \times_{\mathrm{Tr}_\Sigma} \wp_\Sigma(\mathcal{B}, M)] \times_{\mathrm{Tr}_\Sigma} [\wp_\Sigma(\mathcal{C}, N) \times_{\mathrm{Tr}_\Sigma} \wp_\Sigma(\mathcal{D}, N)]$$

In other words:

$$(\mathrm{SP} \times \mathrm{SP}) \circ \mathrm{HS}(\sigma \circledast \tau) \quad : \quad \wp_\Sigma(\mathcal{A}, M) \otimes \wp_\Sigma(\mathcal{B}, M) \quad \to\!\!\!\!\!+\!\!\!\to_{/\mathrm{Tr}_\Sigma} \quad \wp_\Sigma(\mathcal{C}, M) \otimes \wp_\Sigma(\mathcal{B}, N)$$

On the other hand, we have

$$\mathrm{HS}(\sigma) \quad : \quad \wp_\Sigma(\mathcal{A}, M) \quad \to\!\!\!\!\!+\!\!\!\to_{/\mathrm{Tr}_\Sigma} \quad \wp_\Sigma(\mathcal{C}, M) \qquad \text{and} \qquad \mathrm{HS}(\tau) \quad : \quad \wp_\Sigma(\mathcal{B}, M) \quad \to\!\!\!\!\!+\!\!\!\to_{/\mathrm{Tr}_\Sigma} \quad \wp_\Sigma(\mathcal{D}, M)$$

hence

$$\mathrm{HS}(\sigma) \otimes \mathrm{HS}(\tau) \quad : \quad \wp_\Sigma(\mathcal{A}, M) \otimes \wp_\Sigma(\mathcal{B}, M) \quad \to\!\!\!\!\!+\!\!\!\to_{/\mathrm{Tr}_\Sigma} \quad \wp_\Sigma(\mathcal{C}, M) \otimes \wp_\Sigma(\mathcal{B}, N)$$

▶ **Lemma 10.18.** $m^{-1} \circ (\mathrm{HS} \times \mathrm{HS}) \circ \mathrm{SP}_{-\circledast}(\sigma \circledast \tau) \quad = \quad \mathrm{HS}(\sigma) \otimes \mathrm{HS}(\tau)$

Proof. Recall that

$$\mathrm{HS}(\sigma) \otimes \mathrm{HS}(\tau) \quad \subseteq \quad (\wp_\Sigma(\mathcal{A}, M) \otimes \wp_\Sigma(\mathcal{B}, M)) \times_{\mathrm{Tr}_\Sigma} (\wp_\Sigma(\mathcal{C}, N) \otimes \wp_\Sigma(\mathcal{D}, N))$$

and consider

$$((s,t),(u,v)) \in (\wp_\Sigma(\mathcal{A}, M) \otimes \wp_\Sigma(\mathcal{B}, M)) \times_{\mathrm{Tr}_\Sigma} (\wp_\Sigma(\mathcal{C}, N) \otimes \wp_\Sigma(\mathcal{D}, N))$$

Note that

$$\mathrm{tr}(s) \quad = \quad \mathrm{tr}(t) \quad = \quad \mathrm{tr}(u) \quad = \quad \mathrm{tr}(v)$$

By definition of the action of $\_\otimes\_$ on morphisms of $\mathbf{Rel}(\mathbf{Set}/\mathrm{Tr}_\Sigma)$, we have

$$((s,t),(u,v)) \in \mathrm{HS}(\sigma) \otimes \mathrm{HS}(\tau)$$

if and only if

$$(s,u) \in \mathrm{HS}(\sigma) \qquad \text{and} \qquad (t,v) \in \mathrm{HS}(\tau)$$

if and only if there are

$$a \in \wp_\Sigma(\mathcal{G}(\mathcal{A}, M) \multimap\!\!\circledast\; \mathcal{G}(\mathcal{C}, N)) \quad \text{and} \quad b \in \wp_\Sigma(\mathcal{G}(\mathcal{B}, M) \multimap\!\!\circledast\; \mathcal{G}(\mathcal{D}, N))$$

such that

$$\mathrm{HS}(a) = (s, u) \quad \text{and} \quad \mathrm{HS}(b) = (t, v) \quad \text{and} \quad a \in \sigma \quad \text{and} \quad b \in \tau$$

if and only if (by Prop. 10.17, by definition of $\sigma \circledast \tau$, and since $\mathrm{tr}(s) = \mathrm{tr}(t) = \mathrm{tr}(u) = \mathrm{tr}(v)$) there is

$$c \in \sigma \circledast \tau \subseteq \wp^{\mathsf{P}}_\Sigma(\mathcal{G}(\mathcal{A} \circledast \mathcal{B}, M) \multimap\!\!\circledast\; \mathcal{G}(\mathcal{C} \circledast \mathcal{D}, N))$$

such that $\mathrm{SP}_{\multimap\circledast}(c) = (a, b)$. ◀

▶ **Proposition 10.19.** $(\mathrm{SP} \times \mathrm{SP}) \circ \mathrm{HS}(\sigma \circledast \tau) = \mathrm{HS}(\sigma) \otimes \mathrm{HS}(\tau)$.

Proof. By Let. 10.18 we have

$$(\mathrm{HS} \times \mathrm{HS}) \circ \mathrm{SP}_{\multimap\circledast}(\sigma \circledast \tau) \quad = \quad m \circ (\mathrm{HS}(\sigma) \otimes \mathrm{HS}(\tau))$$

On the other hand, by Lem. 10.11 we have

$$(\mathrm{HS} \times \mathrm{HS}) \circ \mathrm{SP}_{\multimap\circledast}(\sigma \circledast \tau) \quad = \quad m \circ (\mathrm{SP} \times \mathrm{SP}) \circ \mathrm{HS}(\sigma \circledast \tau)$$

and we are done since m is a bijection. ◀

10.9 Uniform Bifunctoriality

▶ **Proposition 10.20** (Uniform Bifunctoriality of $\_ \circledast \_$ in $\widehat{\mathbf{SAG}}^{(\mathrm{W/R})}_\Sigma$). *In $\widehat{\mathbf{SAG}}^{(\mathrm{W/R})}_\Sigma$:*

(i) Given $\Sigma \vdash \mathcal{G}(\mathcal{A}, M)$ and $\Sigma \vdash \mathcal{G}(\mathcal{B}, M)$, we have

$$\mathrm{id}_{(\mathcal{A},M)} \circledast \mathrm{id}_{(\mathcal{B},M)} \quad = \quad \mathrm{id}_{(\mathcal{A} \circledast \mathcal{B}, M)}$$

in

$$\Sigma \vdash \mathcal{G}(\mathcal{A} \circledast \mathcal{B}, M) \quad \longrightarrow \quad \Sigma \vdash \mathcal{G}(\mathcal{A} \circledast \mathcal{B}, M)$$

(ii) Given

$$\Sigma \vdash \mathcal{G}(\mathcal{A}_0, M) \quad \xrightarrow{\sigma_0} \quad \Sigma \vdash \mathcal{G}(\mathcal{A}_1, N) \quad \xrightarrow{\sigma_1} \quad \Sigma \vdash \mathcal{G}(\mathcal{A}_2, P)$$

and

$$\Sigma \vdash \mathcal{G}(\mathcal{B}_0, M) \quad \xrightarrow{\tau_0} \quad \Sigma \vdash \mathcal{G}(\mathcal{B}_1, N) \quad \xrightarrow{\tau_1} \quad \Sigma \vdash \mathcal{G}(\mathcal{B}_2, P)$$

we have

$$(\sigma_1 \circ \sigma_0) \circledast (\tau_1 \circ \tau_0) \quad = \quad (\sigma_1 \circledast \tau_1) \circ (\sigma_0 \circledast \tau_0)$$

in

$$\Sigma \vdash \mathcal{G}(\mathcal{A}_0 \circledast \mathcal{B}_0, M) \quad \longrightarrow \quad \Sigma \vdash \mathcal{G}(\mathcal{A}_2 \circledast \mathcal{B}_2, P)$$

Proof. (i) By Prop. 10.19, we have

$$\mathrm{SP}^2 \circ \mathrm{HS}(\mathrm{id}_{(\mathcal{A},M)} \circledast \mathrm{id}_{(\mathcal{B},M)}) \quad = \quad \mathrm{HS}(\mathrm{id}_{(\mathcal{A},M)}) \otimes \mathrm{HS}(\mathrm{id}_{(\mathcal{B},M)})$$

By functoriality of HS (Prop. 4.11) we deduce

$$\mathrm{SP}^2 \circ \mathrm{HS}(\mathrm{id}_{(\mathcal{A},M)} \circledast \mathrm{id}_{(\mathcal{B},M)}) \quad = \quad 1_{\wp_\Sigma(\mathcal{A},M)} \otimes 1_{\wp_\Sigma(\mathcal{B},M)}$$

where $1_{\wp_\Sigma(\mathcal{A},M)}$ and $1_{\wp_\Sigma(\mathcal{B},M)}$ are the identity relations on $\wp_\Sigma(\mathcal{A},M)$ and $\wp_\Sigma(\mathcal{B},M)$. By bifunctoriality of $\_ \otimes \_$ in $\mathbf{Rel}(\mathbf{Set}/\mathrm{Tr}_\Sigma)$ (Prop. 10.1) we get

$$\mathrm{SP}^2 \circ \mathrm{HS}(\mathrm{id}_{(\mathcal{A},M)} \circledast \mathrm{id}_{(\mathcal{B},M)}) \quad = \quad 1_{\wp_\Sigma(\mathcal{A},M) \otimes \wp_\Sigma(\mathcal{B},M)}$$

On the other hand, again by functoriality of HS (Prop. 4.11) we have

$$\mathrm{SP}^2 \circ \mathrm{HS}(\mathrm{id}_{(\mathcal{A} \circledast \mathcal{B},M)}) \quad = \quad \mathrm{SP}^2(1_{\wp(\mathcal{A} \circledast \mathcal{B},M)})$$

where

$$1_{\wp(\mathcal{A} \circledast \mathcal{B},M)} \subseteq \wp(\mathcal{A} \circledast \mathcal{B}, M) \times_{\mathrm{Tr}_\Sigma} \wp(\mathcal{A} \circledast \mathcal{B}, M)$$

is the identity relation.
Note that by definition of SP and of $1_{\wp(\mathcal{A} \circledast \mathcal{B},M)}$ we have

$$((s,t),(u,v)) \quad \in \quad \mathrm{SP}^2(1_{\wp(\mathcal{A} \circledast \mathcal{B},M)})$$

if and only if there is $a \in \wp(\mathcal{A} \circledast \mathcal{B}, M)$ such that $\mathrm{SP}(a) = (s,t) = (u,v)$. It then follows from Prop. 10.9 that

$$\begin{array}{rcl} \mathrm{SP}^2(1_{\wp(\mathcal{A} \circledast \mathcal{B},M)}) & = & \{((s,t),(s,t)) \mid (s,t) \in \wp(\mathcal{A},M) \times_{\mathrm{Tr}_\Sigma} \wp(\mathcal{B},M)\} \\ & = & \{((s,t),(s,t)) \mid (s,t) \in \wp(\mathcal{A},M) \otimes \wp(\mathcal{B},M)\} \\ & = & 1_{\wp(\mathcal{A},M) \otimes \wp(\mathcal{B},M)} \end{array}$$

Hence we are done since

$$\mathrm{SP}^2(1_{\wp(\mathcal{A} \circledast \mathcal{B},M)}) \quad = \quad 1_{\wp(\mathcal{A},M) \otimes \wp(\mathcal{B},M)}$$

(ii) By Prop. 10.19, we have

$$\mathrm{SP}^2 \circ \mathrm{HS}\left((\sigma_1 \circ \sigma_0) \circledast (\tau_1 \circ \tau_0)\right) \quad = \quad \mathrm{HS}(\sigma_1 \circ \sigma_0) \otimes \mathrm{HS}(\tau_1 \circ \tau_0)$$

By functoriality of HS (Prop. 4.10) we deduce

$$\mathrm{SP}^2 \circ \mathrm{HS}\left((\sigma_1 \circ \sigma_0) \circledast (\tau_1 \circ \tau_0)\right) \quad = \quad (\mathrm{HS}(\sigma_1) \circ \mathrm{HS}(\sigma_0)) \otimes (\mathrm{HS}(\tau_1) \circ \mathrm{HS}(\tau_0))$$

and by bifunctoriality of $\_\otimes\_$ in $\mathbf{Rel}(\mathbf{Set}/\mathrm{Tr}_\Sigma)$ (Prop. 10.1) we get (where composition on the right is in $\mathbf{Rel}(\mathbf{Set}/\mathrm{Tr}_\Sigma)$):

$$\mathrm{SP}^2 \circ \mathrm{HS}\left((\sigma_1 \circ \sigma_0) \circledast (\tau_1 \circ \tau_0)\right) \quad = \quad (\mathrm{HS}(\sigma_1) \otimes \mathrm{HS}(\tau_1)) \circ (\mathrm{HS}(\sigma_0) \otimes \mathrm{HS}(\tau_0))$$

By Prop. 10.19 again, we deduce

$$\mathrm{SP}^2 \circ \mathrm{HS}\left((\sigma_1 \circ \sigma_0) \circledast (\tau_1 \circ \tau_0)\right) \quad = \quad \left[\mathrm{SP}^2(\mathrm{HS}(\sigma_1 \circledast \tau_1))\right] \circ \left[\mathrm{SP}^2(\mathrm{HS}(\sigma_0 \circledast \tau_0))\right]$$

Since the maps SP are bijections (Prop. 10.9), it follows from Lem. 10.6 the family of maps $\mathrm{SP}^2 = \mathrm{SP} \times \mathrm{SP}$ preserves relational composition, hence

$$\mathrm{SP}^2 \circ \mathrm{HS}\left((\sigma_1 \circ \sigma_0) \circledast (\tau_1 \circ \tau_0)\right) \quad = \quad \mathrm{SP}^2\left(\mathrm{HS}(\sigma_1 \circledast \tau_1) \circ \mathrm{HS}(\sigma_0 \circledast \tau_0)\right)$$

and again by functoriality of HS (Prop. 4.10) we obtain

$$\mathrm{SP}^2 \circ \mathrm{HS}\left((\sigma_1 \circ \sigma_0) \circledast (\tau_1 \circ \tau_0)\right) \quad = \quad \mathrm{SP}^2 \circ \mathrm{HS}\left((\sigma_1 \circledast \tau_1) \circ (\sigma_0 \circledast \tau_0)\right)$$

Now we are done since $\mathrm{SP}^2 \circ \mathrm{HS}$ is bijective on plays of strategies, thanks to Lem. 4.6.(i) and Prop. 10.9. ◀

10.10 Structure Maps and Coherence

We now provide the synchronous product with symmetric monoidal structure.

As explained above, for the moment we only discuss the first partial variant of $\_ \multimap\!\circledast \_$, which only deals-with acceptance games $\Sigma \vdash \mathcal{G}(\mathcal{A}, M)$ and $\Sigma \vdash \mathcal{G}(\mathcal{B}, M)$ with the same substituted tree M. As above, we only consider *complete* automata.

We build on the monoidal structure of $\mathbf{Rel}(\mathbf{Set}/J)$ (see Sect. 10.1, in part. Lem. 10.3 and Prop. 10.4).

The natural structure maps of $\_ \circledast \_$ will be defined from those of $\mathbf{Rel}(\mathbf{Set}/J)$ in a way similar to the definition of the identity strategy *via* from the identity synchronuous relation in Prop. 4.11.

10.10.1 Monoidal Unit.

For the monoidal unit of alphabet Σ we will take the game

$$\Sigma \vdash \mathcal{G}(\mathcal{I}, \mathrm{Id}_\Sigma)$$

Recall that

$$\mathcal{I} = (Q_\mathcal{I}, q^\imath_\mathcal{I}, \delta_\mathcal{I}, \Omega_\mathcal{I})$$

where the state set if $Q_\mathcal{I} = \mathbf{1} = \{\bullet\}$, the initial state is $q^\imath_\mathcal{I} = \bullet$, the transition function is

$$\delta_\mathcal{I}(q^\imath_\mathcal{I}, a) \quad = \quad \{\{(q^\imath_\mathcal{I}, d) \mid d \in D\}\} \qquad \text{for all } a \in \Sigma$$

and the acceptance condition is $\Omega_\mathcal{I} := \{q^\imath_\mathcal{I}\}^\omega$.

In the following, it will be at some point notationally convenient to use the game $\Sigma \vdash \mathcal{G}(\mathcal{I}, M)$ as monoidal unit (rather than $\Sigma \vdash \mathcal{G}(\mathcal{I}, \mathrm{Id}_\Sigma)$. We actually have:

▶ **Lemma 10.21.** $\Sigma \vdash \mathcal{G}(\mathcal{I}, M) = \Sigma \vdash \mathcal{G}(\mathcal{I}, \mathrm{Id})$

Proof. The positions of the two games are the same, as well as the O-labelled edges. The same holds for the P-labelled edges, since the transition function of $\mathcal{I}$ is constant. ◀

Recall from Sect. 10.1 that in $\mathbf{Rel}(\mathbf{Set}/\mathrm{Tr}_\Sigma)$, the unit of the monoidal product $\_ \otimes \_$ must be an object of the form $I \xrightarrow{\simeq} \mathrm{Tr}_\Sigma$. The corresponding property holds for $\mathcal{I}$, namely $\mathrm{tr} : \wp_\Sigma(\mathcal{I}, M) \simeq \mathrm{Tr}_\Sigma$ in $\mathbf{Set}$.

▶ **Proposition 10.22.** *Given $M \in \mathbf{Tree}[\Sigma, \Gamma]$, we have, in* $\mathbf{Set}$*,*

$$\mathrm{tr} \quad : \quad \wp_\Sigma(\mathcal{I}, M) \quad \xrightarrow{\simeq} \quad \mathrm{Tr}_\Sigma$$

Proof. We first show that tr is injective. Note that if $\mathrm{tr}(s) = \mathrm{tr}(t)$, then s and t must have the same length and end by the same kind of moves. We thus show by induction on $|s| = |t|$ that $\mathrm{tr}(s) = \mathrm{tr}(t)$ implies $s = t$.

In the case case, we have $|s| = |t| = 0$ and we are done since we must have

$$s \quad = \quad (\varepsilon, q^\imath_\mathcal{I}) \quad = \quad t$$

For the inductive step, there are two cases:

Case of $s = s' \to (p, a, \gamma)$ and $t = t' \to (p', a', \gamma')$

Then $\mathrm{tr}(s) = \mathrm{tr}(t)$ implies $a = a'$, and by Lem. 6.1 we also get $p = p'$.

We moreover have $\gamma = \gamma'$ since $\delta_\mathcal{I}(q^\imath_\mathcal{I}, b)$ is always a singletton, and we conclude by induction hypothesis, since $\mathrm{tr}(s) = \mathrm{tr}(t)$ implies $\mathrm{tr}(s') = \mathrm{tr}(t')$.

Case of $s = s' \to (p \cdot d, q^{\imath}_{\mathcal{I}})$ and $t = t' \to (p' \cdot d', q^{\imath}_{\mathcal{I}})$

Then $\mathrm{tr}(s) = \mathrm{tr}(t)$ implies $d = d'$ and by Lem. 6.1 we moreover get $p = p'$.
We can then conclude by induction hypothesis, since $\mathrm{tr}(s) = \mathrm{tr}(t)$ implies $\mathrm{tr}(s') = \mathrm{tr}(t')$.

We now show that tr is surjective, *i.e.* that for all trace

$$t \in \mathrm{Tr}_\Sigma \quad = \quad (\Sigma \cdot D)^* + (\Sigma \cdot D)^* \cdot \Sigma$$

there is $s \in \wp_\Sigma(\mathcal{I}, M)$ such that $\mathrm{tr}(s) = t$. We reason by induction on t. For the base case $t = \varepsilon$, we take $s := (\varepsilon, q^{\imath}_{\mathcal{I}})$. For the induction step, there are two cases:

Case of $t = t' \cdot a$. By induction hypothesis, there is s' such that $\mathrm{tr}(s') = t'$. But in this case, s' must be of the form

$$s' \quad : \quad * \to^* (p, q^{\imath}_{\mathcal{I}})$$

Since $\delta_{\mathcal{I}}$ is constant, it follows that we are done by taking

$$s \quad := \quad s' \to (p, a, \gamma) \in \wp_\Sigma(\mathcal{I}, M)$$

Case of $t = t' \cdot d$. By induction hypothesis, there is s' such that $\mathrm{tr}(s') = t'$. Then s' is of the form

$$s' \quad : \quad * \to^* (p, a, \gamma)$$

Now, by definition of $\delta_{\mathcal{I}}$, we have $(q^{\imath}_{\mathcal{I}}, d) \in \gamma$, hence we are done by taking:

$$s \quad := \quad s' \to (p \cdot d, q^{\imath}_{\mathcal{I}}) \in \wp_\Sigma(\mathcal{I}, M)$$

◀

▶ Remark. Note that Prop. 10.22 fails for the unit automata $\Sigma \vdash \perp\!\!\!\!\!\!\!\perp$ (see also Sect. 13.2): tr is not injective since we can have two distinct γ and γ' in $\delta_{\perp\!\!\!\perp}(\_, \_)$.

Moreover, w.r.t. surjectivity, for the case of $t = t' \cdot d$, not every play s' of $\perp\!\!\!\perp$ with $\mathrm{tr}(s') = t$ can be extended to a play s such that $\mathrm{tr}(s) = t$.

10.10.2 Symmetric Monoidal Structure Maps.

Consider $\Sigma \vdash \mathcal{G}(\mathcal{A}, M)$, $\Sigma \vdash \mathcal{G}(\mathcal{B}, M)$, and $\Sigma \vdash \mathcal{G}(\mathcal{C}, M)$, where $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are complete auotmata.

Proposition 10.4 provides us with a symmetric monoidal structure in $\mathbf{Rel}(\mathbf{Set}/\mathrm{Tr}_\Sigma)$ for $\_ \otimes \_$ with (thanks to Prop. 10.22) unit $\wp(\mathcal{I}, M)$:

$$\begin{array}{rcrcl}
\mathring{\alpha}_{\mathcal{A},\mathcal{B},\mathcal{C}} & : & (\wp(\mathcal{A}, M) \otimes \wp(\mathcal{B}, M)) \otimes \wp(\mathcal{C}, M) & \twoheadrightarrow_{\mathrm{Tr}_\Sigma} & \wp(\mathcal{A}, M) \otimes (\wp(\mathcal{B}, M) \otimes \wp(\mathcal{C}, M)) \\
\mathring{\lambda}_{\mathcal{A}} & : & \wp(\mathcal{I}, M) \otimes \wp(\mathcal{A}, M) & \twoheadrightarrow_{\mathrm{Tr}_\Sigma} & \wp(\mathcal{A}, M) \\
\mathring{\rho}_{\mathcal{A}} & : & \wp(\mathcal{A}, M) \otimes \wp(\mathcal{I}, M) & \twoheadrightarrow_{\mathrm{Tr}_\Sigma} & \wp(\mathcal{A}, M) \\
\mathring{\gamma}_{\mathcal{A},\mathcal{B}} & : & \wp(\mathcal{A}, M) \otimes \wp(\mathcal{B}, M) & \twoheadrightarrow_{\mathrm{Tr}_\Sigma} & \wp(\mathcal{B}, M) \otimes \wp(\mathcal{A}, M)
\end{array}$$

Recall that by construction we have

$$([(\Sigma \vdash \mathcal{G}(\mathcal{A}, M)) \circledast (\Sigma \vdash \mathcal{G}(\mathcal{B}, M))] \circledast (\Sigma \vdash \mathcal{G}(\mathcal{C}, M))) \quad = \quad \Sigma \vdash \mathcal{G}((\mathcal{A} \circledast \mathcal{B}) \circledast \mathcal{C} \ , \ M)$$

and similarly for $\mathcal{G}(\mathcal{A} \circledast (\mathcal{B} \circledast \mathcal{C}), M)$.

Moreover by Prop. 10.9 we have bijections

$$
\begin{array}{l}
((\mathrm{SP} \times 1) \times (1 \times \mathrm{SP})) \circ (\mathrm{SP} \times \mathrm{SP}) \qquad : \\
\wp\,(\mathcal{G}((\mathcal{A} \circledast \mathcal{B}) \circledast \mathcal{C}, M)) \quad \times \quad \wp\,(\mathcal{G}(\mathcal{A} \circledast (\mathcal{B} \circledast \mathcal{C}), M)) \qquad \longrightarrow \\
\qquad\qquad (\wp(\mathcal{A}, M) \otimes \wp(\mathcal{B}, M)) \otimes \wp(\mathcal{C}, M) \quad \times \quad \wp(\mathcal{A}, M) \otimes (\wp(\mathcal{B}, M) \otimes \wp(\mathcal{C}, M))
\end{array}
$$

$$
\begin{array}{l}
(\mathrm{SP} \times 1) \qquad : \\
\wp\,(\mathcal{G}(\mathcal{I} \circledast \mathcal{A}, M)) \quad \times \quad \wp\,(\mathcal{G}(\mathcal{A}, M)) \qquad \longrightarrow \\
\qquad\qquad\qquad\qquad \wp(\mathcal{I}, M) \otimes \wp(\mathcal{A}, M) \quad \times \quad \wp(\mathcal{A}, M)
\end{array}
$$

$$
\begin{array}{l}
(\mathrm{SP} \times 1) \qquad : \\
\wp\,(\mathcal{G}(\mathcal{A} \circledast \mathcal{I}, M)) \quad \times \quad \wp\,(\mathcal{G}(\mathcal{A}, M)) \qquad \longrightarrow \\
\qquad\qquad\qquad\qquad \wp(\mathcal{A}, M) \otimes \wp(\mathcal{I}, M) \quad \times \quad \wp(\mathcal{A}, M)
\end{array}
$$

$$
\begin{array}{l}
(\mathrm{SP} \times \mathrm{SP}) \qquad : \\
\wp\,(\mathcal{G}(\mathcal{A} \circledast \mathcal{B}, M)) \quad \times \quad \wp\,(\mathcal{G}(\mathcal{B} \circledast \mathcal{A}, M)) \qquad \longrightarrow \\
\qquad\qquad\qquad \wp(\mathcal{A}, M) \otimes \wp(\mathcal{B}, M) \quad \times \quad \wp(\mathcal{B}, M) \otimes \wp(\mathcal{A}, M)
\end{array}
$$

Now, thanks to Lem. 10.2, Prop. 6.12 gives us total isomorphisms

$$
\begin{array}{r c r c l}
\alpha_{\mathcal{A},\mathcal{B},\mathcal{C}} & : & \mathcal{G}((\mathcal{A} \circledast \mathcal{B}) \circledast \mathcal{C}, M) & \longrightarrow_{\widehat{\mathbf{SAG}}_\Sigma} & \mathcal{G}(\mathcal{A} \circledast (\mathcal{B} \circledast \mathcal{C}), M) \\
\lambda_{\mathcal{A}} & : & \mathcal{G}(\mathcal{I} \circledast \mathcal{A}, M) & \longrightarrow_{\widehat{\mathbf{SAG}}_\Sigma} & \mathcal{G}(\mathcal{A}, M) \\
\rho_{\mathcal{A}} & : & \mathcal{G}(\mathcal{A} \circledast \mathcal{I}, M) & \longrightarrow_{\widehat{\mathbf{SAG}}_\Sigma} & \mathcal{G}(\mathcal{A}, M) \\
\gamma_{\mathcal{A},\mathcal{B}} & : & \mathcal{G}(\mathcal{A} \circledast \mathcal{B}, M) & \longrightarrow_{\widehat{\mathbf{SAG}}_\Sigma} & \mathcal{G}(\mathcal{B} \circledast \mathcal{A}, M)
\end{array}
\tag{20}
$$

such that

$$
\begin{array}{r c l}
\mathring{\alpha}_{\mathcal{A},\mathcal{B},\mathcal{C}} & = & ((\mathrm{SP} \times 1) \times (1 \times \mathrm{SP})) \circ (\mathrm{SP} \times \mathrm{SP}) \circ \mathrm{HS}(\alpha_{\mathcal{A},\mathcal{B},\mathcal{C}}) \\
\mathring{\lambda}_{\mathcal{A}} & = & (\mathrm{SP} \times 1) \circ \mathrm{HS}(\lambda_{\mathcal{A}}) \\
\mathring{\rho}_{\mathcal{A}} & = & (\mathrm{SP} \times 1) \circ \mathrm{HS}(\rho_{\mathcal{A}}) \\
\mathring{\gamma}_{\mathcal{A},\mathcal{B}} & = & (\mathrm{SP} \times \mathrm{SP}) \circ \mathrm{HS}(\gamma_{\mathcal{A},\mathcal{B}})
\end{array}
$$

▶ **Lemma 10.23.** *The above strategies are isomorphisms of* $\widehat{\mathbf{SAG}}_\Sigma^{(\mathrm{W/R})}$.

Proof. Since totality was ensured by Prop. 6.12, we only have to check that these strategies are winning w.r.t. their respective winning conditions in $\widehat{\mathbf{SAG}}_\Sigma^{\mathrm{W}}$ and $\widehat{\mathbf{SAG}}_\Sigma^{\mathrm{R}}$. Similarly as with Prop. 10.13, expanding the winning conditions gives propositionnal tautologies. Note that for the associativity maps $\mathring{\alpha}$, winning w.r.t. $\widehat{\mathbf{SAG}}_\Sigma^{\mathrm{R}}$ requires classical logic. ◀

10.10.3 Naturality and Coherence.

▶ **Lemma 10.24.** *The maps* $\alpha_{\mathcal{A},\mathcal{B},\mathcal{C}}$*,* $\lambda_{\mathcal{A}}$*,* $\rho_{\mathcal{A}}$ *and* $\gamma_{\mathcal{A}}$ *are natural in* $\mathcal{A}$*,* $\mathcal{B}$ *and* $\mathcal{C}$ *(when applicable).*

Proof. For

$$\alpha_{\mathcal{A},\mathcal{B},\mathcal{C}} \quad : \quad \mathcal{G}((\mathcal{A} \circledast \mathcal{B}) \circledast \mathcal{C}, M) \quad \longrightarrow \quad \mathcal{G}(\mathcal{A} \circledast (\mathcal{B} \circledast \mathcal{C}), M)$$

given

$$\sigma : \mathcal{G}(\mathcal{A}, M) \longrightarrow \mathcal{G}(\mathcal{A}', M') \qquad \tau : \mathcal{G}(\mathcal{B}, M) \longrightarrow \mathcal{G}(\mathcal{B}', M') \qquad \theta : \mathcal{G}(\mathcal{C}, M) \longrightarrow \mathcal{G}(\mathcal{C}', M')$$

we have to show that

$$\alpha_{\mathcal{A}',\mathcal{B}',\mathcal{C}'} \circ ((\sigma \circledast \tau) \circledast \theta) \quad = \quad (\sigma \circledast (\tau \circledast \theta)) \circ \alpha_{\mathcal{A},\mathcal{B},\mathcal{C}}$$

By Cor. 6.7 and Prop. 10.9 we have a bijection

$$((\mathrm{SP} \times 1) \times (1 \times \mathrm{SP})) \circ (\mathrm{SP} \times \mathrm{SP}) \circ \mathrm{HS}$$

Hence, by Prop. 4.10 it suffices to show (where the outer compositions are taken in $\mathbf{Set}/J$ and the inner in $\mathbf{Rel}(\mathbf{Set}/J)$)

$$\begin{aligned} &((\mathrm{SP} \times 1) \times (1 \times \mathrm{SP})) \circ (\mathrm{SP} \times \mathrm{SP})\,(\mathrm{HS}(\alpha_{\mathcal{A}',\mathcal{B}',\mathcal{C}'}) \circ \mathrm{HS}((\sigma \circledast \tau) \circledast \theta)) \quad = \\ &\qquad ((\mathrm{SP} \times 1) \times (1 \times \mathrm{SP})) \circ (\mathrm{SP} \times \mathrm{SP})\,(\mathrm{HS}(\sigma \circledast (\tau \circledast \theta)) \circ \mathrm{HS}(\alpha_{\mathcal{A},\mathcal{B},\mathcal{C}})) \end{aligned}$$

We now perform four applications of Lem. 10.6 (relational composition by maps with central bijections), with central bijections. The first two ones are consecutively performed in the l.-h.s., with central bijections SP and then $(\mathrm{SP} \times 1)$. The last two are consecutively performed in the r.-h.s., with central bijections SP and then $(1 \times \mathrm{SP})$.
By defintion of $\alpha_{\_,\_,\_}$, we are left to show (where the outer compositions are taken in $\mathbf{Rel}(\mathbf{Set}/J)$ and the inner in $\mathbf{Set}/J$)

$$\begin{aligned} &\alpha_{\mathcal{A}',\mathcal{B}',\mathcal{C}'} \circ [((\mathrm{SP} \times 1) \times (\mathrm{SP} \times 1)) \circ (\mathrm{SP} \times \mathrm{SP}) \circ \mathrm{HS}((\sigma \circledast \tau) \circledast \theta)] \quad = \\ &\qquad [((1 \times \mathrm{SP}) \times (1 \times \mathrm{SP})) \circ (\mathrm{SP} \times \mathrm{SP}) \circ \mathrm{HS}(\sigma \circledast (\tau \circledast \theta))] \circ \alpha_{\mathcal{A},\mathcal{B},\mathcal{C}} \end{aligned}$$

But now, by Prop. 10.19, and because Cartesian products of maps of $\mathbf{Set}/\mathrm{Tr}_\Sigma$ commute over $\_ \otimes \_$ (Lem. 10.5) we have

$$\begin{aligned} & ((\mathrm{SP} \times 1) \times (\mathrm{SP} \times 1)) \circ (\mathrm{SP} \times \mathrm{SP}) \circ \mathrm{HS}((\sigma \circledast \tau) \circledast \theta) \\ = \quad & ((\mathrm{SP} \times 1) \times (\mathrm{SP} \times 1))\,[\mathrm{HS}(\sigma \circledast \tau) \otimes \mathrm{HS}(\theta)] \\ = \quad & (\mathrm{SP} \times \mathrm{SP}) \circ \mathrm{HS}(\sigma \circledast \tau) \otimes (1 \times 1) \circ \mathrm{HS}(\theta) \\ = \quad & (\mathrm{HS}(\sigma) \otimes \mathrm{HS}(\tau)) \otimes \mathrm{HS}(\theta) \end{aligned}$$

and

$$\begin{aligned} & ((1 \times \mathrm{SP}) \times (1 \times \mathrm{SP})) \circ (\mathrm{SP} \times \mathrm{SP}) \circ \mathrm{HS}(\sigma \circledast (\tau \circledast \theta)) \\ = \quad & ((1 \times \mathrm{SP}) \times (1 \times \mathrm{SP}))\,[\mathrm{HS}(\sigma) \otimes \mathrm{HS}(\tau \circledast \theta)] \\ = \quad & (1 \times 1) \circ \mathrm{HS}(\sigma) \otimes (\mathrm{SP} \times \mathrm{SP}) \circ \mathrm{HS}(\tau \circledast \theta) \\ = \quad & \mathrm{HS}(\sigma) \otimes (\mathrm{HS}(\tau) \otimes \mathrm{HS}(\theta)) \end{aligned}$$

and we conclude by Lem. 10.3.
The case of

$$\gamma_{\mathcal{A},\mathcal{B}} \quad : \quad \mathcal{G}(\mathcal{A} \circledast \mathcal{B}, M) \quad \longrightarrow \quad \mathcal{G}(\mathcal{B} \circledast \mathcal{A}, M)$$

can be handled similarly.

For

$$\lambda_{\mathcal{A}} \quad : \quad \mathcal{G}(\mathcal{I} \circledast \mathcal{A}, M) \quad \longrightarrow \quad \mathcal{G}(\mathcal{A}, M)$$

given

$$\sigma : \mathcal{G}(\mathcal{A}, M) \longrightarrow \mathcal{G}(\mathcal{A}', M')$$

we have to show

$$\lambda_{\mathcal{A}'} \circ (\mathrm{id}_{\mathcal{I}} \circledast \sigma) \quad = \quad \sigma \circ \lambda_{\mathcal{A}'}$$

We proceed similarly as with α above. Thanks to the bijection

$$(\mathrm{SP} \times 1) \circ \mathrm{HS}$$

it is sufficient to show

$$(\mathrm{SP} \times 1) \circ \mathrm{HS}(\lambda_{\mathcal{A}'} \circ (\mathrm{id}_{\mathcal{I}} \circledast \sigma)) \quad = \quad (\mathrm{SP} \times 1) \circ \mathrm{HS}(\sigma \circ \lambda_{\mathcal{A}'})$$

that is

$$(\mathrm{SP} \times 1)(\mathrm{HS}(\lambda_{\mathcal{A}'}) \circ \mathrm{HS}(\mathrm{id}_{\mathcal{I}} \circledast \sigma)) \quad = \quad (\mathrm{SP} \times 1)(\mathrm{HS}(\sigma) \circ \mathrm{HS}(\lambda_{\mathcal{A}'}))$$

We now apply Lem. 10.6, with, as central bijections, SP on the l.-h.s. and 1 on the r.-h.s.
By defintion of λ_{-}, we are left to show

$$\lambda_{\mathcal{A}'} \circ (\mathrm{HS}(\mathrm{id}_{\mathcal{I}}) \otimes \mathrm{HS}(\sigma)) \quad = \quad \mathrm{HS}(\sigma) \circ \mathring{\lambda}_{\mathcal{A}'}$$

And we are done by Prop. 4.11 and Lem. 10.3.
The case of $\mathring{\rho}_{\mathcal{A}}$ can be dealt-with similarly. ◀

Can be shown using the same techniques as for Lem. 10.24, but relying on Prop. 10.4 instead of Lem. 10.3, we can show that the expected coherence digrams are satisfied by the maps $\alpha_{(-),(-),(-)}$, $\lambda_{(-)}$, $\rho_{(-)}$ and $\gamma_{(-),(-)}$. As in Prop 10.4 above, we only discuss the diagrams required by [16].

▶ **Lemma 10.25.**

(i) $\alpha_{\mathcal{A},\mathcal{B},(\mathcal{C}\circledast\mathcal{D})} \circ \alpha_{\mathcal{A}\circledast\mathcal{B},\mathcal{C},\mathcal{D}} = (\mathrm{id}_{\mathcal{A}} \circledast \alpha_{\mathcal{B},\mathcal{C},\mathcal{D}}) \circ \alpha_{\mathcal{A},(\mathcal{B}\circledast\mathcal{C}),\mathcal{D}} \circ (\alpha_{\mathcal{A},\mathcal{B},\mathcal{C}} \circledast \mathrm{id}_{\mathcal{D}})$
in

$$\mathcal{G}((((\mathcal{A} \circledast \mathcal{B}) \circledast \mathcal{C}) \circledast \mathcal{D}), M) \quad \longrightarrow_{\widehat{\mathbf{SAG}}_{\Sigma}^{(\mathrm{W})}} \quad \mathcal{G}(\mathcal{A} \circledast (\mathcal{B} \circledast (\mathcal{C} \circledast \mathcal{D})), M)$$

(ii) $(\mathrm{id}_{\mathcal{A}} \circledast \lambda_{\mathcal{A}}) \circ \alpha_{\mathcal{A},\mathcal{I},\mathcal{B}} = \rho_{\mathcal{A}} \circledast \mathrm{id}_{\mathcal{B}}$
in

$$\mathcal{G}((\mathcal{A} \circledast \mathcal{I}) \circledast \mathcal{B}, M) \quad \longrightarrow_{\widehat{\mathbf{SAG}}_{\Sigma}^{(\mathrm{W})}} \quad \mathcal{G}(\mathcal{A} \circledast \mathcal{B}, M)$$

(iii) $\alpha_{\mathcal{B},\mathcal{C},\mathcal{A}} \circ (\gamma_{\mathcal{A},(\mathcal{B}\circledast\mathcal{C})}) \circ \alpha_{\mathcal{A},\mathcal{B},\mathcal{C}} = (\mathrm{id}_{\mathcal{B}} \circledast \gamma_{\mathcal{A},\mathcal{C}}) \circ \alpha_{\mathcal{B},\mathcal{A},\mathcal{C}} \circ (\gamma_{\mathcal{A},\mathcal{B}} \circledast \mathrm{id}_{\mathcal{C}})$
in

$$\mathcal{G}((\mathcal{A} \circledast \mathcal{B}) \circledast \mathcal{C}, M) \quad \longrightarrow_{\widehat{\mathbf{SAG}}_{\Sigma}^{(\mathrm{W})}} \quad \mathcal{G}(\mathcal{B} \circledast (\mathcal{C} \circledast \mathcal{A}), M)$$

(iv) $\gamma_{\mathcal{B},\mathcal{A}} = (\gamma_{\mathcal{A},\mathcal{B}})^{-1}$
in

$$\mathcal{G}(\mathcal{B} \circledast \mathcal{A}, M) \quad \longrightarrow_{\widehat{\mathbf{SAG}}_{\Sigma}^{(\mathrm{W})}} \quad \mathcal{G}(\mathcal{A} \circledast \mathcal{B}, M)$$

11 Symmetric Monoidal Fibrations of Games and Automata

In this section, we define a synchronous product $\_ \circledast \_$ and show that substitutions restrict to split *indexed symmetric monoidal categories* (in the sense of [21]):

$$\begin{array}{rcccl} (-)^* & : & \mathbf{Alph}^{\mathrm{op}} & \to & \mathbf{SymMonCat} \\ (-)^* & : & \mathbf{Tree}^{\mathrm{op}} & \to & \mathbf{SymMonCat} \end{array}$$

leading to symmetric monoidal fibrations.

11.1 Symmetric Monoidal Categories of Automata

We first discuss the symmetric monoidal structure of the categories $\widehat{\mathbf{Aut}}_\Sigma^{(\mathrm{W})}$.

The bifunctoriality of $\_ \circledast \_$ directly follows from Prop. 10.13 and Prop. 10.20. The monoidal unit of $\widehat{\mathbf{Aut}}_\Sigma^{(\mathrm{W})}$ is $\Sigma \vdash \mathcal{I}$, and the symmetric monoidal structure maps are as follows, following Sect. 10.10.2:

$$\begin{array}{rcrcl} \alpha_{\mathcal{A},\mathcal{B},\mathcal{C}} & : & \mathcal{G}((\mathcal{A} \circledast \mathcal{B}) \circledast \mathcal{C}, M) & \longrightarrow_{\widehat{\mathbf{SAG}}_\Sigma} & \mathcal{G}(\mathcal{A} \circledast (\mathcal{B} \circledast \mathcal{C}), M) \\ \lambda_{\mathcal{A}} & : & \mathcal{G}(\mathcal{I} \circledast \mathcal{A}, M) & \longrightarrow_{\widehat{\mathbf{SAG}}_\Sigma} & \mathcal{G}(\mathcal{A}, M) \\ \rho_{\mathcal{A}} & : & \mathcal{G}(\mathcal{A} \circledast \mathcal{I}, M) & \longrightarrow_{\widehat{\mathbf{SAG}}_\Sigma} & \mathcal{G}(\mathcal{A}, M) \\ \gamma_{\mathcal{A},\mathcal{B}} & : & \mathcal{G}(\mathcal{A} \circledast \mathcal{B}, M) & \longrightarrow_{\widehat{\mathbf{SAG}}_\Sigma} & \mathcal{G}(\mathcal{B} \circledast \mathcal{A}, M) \end{array}$$

Hence, using Lem. 10.23, Lem. 10.24 and Lem. 10.25 we get

▶ **Proposition 11.1** (Symmetric Monoidal Structure in $\widehat{\mathbf{Aut}}_\Sigma$). *The categories $\widehat{\mathbf{Aut}}_\Sigma^{(\mathrm{W})}$, equipped with the above data, are symmetric monoidal.*

11.2 Compatibility of the Synchronuous Product with Substitution

In this Section, we discuss the compatibility with substitution of the partial version of the monoidal product $\_ \circledast \_$. This will give the symmetric monoidal structure of the categories of acceptance games, as well as the strict symmetric monoidality of substitution.

Recall that a (strong) symmetric monoidal functor $F : (\mathbb{C}, \otimes, u) \to (\mathbb{D}, \bullet, e)$ comes with natural isomorphisms

$$m^0 : e \longrightarrow F(u) \quad \text{and} \quad m^2_{A,B} : F(A) \bullet F(B) \longrightarrow F(A \otimes B)$$

subject to some coherence diagram (see e.g. [16]). Note that naturality means that for all $f \in \mathbb{C}[A, C]$, $g \in \mathbb{C}[B, D]$, we have

$$\begin{tikzcd} F(A) \bullet F(B) \arrow[r, "m^2_{A,B}"] \arrow[d, "F(f)\bullet F(g)"'] & F(A \otimes B) \arrow[d, "F(f\otimes g)"] \\ F(C) \bullet F(D) \arrow[r, "m^2_{C,D}"] & F(C \otimes D) \end{tikzcd}$$

In our cases, we will define, for $\Sigma \vdash \mathcal{G}(\mathcal{A}, M)$ and $\Sigma \vdash \mathcal{G}(\mathcal{B}, M)$, the action of synchronous product as

$$(\Sigma \vdash \mathcal{G}(\mathcal{A}, M)) \circledast (\Sigma \vdash \mathcal{G}(\mathcal{B}, M)) \quad := \quad \Sigma \vdash \mathcal{G}(\mathcal{A} \circledast \mathcal{B}, M)$$

from which we immediatelly get, for $L \in \mathbf{Tree}[\Gamma, \Sigma]$:

$$L^*(\Sigma \vdash \mathcal{G}(\mathcal{A}, M) \circledast \Sigma \vdash \mathcal{G}(\mathcal{B}, M)) \quad = \quad L^*(\Sigma \vdash \mathcal{G}(\mathcal{A}, M)) \circledast L^*(\Sigma \vdash \mathcal{G}(\mathcal{B}, M))$$

It will follow that the mediating maps $m^2_{\_,\_}$ are identities.

As for the units, recall from Lem. 10.21 that we have

$$L^*(\Sigma \vdash \mathcal{G}(\mathcal{I}, M)) \quad = \quad \Gamma \vdash \mathcal{G}(\mathcal{I}, M \circ L) \quad = \quad \Gamma \vdash \mathcal{G}(\mathcal{I}, \mathrm{Id}_\Gamma)$$

Moreover, the strategies $\alpha_{\_,\_,\_}$, $\gamma_{\_,\_}$, $\lambda_{\_}$ and $\rho_{\_}$, from which structure maps will be defined, are preserved by substitution.

The following gathers all the relevant properties we will need to obtain strong symmetric monoidal fibrations:

▶ **Proposition 11.2.** *Consider $L \in \mathbf{Tree}[\Gamma, \Sigma]$.*

(i) *We have*

$$L^*(\Sigma \vdash \mathcal{G}(\mathcal{I}, \mathrm{Id})) \quad = \quad \Gamma \vdash \mathcal{G}(\mathcal{I}, \mathrm{Id}_\Gamma)$$

(ii) *We have*

$$L^*((\Sigma \vdash \mathcal{G}(\mathcal{A}, M)) \circledast (\Sigma \vdash \mathcal{G}(\mathcal{B}, M))) \quad = \quad L^*(\Sigma \vdash \mathcal{G}(\mathcal{A}, M)) \circledast L^*(\Sigma \vdash \mathcal{G}(\mathcal{B}, M))$$

(iii) *Given*

$$\Sigma \vdash \sigma : \mathcal{G}(\mathcal{A}, M) \multimap\!\!\circledast\ \mathcal{G}(\mathcal{C}, N) \qquad \text{and} \qquad \Sigma \vdash \theta : \mathcal{G}(\mathcal{B}, M) \multimap\!\!\circledast\ \mathcal{G}(\mathcal{D}, N)$$

we have

$$L^*(\sigma \circledast \theta) \quad = \quad L^*(\sigma) \circledast L^*(\theta)$$

(iv) *Given*

$$\Sigma \vdash \mathcal{G}(\mathcal{A}, M) \qquad \Sigma \vdash \mathcal{G}(\mathcal{B}, M) \qquad \Sigma \vdash \mathcal{G}(\mathcal{C}, M)$$

we have

$$\begin{array}{rcl} L^*(\alpha_{(\mathcal{A},M),(\mathcal{B},M),(\mathcal{C},M)}) & = & \alpha_{(\mathcal{A},M\circ L),(\mathcal{B},M\circ L),(\mathcal{C},M\circ L)} \\ L^*(\lambda_{(\mathcal{A},M)}) & = & \lambda_{(\mathcal{A},M\circ L)} \\ L^*(\rho_{(\mathcal{A},M)}) & = & \rho_{(\mathcal{A},M\circ L)} \\ L^*(\gamma_{(\mathcal{A},M),(\mathcal{B},M)}) & = & \gamma_{(\mathcal{A},M\circ L),(\mathcal{B},M\circ L)} \end{array}$$

Proof of Prop. 11.2.(i). Recall from Lem. 10.21 that we have

$$L^*(\Sigma \vdash \mathcal{G}(\mathcal{I}, M)) \quad = \quad \Gamma \vdash \mathcal{G}(\mathcal{I}, M \circ L) \quad = \quad \Gamma \vdash \mathcal{G}(\mathcal{I}, \mathrm{Id}_\Gamma)$$

◀

Proof of Prop. 11.2.(ii). By definition we have

$$\begin{array}{rcl} L^*((\Sigma \vdash \mathcal{G}(\mathcal{A}, M)) \circledast (\Sigma \vdash \mathcal{G}(\mathcal{B}, M))) & = & L^*(\Sigma \vdash \mathcal{G}(\mathcal{A} \circledast \mathcal{B}, M)) \\ & = & \Gamma \vdash \mathcal{G}(\mathcal{A} \circledast \mathcal{B}, M \circ L) \\ & = & (\Gamma \vdash \mathcal{G}(\mathcal{A}, M \circ L)) \circledast (\Gamma \vdash \mathcal{G}(\mathcal{B}, M \circ L)) \\ & = & L^*(\Sigma \vdash \mathcal{G}(\mathcal{A}, M \circ L)) \circledast L^*(\Sigma \vdash \mathcal{G}(\mathcal{B}, M \circ L)) \end{array}$$

◀

11.2.0.1 Poof of Prop. 11.2.(iii).

For Prop. 11.2.(iii) we rely on the following simple property:

▶ **Lemma 11.3.** *We have in* **Set** *(and similarly for ϖ_2):*

$$\begin{tikzcd}
\wp_\Gamma\left((\mathcal{A} \circledast \mathcal{B}, M \circ L) \multimap\!\!\circledast (\mathcal{C} \circledast \mathcal{D}, M \circ L)\right) \arrow[d, "\wp(L)_{\multimap\circledast}"'] \arrow[dr, "\varpi_1"] & \\
& \wp_\Gamma\left((\mathcal{A}, M \circ L) \multimap\!\!\circledast (\mathcal{C}, M \circ L)\right) \arrow[d, "\wp(L)_{\multimap\circledast}"'] \\
\wp_\Sigma\left((\mathcal{A} \circledast \mathcal{B}, M) \multimap\!\!\circledast (\mathcal{C} \circledast \mathcal{D}, M)\right) \arrow[dr, "\varpi_1"] & \\
& \wp_\Sigma\left((\mathcal{A}, M) \multimap\!\!\circledast (\mathcal{C}, M)\right)
\end{tikzcd}$$

Proof. By induction on

$$s \in \wp_\Gamma\left((\mathcal{A} \circledast \mathcal{B}, M \circ L) \multimap\!\!\circledast (\mathcal{C} \circledast \mathcal{D}, M \circ L)\right)$$

For the base case, we have

$$s \quad = \quad ((\varepsilon, (q^\imath_\mathcal{A}, q^\imath_\mathcal{B}))\ ,\ (\varepsilon, (q^\imath_\mathcal{C}, q^\imath_\mathcal{D})))$$

and we are done since

$$\varpi_1 \circ \wp(L)_{\multimap\circledast}(s) \quad = \quad ((\varepsilon, q^\imath_\mathcal{A})\ ,\ (\varepsilon, q^\imath_\mathcal{C})) \quad = \quad \wp(L)_{\multimap\circledast} \circ \varpi_1(s)$$

For the inductive step, there are four cases:

Case of $s = t \xrightarrow{\mathsf{O}} ((p, a, \gamma_\mathcal{A} \circledast \gamma_\mathcal{B})\ ,\ (p, (q_\mathcal{C}, q_\mathcal{D})))$.

We have

$$\begin{array}{rcl}
\wp(L)_{\multimap\circledast}(s) & = & \wp(L)_{\multimap\circledast}(t) \to ((p, L(p)(a), \gamma_\mathcal{A} \circledast \gamma_\mathcal{B})\ ,\ (p, (q_\mathcal{C}, q_\mathcal{D}))) \\
\varpi_1 \circ \wp(L)_{\multimap\circledast}(s) & = & \varpi_1 \circ \wp(L)_{\multimap\circledast}(t) \to ((p, L(p)(a), \gamma_\mathcal{A})\ ,\ (p, q_\mathcal{C}))
\end{array}$$

and

$$\begin{array}{rcl}
\varpi_1(s) & = & \varpi_1(t) \to ((p, a, \gamma_\mathcal{A})\ ,\ (p, q_\mathcal{C})) \\
\wp(L)_{\multimap\circledast} \circ \varpi_1(s) & = & \wp(L)_{\multimap\circledast} \circ \varpi_1(t) \to ((p, L(p)(a), \gamma_\mathcal{A})\ ,\ (p, q_\mathcal{C}))
\end{array}$$

and we conclude by induction hypothesis.

Case of $s = t \xrightarrow{\mathsf{P}} ((p, a, \gamma_\mathcal{A} \circledast \gamma_\mathcal{B})\ ,\ (p, a, \gamma_\mathcal{C} \circledast \gamma_\mathcal{D}))$.

We have

$$\begin{array}{rcl}
\wp(L)_{\multimap\circledast}(s) & = & \wp(L)_{\multimap\circledast}(t) \to ((p, L(p)(a), \gamma_\mathcal{A} \circledast \gamma_\mathcal{B})\ ,\ (p, L(p)(a), \gamma_\mathcal{C} \circledast \gamma_\mathcal{D})) \\
\varpi_1 \circ \wp(L)_{\multimap\circledast}(s) & = & \varpi_1 \circ \wp(L)_{\multimap\circledast}(t) \to ((p, L(p)(a), \gamma_\mathcal{A})\ ,\ (p, L(p)(a), \gamma_\mathcal{C}))
\end{array}$$

and

$$\begin{array}{rcl}
\varpi_1(s) & = & \varpi_1(t) \to ((p, a, \gamma_\mathcal{A})\ ,\ (p, a, \gamma_\mathcal{C})) \\
\wp(L)_{\multimap\circledast} \circ \varpi_1(s) & = & \wp(L)_{\multimap\circledast} \circ \varpi_1(t) \to ((p, L(p)(a), \gamma_\mathcal{A})\ ,\ (p, L(p)(a), \gamma_\mathcal{C}))
\end{array}$$

and we conclude by induction hypothesis.

Case of $s = t \xrightarrow{\mathsf{O}} ((p, a, \gamma_{\mathcal{A}} \circledast \gamma_{\mathcal{B}}) \ , \ (p \cdot d, (q_{\mathcal{C}}, q_{\mathcal{D}})))$**.**
We have

$$\begin{array}{rcl} \wp(L)_{\multimap\circledast}(s) & = & \wp(L)_{\multimap\circledast}(t) \to ((p, L(p)(a), \gamma_{\mathcal{A}} \circledast \gamma_{\mathcal{B}}) \ , \ (p \cdot d, (q_{\mathcal{C}}, q_{\mathcal{D}}))) \\ \varpi_1 \circ \wp(L)_{\multimap\circledast}(s) & = & \varpi_1 \circ \wp(L)_{\multimap\circledast}(t) \to ((p, L(p)(a), \gamma_{\mathcal{A}}) \ , \ (p \cdot d, q_{\mathcal{C}})) \end{array}$$

and

$$\begin{array}{rcl} \varpi_1(s) & = & \varpi_1(t) \to ((p, a, \gamma_{\mathcal{A}}) \ , \ (p \cdot d, q_{\mathcal{C}})) \\ \wp(L)_{\multimap\circledast} \circ \varpi_1(s) & = & \wp(L)_{\multimap\circledast} \circ \varpi_1(t) \to ((p, L(p)(a), \gamma_{\mathcal{A}}) \ , \ (p \cdot d, q_{\mathcal{C}})) \end{array}$$

and we conclude by induction hypothesis.

Case of $s = t \xrightarrow{\mathsf{P}} ((p \cdot d, (q_{\mathcal{A}}, q_{\mathcal{B}})) \ , \ (p \cdot d, (q_{\mathcal{C}}, q_{\mathcal{D}})))$**.**
We have

$$\begin{array}{rcl} \wp(L)_{\multimap\circledast}(s) & = & \wp(L)_{\multimap\circledast}(t) \to ((p \cdot d, (q_{\mathcal{A}}, q_{\mathcal{B}})) \ , \ (p \cdot d, (q_{\mathcal{C}}, q_{\mathcal{D}}))) \\ \varpi_1 \circ \wp(L)_{\multimap\circledast}(s) & = & \varpi_1 \circ \wp(L)_{\multimap\circledast}(t) \to ((p \cdot d, q_{\mathcal{A}}) \ , \ (p \cdot d, q_{\mathcal{C}})) \end{array}$$

and

$$\begin{array}{rcl} \varpi_1(s) & = & \varpi_1(t) \to ((p \cdot d, q_{\mathcal{A}}) \ , \ (p \cdot d, q_{\mathcal{C}})) \\ \wp(L)_{\multimap\circledast} \circ \varpi_1(s) & = & \wp(L)_{\multimap\circledast} \circ \varpi_1(t) \to ((p \cdot d, q_{\mathcal{A}}) \ , \ (p \cdot d, q_{\mathcal{C}})) \end{array}$$

and we conclude by induction hypothesis. ◀

Proof of Prop. 11.2.(iii). We show that

$$s \in L^*(\sigma \circledast \theta) \quad \text{iff} \quad s \in L^*(\sigma) \circledast L^*(\theta)$$

for all

$$s \in \wp^{\mathsf{P}}_{\Sigma}\left(\mathcal{G}(\mathcal{A} \circledast \mathcal{B}, M \circ L) \multimap\circledast \mathcal{G}(\mathcal{C} \circledast \mathcal{D}, N \circ L)\right)$$

By definition of $L^*(\sigma \circledast \tau)$, we have

$$\begin{array}{rcl} s \in L^*(\sigma \circledast \theta) & \text{iff} & \wp(L)_{\multimap\circledast}(s) \in \sigma \circledast \theta \\ & \text{iff} & \mathrm{SP} \circ \wp(L)_{\multimap\circledast}(s) \in (\sigma, \theta) \end{array}$$

On the other hand, we have:

$$\begin{array}{rcl} s \in L^*(\sigma) \circledast L^*(\theta) & \text{iff} & \mathrm{SP}(s) \in (L^*(\sigma), L^*(\theta)) \\ & \text{iff} & \wp(L)^2_{\multimap\circledast} \circ \mathrm{SP}(s) \in (\sigma, \theta) \end{array}$$

and we are done since it follows from Lem. 11.3 that

$$\mathrm{SP} \circ \wp(L)_{\multimap\circledast}(s) \quad = \quad \wp(L)^2_{\multimap\circledast} \circ \mathrm{SP}(s)$$

◀

11.2.0.2 Poof of Prop. 11.2.(iv).

We rely on the following simple commutation lemma:

▶ **Lemma 11.4.** *Consider* $L \in \mathbf{Tree}[\Gamma, \Sigma]$.

$$
\begin{tikzcd}
\wp_\Gamma(\mathcal{A} \circledast \mathcal{B}, M \circ L) \arrow[r, "\varpi_2"] \arrow[d, "\varpi_1"'] \arrow[dr, "\wp(L)"] \arrow[dr, phantom, very near start, "\lrcorner"] & \wp_\Gamma(\mathcal{B}, M \circ L) \arrow[dr, "\wp(L)"] & \\
\wp_\Gamma(\mathcal{A}, M \circ L) \arrow[dr, "\wp(L)"'] & \wp_\Sigma(\mathcal{A} \circledast \mathcal{B}, M) \arrow[r, "\varpi_2"] \arrow[d, "\varpi_1"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \wp_\Sigma(\mathcal{B}, M) \arrow[d, "\mathrm{tr}"'] \\
 & \wp_\Sigma(\mathcal{A}, M) \arrow[r, "\mathrm{tr}"] & \mathrm{Tr}_\Sigma
\end{tikzcd}
$$

Proof. We show

$$
\begin{tikzcd}
\wp_\Gamma(\mathcal{A} \circledast \mathcal{B}, M \circ L) \arrow[r, "\wp(L)"] \arrow[d, "\varpi_1"'] & \wp_\Sigma(\mathcal{A} \circledast \mathcal{B}, M) \arrow[d, "\varpi_1"'] \\
\wp_\Gamma(\mathcal{A}, M \circ L) \arrow[r, "\wp(L)"'] & \wp_\Sigma(\mathcal{A}, M)
\end{tikzcd}
$$

The other diagram (involving $\mathcal{B}$ instead of $\mathcal{A}$) is shown similarly. Commutation of the lower-left diagramm, which is not essential here, follows from Prop10.9.

We reason by induction on

$$s \in \wp_\Gamma(\mathcal{A} \circledast \mathcal{B}, M \circ L)$$

The base case

$$s \quad = \quad (\varepsilon, (q^\imath_{\mathcal{A}}, q^\imath_{\mathcal{B}}))$$

is trivial since

$$\wp(L) \circ \varpi_1(s) \quad = \quad (\varepsilon, q^\imath_{\mathcal{A}}) \quad = \quad \varpi_1 \circ \wp(L)(s)$$

For the induction step, there are two cases:

Case of $s \;=\; t \xrightarrow{\mathsf{O}} (p, a, \gamma_{\mathcal{A}} \circledast \gamma_{\mathcal{B}})$.

By definition of $\mathcal{A} \circledast \mathcal{B}$, we have

$$\gamma_{\mathcal{A}} \in \delta_{\mathcal{A}}(q_{\mathcal{A}}, M \circ L(a)) \quad \text{and} \quad \gamma_{\mathcal{B}} \in \delta_{\mathcal{B}}(q_{\mathcal{B}}, M \circ L(a))$$

with

$$t \quad = \quad \varepsilon \to^* (p, (q_{\mathcal{A}}, q_{\mathcal{B}}))$$

It follows that

$$\gamma_{\mathcal{A}} \circledast \gamma_{\mathcal{B}} \in \delta_{\mathcal{A} \circledast \mathcal{B}}((q_{\mathcal{A}}, q_{\mathcal{B}}), M(p)(L(p)(a)))$$

hence

$$\wp(L)(s) \quad = \quad \wp(L)(t) \xrightarrow{\mathsf{O}} (p, L(p)(a), \gamma_{\mathcal{A}} \circledast \gamma_{\mathcal{B}})$$

and

$$\varpi_1 \circ \wp(L)(s) \quad = \quad \varpi_1 \circ \wp(L)(t) \stackrel{\mathsf{O}}{\to} (p, L(p)(a), \gamma_{\mathcal{A}})$$

On the other hand, we have

$$\wp(L) \circ \varpi_1(s) \quad = \quad \wp(L) \circ \varpi_1(t) \stackrel{\mathsf{O}}{\to} (p, L(a), q_{\mathcal{A}})$$

and we are done since by induction hypothesis we have

$$\varpi_1 \circ \wp(L)(t) \quad = \quad \wp(L) \circ \varpi_1(s)$$

Case of $s \; = \; t \stackrel{\mathsf{P}}{\to} (p \cdot d, (q_{\mathcal{A}}, q_{\mathcal{B}}))$.
We directly conclude from the induction hypothesis since

$$\varpi_1 \circ \wp(L)(s) \quad = \quad \varpi_1 \circ \wp(L)(t) \stackrel{\mathsf{O}}{\to} (p \cdot d, q_{\mathcal{A}})$$

and

$$\wp(L) \circ \varpi_1(s) \quad = \quad \wp(L) \circ \varpi_1(t) \stackrel{\mathsf{O}}{\to} (p \cdot d, q_{\mathcal{A}})$$

◀

Proof of Prop. 11.2.(iv). We have to show that given

$$\Sigma \vdash \mathcal{G}(\mathcal{A}, M) \qquad \Sigma \vdash \mathcal{G}(\mathcal{B}, M) \qquad \Sigma \vdash \mathcal{G}(\mathcal{C}, M)$$

we have

$$\begin{array}{rcl} L^*(\alpha_{(\mathcal{A},M),(\mathcal{B},M),(\mathcal{C},M)}) & = & \alpha_{(\mathcal{A},M\circ L),(\mathcal{B},M\circ L),(\mathcal{C},M\circ L)} \\ L^*(\lambda_{(\mathcal{A},M)}) & = & \lambda_{(\mathcal{A},M\circ L)} \\ L^*(\rho_{(\mathcal{A},M)}) & = & \rho_{(\mathcal{A},M\circ L)} \\ L^*(\gamma_{(\mathcal{A},M),(\mathcal{B},M)}) & = & \gamma_{(\mathcal{A},M\circ L),(\mathcal{B},M\circ L)} \end{array}$$

By Lem.4.6.(ii) (or Cor. 6.7) it is sufficient to show

$$\begin{array}{rcl} \mathrm{HS}(L^*(\alpha_{(\mathcal{A},M),(\mathcal{B},M),(\mathcal{C},M)})) & = & \mathrm{HS}(\alpha_{(\mathcal{A},M\circ L),(\mathcal{B},M\circ L),(\mathcal{C},M\circ L)}) \\ \mathrm{HS}(L^*(\lambda_{(\mathcal{A},M)})) & = & \mathrm{HS}(\lambda_{(\mathcal{A},M\circ L)}) \\ \mathrm{HS}(L^*(\rho_{(\mathcal{A},M)})) & = & \mathrm{HS}(\rho_{(\mathcal{A},M\circ L)}) \\ \mathrm{HS}(L^*(\gamma_{(\mathcal{A},M),(\mathcal{B},M)})) & = & \mathrm{HS}(\gamma_{(\mathcal{A},M\circ L),(\mathcal{B},M\circ L)}) \end{array}$$

By Lem. 8.3 this is equivalent to

$$\begin{array}{rcl} ((\wp(L)^{-1} \times \wp(L)^{-1}) \circ \mathrm{HS})(\alpha_{(\mathcal{A},M),(\mathcal{B},M),(\mathcal{C},M)}) & = & \mathrm{HS}(\alpha_{(\mathcal{A},M\circ L),(\mathcal{B},M\circ L),(\mathcal{C},M\circ L)}) \\ ((\wp(L)^{-1} \times \wp(L)^{-1}) \circ \mathrm{HS})(\lambda_{(\mathcal{A},M)}) & = & \mathrm{HS}(\lambda_{(\mathcal{A},M\circ L)}) \\ ((\wp(L)^{-1} \times \wp(L)^{-1}) \circ \mathrm{HS})(\rho_{(\mathcal{A},M)}) & = & \mathrm{HS}(\rho_{(\mathcal{A},M\circ L)}) \\ ((\wp(L)^{-1} \times \wp(L)^{-1}) \circ \mathrm{HS})(\gamma_{(\mathcal{A},M),(\mathcal{B},M)}) & = & \mathrm{HS}(\gamma_{(\mathcal{A},M\circ L),(\mathcal{B},M\circ L)}) \end{array}$$

We now claim that in this situation we have

$$\mathrm{SP} \circ \wp(L)^{-1} \quad = \quad (\wp(L)^{-1} \times \wp(L)^{-1}) \circ \mathrm{SP}$$

Proof. The inclusion $\mathrm{SP} \circ \wp(L)^{-1} \subseteq (\wp(L)^{-1} \times \wp(L)^{-1}) \circ \mathrm{SP}$
directly follows from the commutation of the diagram of Lem. 11.4.
For the other incusion $(\wp(L)^{-1} \times \wp(L)^{-1}) \circ \mathrm{SP} \subseteq \mathrm{SP} \circ \wp(L)^{-1}$ we rely on the universal properties Prop. 7.7 and Prop. 10.9 and the fact that SP preserves the traces. ◀

We can then conclude using the definition of the maps $\alpha_{(-),(-),(-)}$, $\lambda_{(-)}$, $\rho_{(-)}$ and $\gamma_{(-),(-)}$, and again using Prop. 7.7.

◀

11.3 Symmetric Monoidal Fibrations of Acceptance Games

We now give the symmetric monoidal structure to the categories $\widehat{\mathbf{SAG}}_\Sigma^{(\mathrm{W})}$.

On objects, we let

$$(\Sigma \vdash \mathcal{G}(\mathcal{A}, M)) \circledast (\Sigma \vdash \mathcal{G}(\mathcal{B}, N)) \quad := \quad \Sigma \vdash \mathcal{G}(\mathcal{A}[\pi] \circledast \mathcal{B}[\pi'], \langle M, N \rangle)$$

where $\Sigma_\mathcal{A} \vdash \mathcal{A}$ and $\Sigma_\mathcal{B} \vdash \mathcal{B}$ and π and π' are suitable projections:

$$\pi \in \mathbf{Alph}[\Sigma_\mathcal{A} \times \Sigma_\mathcal{B}, \Sigma_\mathcal{A}] \qquad \pi' \in \mathbf{Alph}[\Sigma_\mathcal{A} \times \Sigma_\mathcal{B}, \Sigma_\mathcal{B}]$$

For the action on morphisms, consider

$$\Sigma \vdash \mathcal{G}(\mathcal{A}_0, M_0) \quad \xrightarrow{\sigma} \quad \Sigma \vdash \mathcal{G}(\mathcal{A}_1, M_1)$$

and

$$\Sigma \vdash \mathcal{G}(\mathcal{B}_0, N_0) \quad \xrightarrow{\tau} \quad \Sigma \vdash \mathcal{G}(\mathcal{B}_1, N_1)$$

where, in **Tree**,

$$M_0 : \Sigma \to \Sigma_0 \qquad M_1 : \Sigma \to \Sigma_1 \qquad N_0 : \Sigma \to \Gamma_0 \qquad N_1 : \Sigma \to \Gamma_1$$

We define

$$\sigma \circledast \tau \quad : \quad (\Sigma \vdash \mathcal{G}(\mathcal{A}_0, M_0)) \circledast (\Sigma \vdash \mathcal{G}(\mathcal{B}_0, N_0)) \quad \longrightarrow \quad (\Sigma \vdash \mathcal{G}(\mathcal{A}_1, M_1)) \circledast (\Sigma \vdash \mathcal{G}(\mathcal{B}_1, N_1))$$

as follows.

First, note that we actually have

$$\Sigma \vdash \mathcal{G}(\mathcal{A}_0[\pi_0], \langle M_0, N_0 \rangle) \quad \xrightarrow{\sigma} \quad \Sigma \vdash \mathcal{G}(\mathcal{A}_1[\pi_1], \langle M_1, N_1 \rangle)$$

and

$$\Sigma \vdash \mathcal{G}(\mathcal{B}_0[\pi'_0], \langle M_0, N_0 \rangle) \quad \xrightarrow{\tau} \quad \Sigma \vdash \mathcal{G}(\mathcal{B}_1[\pi'_1], \langle M_1, N_1 \rangle)$$

where π_0, π_1, π'_0 and π'_1 are suitable projections.

On the other hand, we must actually have

$$\sigma \circledast \tau \quad : \quad (\Sigma \vdash \mathcal{G}(\mathcal{A}_0[\pi_0] \circledast \mathcal{B}_0[\pi'_0], \langle M_0, N_0 \rangle)) \quad \longrightarrow \quad (\Sigma \vdash \mathcal{G}(\mathcal{A}_1[\pi_1] \circledast \mathcal{B}_1[\pi'_1], \langle M_1, N_1 \rangle))$$

Hence we are done by defining $\sigma \circledast \tau$ as in Def. 10.12.

For the unit, according to Lem. 10.21 can we take $\Sigma \vdash \mathcal{G}(\mathcal{I}, \mathrm{Id}_\Sigma)$.

For the structure map, we have to give

$$\begin{array}{rclcl}
\alpha_{(\mathcal{A},M),(\mathcal{B}N),(\mathcal{C},L)} & : & (\mathcal{G}(\mathcal{A}, M) \circledast \mathcal{G}(\mathcal{B}, N)) \circledast \mathcal{G}(\mathcal{C}, L) & \longrightarrow_{\widehat{\mathbf{SAG}}_\Sigma} & \mathcal{G}(\mathcal{A}, M) \circledast (\mathcal{G}(\mathcal{B}, N) \circledast \mathcal{G}(\mathcal{C}, L))) \\
\lambda_{(\mathcal{A},M)} & : & \mathcal{G}(\mathcal{I}, \mathrm{Id}_\Sigma) \circledast \mathcal{G}(\mathcal{A}, M) & \longrightarrow_{\widehat{\mathbf{SAG}}_\Sigma} & \mathcal{G}(\mathcal{A}, M) \\
\rho_{(\mathcal{A},M)} & : & \mathcal{G}(\mathcal{A}, M) \circledast \mathcal{G}(\mathcal{I}, \mathrm{Id}_\Sigma) & \longrightarrow_{\widehat{\mathbf{SAG}}_\Sigma} & \mathcal{G}(\mathcal{A}, M) \\
\gamma_{(\mathcal{A},M),(\mathcal{B},N)} & : & \mathcal{G}(\mathcal{A}, M) \circledast \mathcal{G}(\mathcal{B}, N) & \longrightarrow_{\widehat{\mathbf{SAG}}_\Sigma} & \mathcal{G}(\mathcal{B}, N) \circledast \mathcal{G}(\mathcal{A}, M)
\end{array}$$

For

$$\begin{array}{rcl}
\alpha_{(\mathcal{A},M),(\mathcal{B}N),(\mathcal{C},L)} & : & (\mathcal{G}(\mathcal{A}, M) \circledast \mathcal{G}(\mathcal{B}, N)) \circledast \mathcal{G}(\mathcal{C}, L) \quad \longrightarrow \\
& & \qquad \mathcal{G}(\mathcal{A}, M) \circledast (\mathcal{G}(\mathcal{B}, N) \circledast \mathcal{G}(\mathcal{C}, L)))
\end{array}$$

Note that, for suitable projections π_1, π_2, π'_1 and π'_2 we have

$$\begin{array}{rl} & (\mathcal{G}(\mathcal{A}, M) \circledast \mathcal{G}(\mathcal{B}, N)) \circledast \mathcal{G}(\mathcal{C}, L) \\ = & \mathcal{G}\left((\mathcal{A}[\pi_1 \circ \pi_2] \circledast \mathcal{B}[\pi'_1 \circ \pi_2]) \circledast \mathcal{C}[\pi'_2]\ ,\ \langle\langle M, N\rangle, L\rangle\right) \end{array}$$

and for suitable projections π_3, π_3, π'_4 and π'_4 we have

$$\begin{array}{rl} & \mathcal{G}(\mathcal{A}, M) \circledast (\mathcal{G}(\mathcal{B}, N) \circledast \mathcal{G}(\mathcal{C}, L)) \\ = & \mathcal{G}\left(\mathcal{A}[\pi_4] \circledast (\mathcal{B}[\pi'_3 \circ \pi_4] \circledast \mathcal{C}[\pi'_3 \circ \pi'_4])\ ,\ \langle M, \langle N, L\rangle\rangle\right) \end{array}$$

But

$$\mathcal{G}(\mathcal{A}[\pi_1 \circ \pi_2], \langle\langle M, N\rangle, L\rangle) \quad = \quad \mathcal{G}(\mathcal{A}[\pi_4], \langle M, \langle N, L\rangle\rangle \quad = \quad \mathcal{G}(\mathcal{A}, M)$$

and similarly for $\mathcal{B}$ and $\mathcal{C}$. It follows that we actually have

$$\begin{array}{rl} & \mathcal{G}(\mathcal{A}, M) \circledast (\mathcal{G}(\mathcal{B}, N) \circledast \mathcal{G}(\mathcal{C}, L)) \\ = & \mathcal{G}\left(\mathcal{A}[\pi_1 \circ \pi_2] \circledast (\mathcal{B}[\pi'_1 \circ \pi_2] \circledast \mathcal{C}[\pi'_2])\ ,\ \langle\langle M, N\rangle, L\rangle\right) \end{array}$$

and we can take a suitable $\alpha_{(-),(-),(-)}$ as defined in (20) (Sect. 10.10). The same holds for the other structure maps, and we get:

▶ **Proposition 11.5.** *The categories* $\widehat{\mathbf{SAG}}_\Sigma^{(\mathrm{W/R})}$ *are symmetric monoidal.*

11.4 Symmetric Monoidal Fibrations of Acceptance Games and Automata

In order to get symmetric monoidal fibrations, we follow [21, Thm. 12.7], and show that for each $L \in \mathbf{Tree}[\Gamma, \Sigma]$, the functors

$$L^* \quad : \quad \widehat{\mathbf{SAG}}_\Sigma^{(\mathrm{W/R})} \quad \longrightarrow \quad \widehat{\mathbf{SAG}}_\Gamma^{(\mathrm{W/R})}$$

is strong symmetric monoidal. But according to the defintion of the structure maps in Sect. 11.3, this is provided by Prop. 11.2.

We thus get

▶ **Proposition 11.6.** *Given* $L \in \mathbf{Tree}[\Gamma, \Sigma]$*, the functors*

$$L^* \quad : \quad \widehat{\mathbf{SAG}}_\Sigma^{(\mathrm{WR})} \quad \longrightarrow \quad \widehat{\mathbf{SAG}}_\Gamma^{(\mathrm{W/R})}$$

are strict symmetric monoidal.

In particular, we have a split indexed symmetric monoidal category

$$(-)^* : \mathbf{Tree}^{\mathrm{op}} \to \mathbf{SymMonCat}$$

The case of

$$\widehat{\mathsf{aut}}^{(\mathrm{W/R})} \quad : \quad \widehat{\mathbf{Aut}}^{(\mathrm{W/R})} \quad \longrightarrow \quad \mathbf{Alph}$$

is simpler, adn we get

▶ **Proposition 11.7.** *Given* $\beta \in \mathbf{Alph}[\Gamma, \Sigma]$*, the functors*

$$\beta^* \quad : \quad \widehat{\mathbf{Aut}}_\Sigma^{(\mathrm{WR})} \quad \longrightarrow \quad \widehat{\mathbf{Aut}}_\Gamma^{(\mathrm{W/R})}$$

are strict symmetric monoidal.

In particular, we have a split indexed symmetric monoidal category

$$(-)^* : \mathbf{Alph}^{\mathrm{op}} \to \mathbf{SymMonCat}$$

12 Correctness w.r.t. Language Operations

This Section gathers the proofs of Sect. 6.1.

Sections 12.1 and 12.2 assert the correctness of the synchronous arrow $\_ \multimap_{\circledast} \_$ w.r.t. language inclusion. The main point is the proof of Prop. 6.1. Proposition 6.2 is then an immediate corollary of substitution (Prop. 4.3).

12.1 Correspondence with Acceptance Games

Consider $\Sigma \vdash \mathcal{A}$ and $t \in \mathbf{Tree}[\Sigma]$. In this Section, we describe a bijection between total winning strategies $\mathbf{1} \vdash \mathcal{G}(\mathcal{A}, \dot{t})$ and total winning strategies on $\mathbf{1} \vdash \mathcal{G}(\mathcal{I}, \dot{t}) \multimap_{\circledast} \mathcal{G}(\mathcal{A}, \dot{t})$, thus proving Prop 6.1.

The main point is that $\mathcal{G}(\mathcal{I}, \dot{t})$ provides a monoidal unit in the fibered sense. According to Sect. 10.1, in $\mathbf{Rel}(\mathbf{Set}/\mathrm{Tr}_{\mathbf{1}})$ this is provided by Prop. 10.22, that is

$$\mathrm{tr} \quad : \quad \wp_{\mathbf{1}}(\mathcal{I}, \dot{t}) \quad \xrightarrow{\simeq} \quad \mathrm{Tr}_{\mathbf{1}}$$

12.1.1 Monoidal Lifting of Strategies

We define a map taking a total[5] $\mathbf{1} \vdash \tau : \mathcal{G}(\mathcal{A}, \dot{t})$ to a total

$$\mathbf{1} \vdash \mathcal{I} \multimap_{\circledast} \tau : \mathcal{G}(\mathcal{I}, \dot{t}) \multimap_{\circledast} \mathcal{G}(\mathcal{A}, \dot{t})$$

The map is defined by induction on plays. By construction we will have

$$\downarrow((\mathcal{I} \multimap_{\circledast} \tau) \restriction \mathcal{G}(\mathcal{A}, \dot{t})) \quad = \quad \downarrow\tau \tag{21}$$

where $\downarrow$ denotes prefix-closure.
For the base case, let $((\varepsilon, q^{\imath}_{\mathcal{I}}),\ (\varepsilon, q^{\imath}_{\mathcal{A}})) \in \mathcal{I} \multimap_{\circledast} \tau$ and by definition of a strategy, we have

$$((\varepsilon, q^{\imath}_{\mathcal{I}}),\ (\varepsilon, q^{\imath}_{\mathcal{A}})) \restriction \mathcal{G}(\mathcal{A}, \dot{t}) \quad = \quad (\varepsilon, q^{\imath}_{\mathcal{A}}) \quad \in \quad \tau$$

For the induction step, assume given

$$s \quad := \quad ((\varepsilon, q^{\imath}_{\mathcal{I}}),\ (\varepsilon, q^{\imath}_{\mathcal{A}})) \quad \to^* \quad ((p, q^{\imath}_{\mathcal{I}}),\ (p, q_{\mathcal{A}})) \quad \in \quad \mathcal{I} \multimap_{\circledast} \tau$$

with

$$s \restriction \mathcal{G}(\mathcal{A}, \dot{t}) \quad = \quad (\varepsilon, q^{\imath}_{\mathcal{A}}) \quad \to^* \quad (p, q_{\mathcal{A}}) \quad \in \quad \downarrow\tau$$

It is O's turns to play in $\mathcal{G}(\mathcal{I}, \dot{t}) \multimap_{\circledast} \mathcal{G}(\mathcal{A}, \dot{t})$ and P's turns to play in $\mathcal{G}(\mathcal{A}, \dot{t})$.

Since τ is total by assumption, it makes a P-move in $\mathcal{G}(\mathcal{A}, \dot{t})$, say:

$$* \quad \to^* \quad (p, q_{\mathcal{A}}) \quad \xrightarrow{\mathsf{P}} \quad (p, \bullet, \gamma_{\mathcal{A}}) \quad \in \quad \tau$$

On the other hand, the only possible O-move in $\mathcal{G}(\mathcal{I}, \dot{t}) \multimap_{\circledast} \mathcal{G}(\mathcal{A}, \dot{t})$ is

$$* \quad \to^* \quad ((p, q^{\imath}_{\mathcal{I}}),\ (p, q_{\mathcal{A}})) \quad \xrightarrow{\mathsf{O}} \quad ((p, \bullet, \gamma_{\mathcal{I}}),\ (p, q_{\mathcal{A}}))$$

(where $\gamma_{\mathcal{I}} := \{(q^{\imath}_{\mathcal{I}}, d) \mid d \in D\}$), after which we make $\mathcal{I} \multimap_{\circledast} \tau$ copy τ's move, that is we let

$$* \quad \to^* \quad ((p, \bullet, \gamma_{\mathcal{I}}),\ (p, q_{\mathcal{A}})) \quad \xrightarrow{\mathsf{P}} \quad ((p, \bullet, \gamma_{\mathcal{I}}),\ (p, \bullet, \gamma_{\mathcal{A}})) \quad \in \quad \mathcal{I} \multimap_{\circledast} \tau$$

[5] Totality is not required, but makes the presentation simpler.

Note that this P-move of $\mathcal{I} \multimap\!\!\circledast\, \tau$ is completely determined by τ's reaction to the projection of the corresponding play. only depends on
Now, consider an O-move in $\mathcal{G}(\mathcal{I}, \dot{t}) \multimap\!\!\circledast\, \mathcal{G}(\mathcal{A}, \dot{t})$, say

$$* \quad \to^* \quad ((p, \bullet, \gamma_{\mathcal{I}}),\ (p, \bullet, \gamma_{\mathcal{A}})) \quad \overset{\mathsf{O}}{\to} \quad ((p, \bullet, \gamma_{\mathcal{I}}),\ (p \cdot d, q'_{\mathcal{A}}))$$

We make $\mathcal{I} \multimap\!\!\circledast\, \tau$ play the only possible P-move, that is:

$$* \quad \to^* \quad ((p, \bullet, \gamma_{\mathcal{I}}),\ (p \cdot d, q'_{\mathcal{A}})) \quad \overset{\mathsf{P}}{\to} \quad ((p \cdot d, q^{\imath}_{\mathcal{I}}),\ (p \cdot d, q'_{\mathcal{A}})) \quad \in \quad \mathcal{I} \multimap\!\!\circledast\, \tau$$

and we get (21) since by assumption

$$[* \to^* ((p \cdot d, q^{\imath}_{\mathcal{I}}),\ (p \cdot d, q'_{\mathcal{A}}))] \restriction \mathcal{G}(\mathcal{A}, \dot{t}) \quad = \quad * \to^* (p \cdot d, q'_{\mathcal{A}}) \quad \in \quad {\downarrow}\tau$$

This completes the definition of $\mathcal{I} \multimap\!\!\circledast\, \tau$. Note that $\mathcal{I} \multimap\!\!\circledast\, \tau$ is total and P-deterministic.

▶ **Proposition 12.1.** *If $\mathbf{1} \vdash \tau \Vdash \mathcal{G}(\mathcal{A}, \dot{t})$ then $\mathcal{I} \multimap\!\!\circledast\, \tau$ is a morphism of $\mathbf{SAG}_{\mathbf{1}}^{\mathrm{W/R}}$.*

Proof. Given any infinite play π of $\mathcal{I} \multimap\!\!\circledast\, \tau$, it follows from (21) that $\pi \restriction \mathcal{G}(\mathcal{A}, \dot{t})$ is an infinite play of τ, hence $\pi \restriction \mathcal{G}(\mathcal{A}, \dot{t}) \in \mathcal{W}_{\mathcal{G}(\mathcal{A}, \dot{t})}$, so that $\mathcal{I} \multimap\!\!\circledast\, \tau$ is winning w.r.t. $\_ \multimap\!\!\circledast\, \_$.

Moreover, by definition of $\Omega_{\mathcal{I}}$, we have

$$\pi \restriction \mathcal{G}(\mathcal{I}, \dot{t}) \in \mathcal{W}_{\pi \restriction \mathcal{G}(\mathcal{I}, \dot{t})}$$

and it follows that $\mathcal{I} \multimap\!\!\circledast\, \tau$ is also winning w.r.t. $\_ \circledast\!\!\multimap\!\!\circledast\, \_$. ◀

12.1.2 Completeness of the Monoidal Lifting.

▶ **Lemma 12.2.** *The map $\mathcal{I} \multimap\!\!\circledast\, (-)$ is injective.*

Proof. If $\mathcal{I} \multimap\!\!\circledast\, \theta = \mathcal{I} \multimap\!\!\circledast\, \tau$, we have

$${\downarrow}((\mathcal{I} \multimap\!\!\circledast\, \theta) \restriction \mathcal{G}(\mathcal{A}, \dot{t})) \quad = \quad {\downarrow}((\mathcal{I} \multimap\!\!\circledast\, \tau) \restriction \mathcal{G}(\mathcal{A}, \dot{t}))$$

hence $\theta = \tau$ by definition of $\mathcal{I} \multimap\!\!\circledast\, (-)$. ◀

We now define an inverse to $\mathcal{I} \multimap\!\!\circledast\, (-)$.

▶ **Lemma 12.3.** *Given a strategy*

$$\mathbf{1} \vdash \sigma : \mathcal{G}(\mathcal{I}, \dot{t}) \multimap\!\!\circledast\, \mathcal{G}(\mathcal{A}, \dot{t})$$

the set of plays

$$\sigma \restriction \mathcal{G}(\mathcal{A}, \dot{t})^{\mathsf{P}} \quad := \quad \{s \in \wp^{\mathsf{P}}(\mathcal{A}, \dot{t}) \mid s \in \sigma \restriction \mathcal{G}(\mathcal{A}, \dot{t})\} \quad = \quad \sigma \restriction \mathcal{G}(\mathcal{A}, \dot{t}) \cap \wp^{\mathsf{P}}(\mathcal{A}, \dot{t})$$

is a strategy on $\mathbf{1} \vdash \mathcal{G}(\mathcal{A}, \dot{t})$.

Proof. By definition, $\sigma \restriction \mathcal{G}(\mathcal{A}, \dot{t})^{\mathsf{P}}$ is a set of P-plays. Moreover, closure under P-prefix follows from the closure under P-prefix of σ.

It remains to show that $\sigma \restriction \mathcal{G}(\mathcal{A}, \dot{t})^{\mathsf{P}}$ is P-deterministic. This crucially rely on Prop. 10.22. Consider two plays of $\sigma \restriction \mathcal{G}(\mathcal{A}, \dot{t})^{\mathsf{P}}$

$$s \to (p, \bullet, \gamma_{\mathcal{A}}) \qquad \text{and} \qquad s \to (p, \bullet, \gamma'_{\mathcal{A}})$$

and let $u, v \in \sigma$ with

$$u{\restriction}\mathcal{G}(\mathcal{A}, \dot{t}) \quad = \quad s \to (p, \bullet, \gamma_{\mathcal{A}}) \qquad \text{and} \qquad v{\restriction}\mathcal{G}(\mathcal{A}, \dot{t}) \quad = \quad s \to (p, \bullet, \gamma'_{\mathcal{A}})$$

Then, since σ is synchronous, we must have

$$\mathrm{tr}(u{\restriction}\mathcal{G}(\mathcal{I}, \dot{t})) \quad = \quad \mathrm{tr}(s \to (p, \bullet, \gamma_{\mathcal{A}})) \quad = \quad \mathrm{tr}(v{\restriction}\mathcal{G}(\mathcal{I}, \dot{t}))$$

But by Prop. 10.22 this implies

$$u{\restriction}\mathcal{G}(\mathcal{I}, \dot{t}) \quad = \quad v{\restriction}\mathcal{G}(\mathcal{I}, \dot{t})$$

We thus have $\mathrm{HS}(u) = \mathrm{HS}(v)$, which implies $u = v$ by Cor. 6.7 (the more general Lem. i.(i) would also have done the job since $u, v \in \sigma$). ◀

For the preservation of totality we strongly rely on the completeness of $\mathcal{I}$.

▶ **Lemma 12.4.** *If*

$$\sigma{\restriction}\mathcal{G}(\mathcal{A}, \dot{t})^{\mathsf{P}} \quad = \quad \theta{\restriction}\mathcal{G}(\mathcal{A}, \dot{t})^{\mathsf{P}}$$

then $\sigma = \theta$.

Proof. Reasoning as in the proof of Lem. 12.3, by Prop. 10.22 and Cor. 6.7, for every $s \in \wp^{\mathsf{P}}_{\mathbf{1}}(\mathcal{A}, \dot{t})$, there is a unique

$$u \in \wp^{\mathsf{P}}_{\mathbf{1}}(\mathcal{G}(\mathcal{I}, \dot{t}) \multimap\!\!\circledast\, \mathcal{G}(\mathcal{A}, \dot{t}))$$

such that $u = \mathrm{HS}(v, s)$ for some v. ◀

▶ **Lemma 12.5.** $\sigma{\restriction}\mathcal{G}(\mathcal{A}, \dot{t})^{\mathsf{P}}$ *is total if* σ *is total.*

Proof. Write $\tau := \sigma{\restriction}\mathcal{G}(\mathcal{A}, \dot{t})^{\mathsf{P}}$. Let s be an O-interrogation of τ. There are two cases:

Case of $s = (\varepsilon, q^{\imath}_{\mathcal{A}})$. We have

$$((\varepsilon, q^{\imath}_{\mathcal{I}}), (\varepsilon, q^{\imath}_{\mathcal{A}})) \in \sigma$$

Since σ is total, it must answer the O-move given by $\gamma_{\mathcal{I}} \in \delta_{\mathcal{I}}(q_{\mathcal{I}}, \dot{t}(\varepsilon)(\bullet))$. Hence, for some $\gamma_{\mathcal{A}} \in \delta_{\mathcal{A}}(q_{\mathcal{A}}, \dot{t}(\varepsilon)(\bullet))$ we have

$$((\varepsilon, q^{\imath}_{\mathcal{I}}), (\varepsilon, q^{\imath}_{\mathcal{A}})) \xrightarrow{\mathsf{O}} ((\varepsilon, \bullet, \gamma_{\mathcal{I}}), (\varepsilon, q^{\imath}_{\mathcal{A}})) \xrightarrow{\mathsf{P}} ((\varepsilon, \bullet, \gamma_{\mathcal{I}}), (\varepsilon, \bullet, \gamma_{\mathcal{A}})) \in \sigma$$

and it follows that

$$s \xrightarrow{\mathsf{P}} (\varepsilon, \bullet, \gamma_{\mathcal{A}}) \in \tau$$

Case of $s = t \xrightarrow{\mathsf{O}} (p \cdot d, q_{\mathcal{A}})$. We have by assumption $t \in \tau$. Let v be a play such that $u := (v, t) \in \mathrm{HS}(\sigma)$.

By construction of $\mathcal{I}$, the play

$$u \xrightarrow{\mathsf{O}} ((p, \bullet, \gamma_{\mathcal{I}}), (p \cdot d, q_{\mathcal{A}})$$

is an O-interrogation of σ, and by totality of σ, we have

$$u \xrightarrow{\mathsf{O}} ((p, \bullet, \gamma_{\mathcal{I}}), (p \cdot d, q_{\mathcal{A}}) \xrightarrow{\mathsf{P}} ((p \cdot d, q^{\imath}_{\mathcal{I}}), (p \cdot d, q_{\mathcal{A}})) \in \sigma$$

Again by construction of $\mathcal{I}$, the play

$$u \xrightarrow{\mathsf{O}} ((p, \bullet, \gamma_{\mathcal{I}}), (p \cdot d, q_{\mathcal{A}}) \xrightarrow{\mathsf{P}} ((p \cdot d, q^{\imath}_{\mathcal{I}}), (p \cdot d, q_{\mathcal{A}})) \xrightarrow{\mathsf{O}} ((p \cdot d, \bullet, \gamma_{\mathcal{I}}), (p \cdot d, q_{\mathcal{A}}))$$

is an O-interrogation of σ, and by totality of σ again, for some $\gamma_{\mathcal{A}}$ we have

$$u \xrightarrow{\mathsf{O}} ((p, \bullet, \gamma_{\mathcal{I}}), (p \cdot d, q_{\mathcal{A}}) \xrightarrow{\mathsf{P}} ((p \cdot d, q^{\imath}_{\mathcal{I}}), (p \cdot d, q_{\mathcal{A}}))$$
$$\xrightarrow{\mathsf{O}} ((p \cdot d, \bullet, \gamma_{\mathcal{I}}), (p \cdot d, q_{\mathcal{A}})) \xrightarrow{\mathsf{P}} ((p \cdot d, \bullet, \gamma_{\mathcal{I}}), (p \cdot d, \bullet, \gamma_{\mathcal{A}})) \in \sigma$$

It follows that

$$t \xrightarrow{\mathsf{O}} (p \cdot d, q_{\mathcal{A}}) \xrightarrow{\mathsf{P}} (p \cdot d, \bullet, \gamma_{\mathcal{A}}) \in \tau$$

◀

▶ **Proposition 12.6.** *Let*

$$\mathbf{1} \vdash \sigma : \mathcal{G}(\mathcal{I}, \dot{t}) \multimap_{\circledast} \mathcal{G}(\mathcal{A}, \dot{t})$$

If σ is a morphism of $\mathbf{SAG}^{\mathrm{W/R}}$*, then*

$$\mathbf{1} \vdash \sigma \restriction \mathcal{G}(\mathcal{A}, \dot{t})^{\mathsf{P}} \Vdash \mathcal{G}(\mathcal{A}, \dot{t})$$

Proof. The totality part follows from Lem. 12.5.

As for winning note that any infinite play of $\sigma \restriction \mathcal{G}(\mathcal{A}, \dot{t})^{\mathsf{P}}$ is the projection on $\mathcal{G}(\mathcal{A}, \dot{t})$ of an infinite play π of σ. But by definition of $\Omega_{\mathcal{I}}$, for such a π we have

$$\pi \restriction \mathcal{G}(\mathcal{I}, \dot{t}) \in \mathcal{W}_{\pi \restriction \mathcal{G}(\mathcal{I}, \dot{t})}$$

It follows that π is winning for P w.r.t. $\_ \multimap_{\circledast} \_$ iff π is winning for P w.r.t. $\_ \circledast\!\!-\!\!\circledast \_$, iff its projection on $\mathcal{G}(\mathcal{A}, \dot{t})$ is winning for P. ◀

12.1.3 Correspondence with Acceptance Games

We now prove Prop. 6.1:

▶ **Proposition 12.7** (Prop. 6.1)**.** *Given $\Sigma \vdash \mathcal{A}$ and $t \in$ **Tree**$[\Sigma]$, the map $\sigma \mapsto \mathcal{I} \multimap_{\circledast} \sigma$ gives bijections*

$$\begin{array}{rcl} \{\sigma \mid \sigma \Vdash \mathcal{G}(\mathcal{A}, \dot{t})\} & \simeq & \{\theta \mid \mathbf{1} \vdash \theta \Vdash \mathcal{G}(\mathcal{I}, \dot{t}) \multimap_{\circledast} \mathcal{G}(\mathcal{A}, \dot{t})\} \\ & \simeq & \{\theta \mid \mathbf{1} \vdash \theta \Vdash \mathcal{G}(\mathcal{I}, \dot{t}) \circledast\!\!-\!\!\circledast \mathcal{G}(\mathcal{A}, \dot{t})\} \end{array}$$

Proof. We first prove that $\mathcal{I} \multimap_{\circledast} -$ is a bijection. According to Lem. 12.2 it remains to show that $\mathcal{I} \multimap_{\circledast} -$ is surjective. Consider

$$\mathbf{1} \vdash \sigma : \mathcal{G}(\mathcal{I}, \dot{t}) \multimap_{\circledast} \mathcal{G}(\mathcal{A}, \dot{t})$$

and

$$\mathcal{I} \multimap_{\circledast} (\sigma \restriction \mathcal{G}(\mathcal{A}, \dot{t})^{\mathsf{P}})$$

Then by (21) (construction of $\mathcal{I} \multimap_{\circledast} -$), we have

$$\downarrow [\mathcal{I} \multimap_{\circledast} (\sigma \restriction \mathcal{G}(\mathcal{A}, \dot{t})^{\mathsf{P}})] \restriction \mathcal{G}(\mathcal{A}, \dot{t}) \quad = \quad \downarrow (\sigma \restriction \mathcal{G}(\mathcal{A}, \dot{t})^{\mathsf{P}})$$

and in particular

$$[\mathcal{I} \multimap_{\circledast} (\sigma \restriction \mathcal{G}(\mathcal{A}, \dot{t})^{\mathsf{P}})] \restriction \mathcal{G}(\mathcal{A}, \dot{t})^{\mathsf{P}} \quad = \quad \sigma \restriction \mathcal{G}(\mathcal{A}, \dot{t})^{\mathsf{P}}$$

But by Lem. 12.4 this implies

$$\mathcal{I} \multimap_{\circledast} (\sigma \restriction \mathcal{G}(\mathcal{A}, \dot{t})^{\mathsf{P}}) \quad = \quad \sigma$$

The part concerning winning (and totality) follows from Prop. 12.1 and Prop. 12.6. ◀

12.2 Correctness of the Synchronous Arrow w.r.t. Language Inclusion

We can now check that the arrow $\mathcal{G}(\mathcal{A}, M) \multimap_{\circledast} \mathcal{G}(\mathcal{B}, N)$ is correct w.r.t. language inclusion: if

$$\Sigma \Vdash \mathcal{G}(\mathcal{A}, M) \multimap_{\circledast} \mathcal{G}(\mathcal{B}, N)$$

then

$$\forall t \in \mathbf{Tree}[\Sigma], \quad M(t) \in \mathcal{L}(\mathcal{A}) \quad \Longrightarrow \quad N(t) \in \mathcal{L}(\mathcal{B})$$

▶ **Proposition 12.8** (Correctness of the Arrow)**.** *Assume given* $\Sigma \vdash \sigma \Vdash \mathcal{G}(\mathcal{A}, M) \multimap_{\circledast} \mathcal{G}(\mathcal{B}, N)$.

(i) For all $t \in \mathbf{Tree}[\Sigma]$, *we have* $t^*(\sigma) \Vdash \mathcal{G}(\mathcal{A}, M(t)) \multimap_{\circledast} \mathcal{G}(\mathcal{B}, N(t))$.
(ii) If $\theta \Vdash \mathcal{G}(\mathcal{A}, M(t))$ *then* $\sigma \circ (\mathcal{I} \multimap_{\circledast} \theta) \Vdash \mathcal{G}(\mathcal{B}, N(t))$.
(iii) For all tree $t \in \mathbf{Tree}[\Sigma]$, *if* $M(t) \in \mathcal{L}(\mathcal{A})$ *then* $N(t) \in \mathcal{L}(\mathcal{B})$.

Proof. (i) Since $t^* : \mathbf{SAG}^{\mathrm{W}}_{\Sigma} \to \mathbf{SAG}^{\mathrm{W}}_{\mathbf{1}}$ Prop. 8.2.(ii) (Prop. 4.3).
(ii) By (i) and Prop. 12.7 (actually Prop. 12.1).
(iii) By (ii), by definition of $\mathcal{L}(-)$ and Prop. 12.7 (actually Prop. 12.6). ◀

12.3 Correctness of the Synchronous Product

We now check that the synchronous product $\_ \circledast \_$ implements the intersection on the languages recognized by automata. Consider *complete* automata $\Sigma \vdash \mathcal{A}$ and $\Sigma \vdash \mathcal{B}$.

We first show that

$$\mathcal{L}(\mathcal{A}) \cap \mathcal{L}(\mathcal{B}) \quad \subseteq \quad \mathcal{L}(\mathcal{A} \circledast \mathcal{B})$$

Let $t \in \mathcal{L}(\mathcal{A}) \cap \mathcal{L}(\mathcal{B})$. By definition, there are strategies

$$\mathbf{1} \vdash \tau_{\mathcal{A}} \Vdash \mathcal{G}(\mathcal{A}, \dot{t}) \qquad \text{and} \qquad \mathbf{1} \vdash \tau_{\mathcal{B}} \Vdash \mathcal{G}(\mathcal{B}, \dot{t})$$

By Prop. 12.7 (a.k.a. Prop. 6.1), we get

$$\begin{array}{l} \mathbf{1} \vdash \mathcal{I} \multimap_{\circledast} \tau_{\mathcal{A}} \Vdash \mathcal{G}(\mathcal{I}, \dot{t}) \multimap_{\circledast} \mathcal{G}(\mathcal{A}, \dot{t}) \\ \mathbf{1} \vdash \mathcal{I} \multimap_{\circledast} \tau_{\mathcal{A}} \Vdash \mathcal{G}(\mathcal{I}, \dot{t}) \multimap_{\circledast} \mathcal{G}(\mathcal{B}, \dot{t}) \end{array}$$

and since $\mathcal{I}$, $\mathcal{A}$ and $\mathcal{B}$ are *complete*, by Prop. 10.13 and Prop. 10.23 we obtain

$$\mathbf{1} \Vdash \mathcal{G}(\mathcal{I}, \dot{t}) \multimap_{\circledast} \mathcal{G}(\mathcal{A} \circledast \mathcal{B}, \dot{t})$$

and thus, by Prop. 12.7 (Prop. 6.1) again, $t \in \mathcal{L}(\mathcal{A} \circledast \mathcal{B})$.

The converse direction is a bit more technical. We have to go from

$$\mathbf{1} \vdash \tau \Vdash \mathcal{G}(\mathcal{A} \circledast \mathcal{B}, \dot{t})$$

to

$$\mathbf{1} \vdash \tau_{\mathcal{A}} \Vdash \mathcal{G}(\mathcal{A}, \dot{t}) \qquad \mathbf{1} \vdash \tau_{\mathcal{B}} \Vdash \mathcal{G}(\mathcal{B}, \dot{t})$$

We only discuss the case of $\tau_{\mathcal{A}}$. The point is that the direct projection of τ (using the projections of Sect. 10.4) need not be a strategy. It will be total, winning and P-prefix-closed, by P-determinism may fail: since $\mathcal{A}$ and $\mathcal{B}$ need not be non-deterministic, given a

P-play

$$s \quad : \quad * \quad \to^* \quad (p, \bullet, \gamma_{\mathcal{A}} \circledast \gamma_{\mathcal{B}}) \quad \in \quad \wp^{\mathsf{P}}(\mathcal{A} \circledast \mathcal{B}, \dot{t})$$

and an O-move in $\mathcal{G}(\mathcal{A}, \dot{t})$:

$$(p, \bullet, \gamma_{\mathcal{A}}) \quad \stackrel{\mathsf{O}}{\to} \quad (p \cdot d, q_{\mathcal{A}})$$

there might be several $q_{\mathcal{B}} \in Q_{\mathcal{B}}$ such that $(q_{\mathcal{B}}, d) \in \gamma_{\mathcal{A}}$, *i.e.* such that, in $\mathcal{G}(\mathcal{A} \circledast \mathcal{B}, \dot{t})$, the play s can be extended as

$$s \quad \stackrel{\mathsf{O}}{\to} \quad (p \cdot d, (q_{\mathcal{A}}, q_{\mathcal{B}}))$$

In other words, a choice has to be made, for all $(q_{\mathcal{A}}, d) \in Q_{\mathcal{A}} \times D$ and all $\gamma_{\mathcal{B}}$ in the codomain of $\delta_{\mathcal{B}}$, of some $(q_{\mathcal{B}}, d) \in \gamma_{\mathcal{B}}$. Note that this choice is always possible since $\mathcal{B}$ is assumed to be complete. We assume that this choice is made by a map

$$\ell_{\mathcal{A}} \quad : \quad (Q_{\mathcal{A}} \times D) \times \mathcal{P}(Q_{\mathcal{B}} \times D) \quad \longrightarrow \quad Q_{\mathcal{B}}$$

Now equipped with $\ell_{\mathcal{A}}$, given a total

$$\mathbf{1} \vdash \tau : \mathcal{G}(\mathcal{A} \circledast \mathcal{B}, \dot{t})$$

we build

$$\mathbf{1} \Vdash \tau_{\mathcal{A}} : \mathcal{G}(\mathcal{A}, \dot{t})$$

The strategy $\tau_{\mathcal{A}}$ is defined by induction on plays. It is defined together with a map $u \mapsto s_u$ where

$$u \in \tau_{\mathcal{A}} \subseteq \wp^{\mathsf{P}}(\mathcal{A}, \dot{t})$$

and

$$s_u \in \tau \subseteq \wp^{\mathsf{P}}(\mathcal{A} \circledast \mathcal{B}, \dot{t})$$

are such that

$$u \quad = \quad \varpi_1(s_u)$$

and moreover, if v extends u, then s_v extends s_u.

For the base case, we let

$$u \quad := \quad (\varepsilon, q^{\imath}_{\mathcal{A}}) \in \tau_{\mathcal{A}}$$

and

$$s_u \quad := \quad (\varepsilon, (q^{\imath}_{\mathcal{A}}, q^{\imath}_{\mathcal{B}}))$$

For the first step, since τ is total, for some $\gamma_{\mathcal{A}}$, $\gamma_{\mathcal{B}}$ we have

$$s \quad := \quad (\varepsilon, (q^{\imath}_{\mathcal{A}}, q^{\imath}_{\mathcal{B}})) \quad \stackrel{\mathsf{P}}{\to} \quad (\varepsilon, \bullet, \gamma_{\mathcal{A}} \circledast \gamma_{\mathcal{B}}) \quad \in \quad \tau$$

We put

$$u \quad := \quad (\varepsilon, q^{\imath}_{\mathcal{A}}) \quad \stackrel{\mathsf{P}}{\to} \quad (\varepsilon, \bullet, \gamma_{\mathcal{A}}) \quad \in \quad \tau_{\mathcal{A}}$$

Since s is unique (by P-determinism of τ), we let

$$u_s \quad := \quad s$$

For the induction step, by induction hypothesis we have $u \in \tau_{\mathcal{A}}$, of the form

$$u \quad : \quad * \quad \to^* \quad (p, \bullet, \gamma_{\mathcal{A}})$$

and $s_u \in \tau$, with $\varpi_1(s_u) = u$, hence of the form:

$$s_u \quad : \quad * \quad \to^* \quad (p, \bullet, \gamma_{\mathcal{A}} \circledast \gamma_{\mathcal{B}})$$

Consider now some O-move extending u, say

$$u \quad \overset{\mathsf{O}}{\to} \quad (p \cdot d, q_{\mathcal{A}})$$

Since $\mathcal{B}$ is complete we have $(q_{\mathcal{B}}, d) \in \gamma_{\mathcal{B}}$ for

$$q_{\mathcal{B}} \quad := \quad \ell_{\mathcal{A}}((q_{\mathcal{A}}, d), \gamma_{\mathcal{B}})$$

We then extend s_u with the corresponding O-move, and let τ answer (by completeness of $\mathcal{A}$ and totality of τ), say

$$s' \quad := \quad s \quad \overset{\mathsf{O}}{\to} \quad (p \cdot d, (q_{\mathcal{A}, q_{\mathcal{B}}})) \quad \overset{\mathsf{P}}{\to} \quad (p \cdot d, \bullet, \gamma_{\mathcal{A}} \circledast \gamma_{\mathcal{B}}) \quad \in \quad \tau$$

We then put

$$u' \quad := \quad u \quad \overset{\mathsf{O}}{\to} \quad (p \cdot d, q_{\mathcal{A}}) \quad \overset{\mathsf{P}}{\to} \quad (p \cdot d, \bullet, \gamma_{\mathcal{A}}) \quad \in \quad \tau_{\mathcal{A}}$$

Thanks to $\ell_{\mathcal{A}}$, the play s' is uniquely determined from u' and s_u, and moreover extends s_u. We let

$$s'_{u'} \quad := \quad s'$$

This completes the construction of $\tau_{\mathcal{A}}$. Note that $\tau_{\mathcal{A}}$ is total and P-prefix-closed by construction. As for P-determinism given P-plays:

$$\begin{array}{rcll} u & : & v \overset{\mathsf{O}}{\to} (p, q_{\mathcal{A}}) \overset{\mathsf{P}}{\to} (p, \bullet, \gamma_{\mathcal{A}}) & \in \tau_{\mathcal{A}} \\ u' & : & v \overset{\mathsf{O}}{\to} (p, q_{\mathcal{A}}) \overset{\mathsf{P}}{\to} (p, \bullet, \gamma'_{\mathcal{A}}) & \in \tau_{\mathcal{A}} \end{array}$$

by construction we have $\gamma_{\mathcal{A}} = \gamma'_{\mathcal{A}}$.

As for winning, since the map $u \mapsto s_u$ respects the prefix order, infinite plays of $\tau_{\mathcal{A}}$ are projections of infinite plays of τ. Hence $\tau_{\mathcal{A}}$ is winning as soon as τ is winning. We thus have shown:

▶ **Proposition 12.9.** *If $\mathcal{A}$ and $\mathcal{B}$ are* complete *automata, then $\mathcal{L}(\mathcal{A} \circledast \mathcal{B}) = \mathcal{L}(\mathcal{A}) \cap \mathcal{L}(\mathcal{B})$.*

13 Complementation of Alternating Automata

This Section gathers the proofs of Sect. 6.2.

13.1 The Operation of Complementation

Given sets $S, S' \subseteq \mathcal{P}(P)$, write

$$S \perp\!\!\!\perp S' \qquad \text{whenever} \quad \forall a \in S,\ \forall a' \in S',\ a \cap a' \neq \emptyset$$

and let

$$S^{\perp\!\!\!\perp} \quad := \quad \{a' \in \mathcal{P}(P) \mid \forall a \in S,\ a \perp\!\!\!\perp a'\}$$

Given an automaton $\Sigma \vdash \mathcal{A}$ with

$$\mathcal{A} \quad = \quad (Q, q^\imath, \delta, \Omega)$$

define $\Sigma \vdash \sim\!\mathcal{A}$ as

$$\sim\!\mathcal{A} \quad := \quad (Q, q^\imath, \delta_{\sim\mathcal{A}}, \Omega_{\sim\mathcal{A}})$$

where

$$\delta_{\sim\mathcal{A}}(q, a) \quad := \quad \delta(q, a)^{\perp\!\!\!\perp}$$

and

$$\Omega_{\sim\mathcal{A}} \quad := \quad Q^\omega \setminus \Omega$$

Note that if $\mathcal{A}$ is complete, then $\sim\!\mathcal{A}$ is not necesserarily complete. However, $\delta_{\sim\mathcal{A}}$ is always not empty, and so are the γ's in its codomain (given $q \in Q$ and $a \in \Sigma$, $\delta(q, a)$ is not empty, and moreover if $\gamma \in \delta(q, a)$, then γ is not empty as well).

Thanks to Borel determinacy [14], we have:

▶ **Proposition 13.1** ([23]). *Given $\Sigma \vdash \mathcal{A}$ with $\Omega_{\mathcal{A}}$ a Borel set, we have $\mathcal{L}(\sim\!\mathcal{A}) = \mathbf{Tree}[\Sigma] \setminus \mathcal{L}(\mathcal{A})$.*

13.2 The Falsity Automaton $\curlywedge$

We define $\Sigma \vdash \curlywedge$ as

$$\curlywedge \quad := \quad (\{q_\curlywedge\}, q_\curlywedge, \delta_\curlywedge, \Omega_\curlywedge)$$

where:

$$\delta_\curlywedge(q_\curlywedge, a) \quad := \quad \{\{(q_\curlywedge, d)\} \mid d \in D\}$$

and

$$\Omega_\curlywedge \quad := \quad \emptyset$$

Note that $\mathcal{I} = \sim\!\curlywedge$.

13.3 Dialogue Properties

▶ **Proposition 13.2.** *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be* complete *automata on* Σ. *Then*

(i) $\Sigma \Vdash \mathcal{A} \multimap_{\circledast} \widehat{\sim\mathcal{B}}$ *implies* $\Sigma \Vdash \mathcal{A} \circledast \mathcal{B} \multimap_{\circledast} \widehat{\perp}$.
(ii) $\Sigma \Vdash \mathcal{A} \circledast \mathcal{B} \multimap_{\circledast} \widehat{\perp}$ *implies* $\Sigma \Vdash \mathcal{A} \multimap_{\circledast} \widehat{\sim\mathcal{B}}$.

▶ **Corollary 13.3.** *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be* complete *automata on* Σ.

(i) $\Sigma \Vdash \mathcal{A} \multimap_{\circledast} \widehat{\sim\mathcal{B}}$ *iff* $\Sigma \Vdash \mathcal{B} \multimap_{\circledast} \widehat{\sim\mathcal{A}}$.
(ii) $\Sigma \Vdash \mathcal{A} \multimap_{\circledast} \widehat{\sim\widehat{\sim\mathcal{A}}}$.
(iii) *if* $\Sigma \Vdash \mathcal{A} \multimap_{\circledast} \mathcal{B}$ *then* $\Sigma \Vdash \widehat{\sim\mathcal{B}} \multimap_{\circledast} \widehat{\sim\mathcal{A}}$.

(i) Use Prop. 13.2.(i) twice together with the monoidal braiding γ (Lem. 10.23).
(ii) Use (i) twice.
(iii) Apply (i) to $\Sigma \Vdash \mathcal{A} \multimap_{\circledast} \widehat{\sim\widehat{\sim\mathcal{A}}}$ obtained by composition and (ii).

▶ **Corollary 13.4.** *Given a* complete *automaton* $\mathbf{1} \vdash \mathcal{A}$,

$$\mathbf{1} \Vdash {\sim}\mathcal{A} \qquad \Longleftrightarrow \qquad \mathbf{1} \Vdash \mathcal{A} \multimap_{\circledast} \perp$$

Proof. First, by Prop. 10.7 we have

$$\mathbf{1} \Vdash {\sim}\mathcal{A} \qquad \Longleftrightarrow \qquad \mathbf{1} \Vdash \widehat{\sim\mathcal{A}}$$

Then, by Prop. 12.7 we have

$$\mathbf{1} \Vdash {\sim}\mathcal{A} \qquad \Longleftrightarrow \qquad \mathbf{1} \Vdash \mathcal{I} \multimap_{\circledast} {\sim}\mathcal{A}$$

It follows from Prop. 13.2 that

$$\mathbf{1} \Vdash {\sim}\mathcal{A} \qquad \text{iff} \qquad \mathbf{1} \Vdash \mathcal{I} \circledast \mathcal{A} \multimap_{\circledast} \perp$$

and we conclude using the monoidal structure isomorphisms (Lem. 10.23). ◀

13.4 Proof of Prop. 13.2.(i)

13.4.1 Construction of the Strategy

Given a total strategy

$$\Sigma \vdash \sigma : \mathcal{A} \multimap_{\circledast} \widehat{\sim\mathcal{B}}$$

we define a total strategy

$$\Sigma \vdash \tilde{\sigma} : \mathcal{A} \circledast \mathcal{B} \multimap_{\circledast} \widehat{\perp}$$

We inductively map positions

$$((p, (q_{\mathcal{A}}, q_{\mathcal{B}})),\ (p, \widehat{q_{\perp}})) \quad \in \tilde{\sigma} \subseteq \wp(\mathcal{A} \circledast \mathcal{B} \multimap_{\circledast} \widehat{\perp})$$

to positions

$$((p, q_{\mathcal{A}}),\ (p, \widehat{q_{\mathcal{B}}})) \quad \in \sigma \subseteq \wp(\mathcal{A} \multimap_{\circledast} \widehat{\sim\mathcal{B}})$$

such that $\widehat{q_{\mathcal{B}}} = q_{\mathcal{B}}$ whenever $\widehat{q_{\perp}} = q_{\perp}$.

13.4.1.1 Base Case.

For the base case, we let

$$((\varepsilon, (q^\imath_{\mathcal{A}}, q^\imath_{\mathcal{B}})), (\varepsilon, q_\bot)) \in \tilde{\sigma}$$

and we have

$$((\varepsilon, q^\imath_{\mathcal{A}}),\ (\varepsilon, q^\imath_{\mathcal{B}})) \in \sigma$$

13.4.1.2 Inductive Step.

For the induction step, we proceed as follows. Consider some O-interrogation of $\tilde{\sigma}$:

$$
\begin{array}{c|ccc|l}
 & \mathcal{A} \circledast \mathcal{B} & -\!\!\circledast & \widehat{\bot} & \tilde{\sigma} \\
\hline
 & ((p, (q_{\mathcal{A}}, q_{\mathcal{B}})) & , & (p, \widehat{q_\bot})) & \\
\mathsf{O} & \downarrow & & & \\
 & ((p, a, \gamma_{\mathcal{A}} \circledast \gamma_{\mathcal{B}}) & , & (p, \widehat{q_\bot})) & \gamma_{\mathcal{A}} \circledast \gamma_{\mathcal{B}} \in \delta_{\mathcal{A} \circledast \mathcal{B}}((q_{\mathcal{A}}, q_{\mathcal{B}}), a)
\end{array}
$$

There are two cases, according to whether $\widehat{q_\bot} = q_\bot \in Q_\bot$. If $\widehat{q_\bot} = \mathsf{true}$, then we complete $\tilde{\sigma}$ by completeness of $\mathcal{A}$ and $\mathcal{B}$, regardless of σ. Then $\tilde{\sigma}$ will be winning since all its infinite plays will be winning on $\widehat{\bot}$.

Consider now the case of $\widehat{q_\bot} = q_\bot \in Q_\bot$ By induction hypothesis, we have $\widehat{q_{\mathcal{B}}} = q_{\mathcal{B}} \in Q_{\mathcal{B}}$. Since $\mathcal{B}$ is *complete* and σ is total by assumption, we let σ answer some $\gamma_{\sim\mathcal{B}}$ on the corresponding O-interrogation in $\mathcal{A}$:

$$
\begin{array}{c|ccc|l}
 & \mathcal{A} & -\!\!\circledast & \widehat{\sim\mathcal{B}} & \sigma \\
\hline
 & (p, q_{\mathcal{A}}) & , & (p, q_{\mathcal{B}})) & \\
\mathsf{O} & \downarrow & & & \\
 & ((p, a, \gamma_{\mathcal{A}}) & , & (p, q_{\mathcal{B}})) & \gamma_{\mathcal{A}} \in \delta_{\mathcal{A}}(q_{\mathcal{A}}, a) \\
\mathsf{P} & & & \downarrow & \\
 & ((p, a, \gamma_{\mathcal{A}}) & , & (p, a, \widehat{\gamma_{\sim\mathcal{B}}})) & \gamma_{\sim\mathcal{B}} \in \delta_{\sim\mathcal{B}}(q_{\mathcal{B}}, a)
\end{array}
$$

By construction, $\gamma_{\mathcal{B}} \cap \gamma_{\sim\mathcal{B}} \neq \emptyset$. We thus build $\tilde{\sigma}$'s response from some $(q'_{\mathcal{B}}, d) \in \gamma_{\mathcal{B}} \cap \gamma_{\sim\mathcal{B}}$, and consider a further O-interrogation:

$$
\begin{array}{c|ccc|l}
 & \mathcal{A} \circledast \mathcal{B} & -\!\!\circledast & \widehat{\bot} & \tilde{\sigma} \\
\hline
 & ((p, (q_{\mathcal{A}}, q_{\mathcal{B}})) & , & (p, q_\bot)) & \\
\mathsf{O} & \downarrow & & & \\
 & ((p, a, \gamma_{\mathcal{A}} \circledast \gamma_{\mathcal{B}}) & , & (p, q_\bot)) & \\
\mathsf{P} & & & \downarrow & \\
 & ((p, a, \gamma_{\mathcal{A}} \circledast \gamma_{\mathcal{B}}) & , & (p, a, \{\widehat{(q_\bot, d)}\})) & \\
\mathsf{O} & & & \downarrow & \\
 & ((p, a, \gamma_{\mathcal{A}} \circledast \gamma_{\mathcal{B}}) & , & (p \cdot d, \widehat{q'_\bot})) &
\end{array}
$$

But now again there are two cases, again according to whether $\widehat{q'_\bot} = q_\bot \in Q_\bot$.

Assume first that $\widehat{q'_\bot} = q_\bot \in Q_\bot$. In order to build σ's response, we interrogate σ on

$(q'_\mathcal{B}, d)$ (recall that $(q'_\mathcal{B}, d) \in \gamma_\mathcal{B} \cap \gamma_{\sim\mathcal{B}}$):

	$\mathcal{A}$	$-\!\!\circledast$	$\widehat{\sim\mathcal{B}}$	σ
	$(p, q_\mathcal{A})$	,	$(p, q_\mathcal{B}))$	
O	$\downarrow$			
	$((p, a, \gamma_\mathcal{A})$	,	$(p, q_\mathcal{B}))$	$\gamma_\mathcal{A} \in \delta_\mathcal{A}(q_\mathcal{A}, a)$
P			$\downarrow$	
	$((p, a, \gamma_\mathcal{A})$	,	$(p, a, \gamma_{\sim\mathcal{B}}))$	$\gamma_{\sim\mathcal{B}} \in \delta_{\sim\mathcal{B}}(q_\mathcal{B}, a)$
O			$\downarrow$	
	$((p, a, \gamma_\mathcal{A})$	,	$(p \cdot d, q'_\mathcal{B}))$	$(q'_\mathcal{B}, d) \in \gamma_{\sim\mathcal{B}}$
P	$\downarrow$			
	$((p \cdot d, q'_\mathcal{A})$	,	$(p \cdot d, q'_\mathcal{B}))$	$(q'_\mathcal{A}, d) \in \gamma_\mathcal{A}$

Since $(q'_\mathcal{A}, d) \in \gamma_\mathcal{A}$ and $(q'_\mathcal{B}, d) \in \gamma_\mathcal{B}$, we get $((q'_\mathcal{A}, q'_\mathcal{B}), d) \in \gamma_\mathcal{A} \circledast \gamma_\mathcal{B}$ and let $\tilde{\sigma}$ play this move:

	$\mathcal{A} \circledast \mathcal{B}$	$-\!\!\circledast$	$\widehat{\perp}$	$\tilde{\sigma}$
	$((p, (q_\mathcal{A}, q_\mathcal{B}))$	,	$(p, q_\perp))$	
O	$\downarrow$			
	$((p, a, \gamma_\mathcal{A} \circledast \gamma_\mathcal{B})$	,	$(p, q_\perp))$	
P			$\downarrow$	
	$((p, a, \gamma_{\mathcal{A} \circledast \mathcal{B}})$	,	$(p, a, \{\widehat{(q_\perp, d)}\}))$	
O			$\downarrow$	
	$((p, a, \gamma_\mathcal{A} \circledast \gamma_\mathcal{B})$	,	$(p \cdot d, q_\perp))$	
P	$\downarrow$			
	$((p \cdot d, (q'_\mathcal{A}, q'_\mathcal{B}))$	,	$(p \cdot d, q_\perp))\}$	

In the other case, O plays (true, d') for some $d' \neq d$. We can then conplete $\tilde{\sigma}$ by completeness of $\mathcal{A} \circledast \mathcal{B}$, and the play will be winning since it will be winning on $\widehat{\perp}$.

13.4.2 Proof of Correctness

The totality of $\tilde{\sigma}$ follows easily from the totality of σ.

As for winning, if at the some point the state true of $\widehat{\perp}$ is seen in a play of $\tilde{\sigma}$, then all further plays see no other state of $\widehat{\perp}$ than true, and the corresponding infinite play is winning for $\tilde{\sigma}$.

Otherwise, the inductive invariant ensures that given an infinite play of $\tilde{\sigma}$, the projections on the states of $\mathcal{A}$ and $\mathcal{B}$ (which has the same states as $\sim\mathcal{B}$) are the same as those of the corresponding play of σ. Hence if the projection on $\mathcal{A} \circledast \mathcal{B}$ is winning, then the projection on $\widehat{\sim\mathcal{B}}$ is loosing, contradicting the assumption that σ is winning.

13.5 Proof of Prop. 13.2.(ii)

13.5.1 Construction of the Strategy

Given a total strategy

$$\Sigma \vdash \sigma : \mathcal{A} \circledast \mathcal{B} \multimap_\circledast \widehat{\perp}$$

we define a total strategy

$$\Sigma \vdash \tilde{\sigma} : \mathcal{A} \multimap_\circledast \widehat{\sim\mathcal{B}}$$

We inductively map positions

$$((p, q_{\mathcal{A}}),\ (p, \widehat{q_{\mathcal{B}}})) \qquad \in \tilde{\sigma} \subseteq \wp(\mathcal{A} \multimap\!\!\circledast \widehat{\sim\mathcal{B}})$$

to positions

$$((p, (q_{\mathcal{A}}, q_{\mathcal{B}})),\ (p, \widehat{q_{\perp}})) \qquad \in \sigma \subseteq \wp(\mathcal{A} \circledast \mathcal{B} \multimap\!\!\circledast \widehat{\perp})$$

with either $\widehat{q_{\mathcal{B}}} = q_{\mathcal{B}}$ and $\widehat{q_{\perp}} = q_{\perp}$, or $\widehat{q_{\mathcal{B}}} = \widehat{q_{\perp}} = \mathsf{true}$.

13.5.1.1 Base Case.

For the base case, we let

$$((\varepsilon, q^{\imath}_{\mathcal{A}}),\ (\varepsilon, q^{\imath}_{\mathcal{B}})) \in \tilde{\sigma}$$

and we have

$$((\varepsilon, (q^{\imath}_{\mathcal{A}}, q^{\imath}_{\mathcal{B}})), (\varepsilon, q_{\perp})) \in \sigma$$

13.5.1.2 Induction Step.

For the induction step, we proceed as follows. Consider some O-interrogation:

$$\begin{array}{c|ccc|c}
 & \mathcal{A} & \multimap\!\!\circledast & \widehat{\sim\mathcal{B}} & \tilde{\sigma} \\
\hline
 & (p, q_{\mathcal{A}}) & , & (p, \widehat{q_{\mathcal{B}}})) & \\
\mathsf{O} & \downarrow & & & \\
 & ((p, a, \gamma_{\mathcal{A}}) & , & (p, \widehat{q_{\mathcal{B}}})) & \gamma_{\mathcal{A}} \in \delta_{\mathcal{A}}(q_{\mathcal{A}}, a)
\end{array}$$

There are two cases, according to whether $\widehat{q_{\mathcal{B}}} = q_{\mathcal{B}} \in Q_{\mathcal{B}}$. If $\widehat{q_{\mathcal{B}}} = \mathsf{true}$, then we let $\tilde{\sigma}$ play arbitrarily, relying on the completeness of $\mathcal{A}$, and regardless of σ. Then all further plays on $\widehat{\sim\mathcal{B}}$ will be on state true.

Otherwise, we have by induction hypothesis $\widehat{q_{\mathcal{B}}} = q_{\mathcal{B}} \in Q_{\mathcal{B}}$. In order to build $\tilde{\sigma}$'s response, we first build a map

$$\ell \quad : \quad \gamma_{\mathcal{B}} \in \delta_{\mathcal{B}}(q_{\mathcal{B}}, d) \quad \mapsto \quad (q'_{\mathcal{B}}, d) \in \gamma_{\mathcal{B}}$$

Definition of ℓ. Let $\gamma_{\mathcal{B}} \in \delta_{\mathcal{B}}(q_{\mathcal{B}}, d)$, and consider σ's response to the O-interrogation $\gamma_{\mathcal{A}} \circledast \gamma_{\mathcal{B}}$, followed by the unique O-move staying in $\perp$ and σ's response to that second move:

$$\begin{array}{c|ccc|c}
 & \mathcal{A} \circledast \mathcal{B} & \multimap\!\!\circledast & \widehat{\perp} & \sigma \\
\hline
 & ((p, (q_{\mathcal{A}}, q_{\mathcal{B}})) & , & (p, q_{\perp})) & \\
\mathsf{O} & \downarrow & & & \\
 & ((p, a, \gamma_{\mathcal{A}} \circledast \gamma_{\mathcal{B}}) & , & (p, q_{\perp})) & \gamma_{\mathcal{A}} \circledast \gamma_{\mathcal{B}} \in \delta_{\mathcal{A} \circledast \mathcal{B}}((q_{\mathcal{A}}, q_{\mathcal{B}}), a) \\
\mathsf{P} & & & \downarrow & \\
 & ((p, a, \gamma_{\mathcal{A}} \circledast \gamma_{\mathcal{B}}) & , & (p, a, \{\widehat{(q_{\perp}, d)}\})) & \\
\mathsf{O} & & & \downarrow & \\
 & ((p, a, \gamma_{\mathcal{A}} \circledast \gamma_{\mathcal{B}}) & , & (p \cdot d, q_{\perp})) & \\
\mathsf{P} & \downarrow & & & \\
 & ((p \cdot d, (q'_{\mathcal{A}}, q'_{\mathcal{B}})) & , & (p \cdot d, q_{\perp})) &
\end{array} \tag{22}$$

Now, we have $(q'_{\mathcal{B}}, d) \in \gamma_{\mathcal{B}}$ by definition, so we let

$$\ell(\gamma_{\mathcal{B}}) \quad := \quad (q'_{\mathcal{B}}, d)$$

This completes the definition of ℓ. ◀

Write $\gamma_{\sim\mathcal{B}}$ for the image of ℓ. By definition of $\sim\mathcal{B}$, we have $\gamma_{\sim\mathcal{B}} \in \delta_{\sim\mathcal{B}}(q_\mathcal{B}, a)$, and we let $\tilde{\sigma}$ play the corresponding move:

$$
\begin{array}{c|ccc|c}
 & \mathcal{A} & \multimap & \widehat{\sim\mathcal{B}} & \tilde{\sigma} \\
\hline
 & (p, q_\mathcal{A}) & , & (p, q_\mathcal{B})) & \\
\mathsf{O} & \downarrow & & & \\
 & ((p, a, \gamma_\mathcal{A}) & , & (p, q_\mathcal{B})) & \gamma_\mathcal{A} \in \delta_\mathcal{A}(q_\mathcal{A}, a) \\
\mathsf{P} & & & \downarrow & \\
 & ((p, a, \gamma_\mathcal{A}) & , & (p, a, \widehat{\gamma_{\sim\mathcal{B}}})) &
\end{array}
$$

Consider now some further O-move:

$$
\begin{array}{c|ccc|c}
 & \mathcal{A} & \multimap & \widehat{\sim\mathcal{B}} & \tilde{\sigma} \\
\hline
 & (p, q_\mathcal{A}) & , & (p, q_\mathcal{B})) & \\
\mathsf{O} & \downarrow & & & \\
 & ((p, a, \gamma_\mathcal{A}) & , & (p, q_\mathcal{B})) & \gamma_\mathcal{A} \in \delta_\mathcal{A}(q_\mathcal{A}, a) \\
\mathsf{P} & & & \downarrow & \\
 & ((p, a, \gamma_\mathcal{A}) & , & (p, a, \widehat{\gamma_{\sim\mathcal{B}}})) & \\
\mathsf{O} & & & \downarrow & \\
 & ((p, a, \gamma_\mathcal{A}) & , & (p \cdot d, \widehat{q'_\mathcal{B}})) &
\end{array}
$$

Once again there are two cases, according to whether $\widehat{q'_\mathcal{B}} \in Q_\mathcal{B}$.

Case of $\widehat{q'_\mathcal{B}} = q'_\mathcal{B} \in Q_\mathcal{B}$. By construction, $(q'_\mathcal{B}, d)$ is in the image of ℓ, and it follows that there are $\gamma_\mathcal{B}$ and $q'_\mathcal{A}$ such that, in (22), σ answers

$$((p \cdot d, (q'_\mathcal{A}, q'_\mathcal{B}))\ ,\ (p \cdot d, q_\perp))$$

Note that $(q'_\mathcal{A}, d) \in \gamma_\mathcal{A}$. We let $\tilde{\sigma}$ answer the corresponding move:

$$
\begin{array}{c|ccc|c}
 & \mathcal{A} & \multimap & \widehat{\sim\mathcal{B}} & \tilde{\sigma} \\
\hline
 & (p, q_\mathcal{A}) & , & (p, q_\mathcal{B})) & \\
\mathsf{O} & \downarrow & & & \\
 & ((p, a, \gamma_\mathcal{A}) & , & (p, q_\mathcal{B})) & \gamma_\mathcal{A} \in \delta_\mathcal{A}(q_\mathcal{A}, a) \\
\mathsf{P} & & & \downarrow & \\
 & ((p, a, \gamma_\mathcal{A}) & , & (p, a, \widehat{\gamma_{\sim\mathcal{B}}})) & \\
\mathsf{O} & & & \downarrow & \\
 & ((p, a, \gamma_\mathcal{A}) & , & (p \cdot d, q'_\mathcal{B})) & \\
\mathsf{P} & \downarrow & & & \\
 & ((p \cdot d, q'_\mathcal{A}) & , & (p \cdot d, q'_\mathcal{B})) &
\end{array}
$$

Case of $\widehat{q'_\mathcal{B}} = \mathbf{true}$. In this case, by defintion of ℓ, there is no state $q'_\mathcal{B}$ such that, in (22), σ answers

$$((p \cdot d, (q'_\mathcal{A}, q'_\mathcal{B}))\ ,\ (p \cdot d, q_\perp))$$

We therefore interrogate σ on the O-move (true, d):

	$\mathcal{A} \circledast \mathcal{B}$	$\mathrel{-\!\!\circledast}$	$\widehat{\bot}$	σ
	$((p, (q_{\mathcal{A}}, q_{\mathcal{B}}))$	,	$(p, q_{\bot}))$	
O	$\downarrow$			
	$((p, a, \gamma_{\mathcal{A}} \circledast \gamma_{\mathcal{B}})$	,	$(p, q_{\bot}))$	$\gamma_{\mathcal{A}} \circledast \gamma_{\mathcal{B}} \in \delta_{\mathcal{A} \circledast \mathcal{B}}((q_{\mathcal{A}}, q_{\mathcal{B}}), a)$
P			$\downarrow$	
	$((p, a, \gamma_{\mathcal{A}} \circledast \gamma_{\mathcal{B}})$	,	$(p, a, \{\widehat{(q_{\bot}, d')}\}))$	
O			$\downarrow$	
	$((p, a, \gamma_{\mathcal{A}} \circledast \gamma_{\mathcal{B}})$	,	$(p \cdot d, \mathsf{true}))$	
P	$\downarrow$			
	$((p \cdot d, (q'_{\mathcal{A}}, q'_{\mathcal{B}}))$	,	$(p \cdot d, \mathsf{true}))$	

And let $\tilde{\sigma}$ copy σ's response in $\mathcal{A}$:

	$\mathcal{A}$	$\mathrel{-\!\!\circledast}$	$\widehat{\sim\mathcal{B}}$	$\tilde{\sigma}$
	$(p, q_{\mathcal{A}})$	,	$(p, q_{\mathcal{B}}))$	
O	$\downarrow$			
	$((p, a, \gamma_{\mathcal{A}})$	,	$(p, q_{\mathcal{B}}))$	$\gamma_{\mathcal{A}} \in \delta_{\mathcal{A}}(q_{\mathcal{A}}, a)$
P			$\downarrow$	
	$((p, a, \gamma_{\mathcal{A}})$	,	$(p, a, \widehat{\gamma_{\sim\mathcal{B}}}))$	
O			$\downarrow$	
	$((p, a, \gamma_{\mathcal{A}})$	,	$(p \cdot d, \mathsf{true}))$	
P	$\downarrow$			
	$((p \cdot d, q'_{\mathcal{A}})$	,	$(p \cdot d, \mathsf{true}))$	

13.5.2 Proof of Correctness

The totality of $\tilde{\sigma}$ follows easily from the totality of σ. If at some point true is played in $\widehat{\sim\mathcal{B}}$, then no other state of $\widehat{\sim\mathcal{B}}$ than true will be seen on further moves. Then the corresponding infinite play is winning for $\tilde{\sigma}$ since its projection on $\widehat{\sim\mathcal{B}}$ is winning.

Otherwise, the inductive invariant ensures that given an infinite play of $\tilde{\sigma}$, the projections on the states of $\mathcal{A}$ and $\mathcal{B}$ (which has the same states as $\sim\mathcal{B}$) are the same as those of the corresponding play of σ. Hence the projections on $\mathcal{A}$ and $\mathcal{B}$ can not be both winning, ensuring winning for $\tilde{\sigma}$.

14 Existential Quantification on Complete Automata

In this Section, we show that the fibrations $\mathsf{aut}^{(\mathrm{W/R})} : \widehat{\mathbf{Aut}}^{(\mathrm{W/R})} \to \mathbf{Alph}$ have existential quantifications, in the sense of the simple coproducts of [12]: Given alphabets Σ and Γ, the *weakening* functor

$$\pi^* \quad : \quad \widehat{\mathbf{Aut}}_{\Sigma}^{(\mathrm{W/R})} \quad \longrightarrow \quad \widehat{\mathbf{Aut}}_{\Sigma\times\Gamma}^{(\mathrm{W/R})}$$

induced (by Prop. 8.4) by the left projection

$$\pi_{\Sigma}^{\Sigma,\Gamma} \in \mathbf{Alph}[\Sigma \times \Gamma, \Sigma]$$

has a left adjoint:

$$\amalg_{\Sigma,\Gamma} \quad : \quad \widehat{\mathbf{Aut}}_{\Sigma\times\Gamma}^{(\mathrm{W/R})} \quad \longrightarrow \quad \widehat{\mathbf{Aut}}_{\Sigma}^{(\mathrm{W/R})}$$

and the Beck-Chevalley condition holds.

Recall (from e.g. [13]) that an adjunction $\amalg \dashv \pi$ as above is given by a natural isomorphism

$$\phi_{\mathcal{A},\mathcal{B}} \quad : \quad \widehat{\mathbf{Aut}}_{\Sigma}^{(\mathrm{W/R})}[\amalg\mathcal{A}, \mathcal{B}] \quad \xrightarrow{\simeq} \quad \widehat{\mathbf{Aut}}_{\Sigma\times\Gamma}^{(\mathrm{W/R})}[\mathcal{A}, \mathcal{B}[\pi]]$$

(We drop the subscripts and superscript from $\amalg$ and π when convenient.)

Recall also from [13, Thm. IV.1.2.(ii)] that an adjunction as above is completely determined by the functor π^* together with, for each $\Sigma \times \Gamma \vdash \mathcal{A}$, an object

$$\Sigma \vdash \amalg_{\Sigma,\Gamma}\mathcal{A}$$

and a map

$$\eta_{\mathcal{A}} \quad : \quad \Sigma \times \Gamma \vdash \mathcal{A} \quad \longrightarrow \quad \Sigma \times \Gamma \vdash (\amalg_{\Sigma,\Gamma}\mathcal{A})[\pi]$$

satisfying the following universal lifting property: For every

$$\sigma \quad : \quad \Sigma \times \Gamma \vdash \mathcal{A} \quad \longrightarrow \quad \Sigma \times \Gamma \vdash \mathcal{B}[\pi]$$

there is a unique

$$\tau \quad : \quad \Sigma \vdash \amalg_{\Sigma,\Gamma}\mathcal{A} \quad \longrightarrow \quad \Sigma \vdash \mathcal{B}$$

such that in $\widehat{\mathbf{Aut}}_{\Sigma\times\Gamma}^{(\mathrm{W/R})}$:

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\eta_{\mathcal{A}}"] \arrow[dr, "\sigma"'] & (\amalg_{\Sigma,\Gamma}\mathcal{A})[\pi] \arrow[d, "\pi^*(\tau)"] \\ & \mathcal{B}[\pi] \end{tikzcd} \tag{23}$$

In this setting, the natural isomorphisms

$$\phi_{\mathcal{A},\mathcal{B}} \quad : \quad \widehat{\mathbf{Aut}}_{\Sigma}^{(\mathrm{W/R})}[\amalg\mathcal{A}, \mathcal{B}] \quad \xrightarrow{\simeq} \quad \widehat{\mathbf{Aut}}_{\Sigma\times\Gamma}^{(\mathrm{W/R})}[\mathcal{A}, \mathcal{B}[\pi]]$$

are defined as

$$\phi_{\mathcal{A},\mathcal{B}}(\tau) \quad := \quad \pi^*(\tau) \circ \eta_{\mathcal{A}}$$

Using the universal lifting property of $\eta_{\mathcal{A}}$, its inverse $\phi^{-1}(\sigma)$ is the unique τ satifying (23).

The Beck-Chevalley condition reads as follows (see e.g. [12, 1.8.9]). Consider $\beta \in \mathbf{Alph}[\Delta, \Sigma]$, so that

$$(\beta \times \mathrm{Id}_\Gamma)^* \quad : \quad \widehat{\mathbf{Aut}}^{(\mathrm{W/R})}_{\Sigma\times\Gamma} \quad \longrightarrow \quad \widehat{\mathbf{Aut}}^{(\mathrm{W/R})}_{\Delta\times\Gamma}$$

Then we have

$$(\beta \times \mathrm{Id}_\Gamma)^*(\eta_{\mathcal{A}}) \quad : \quad \Delta \times \Gamma \vdash \mathcal{A}[\beta \times \mathrm{Id}_\Gamma] \quad \longrightarrow \quad \Delta \times \Gamma \vdash (\amalg_{\Sigma,\Gamma}\mathcal{A})[\pi][\beta \times \mathrm{Id}_\Gamma]$$

Note that, for $\pi' \in \mathbf{Alph}[\Delta \times \Gamma, \Delta]$,

$$\begin{array}{lll} \Delta \times \Gamma \vdash (\amalg_{\Sigma,\Gamma}\mathcal{A})[\pi][\beta \times \mathrm{Id}_\Gamma] & = & \Delta \times \Gamma \vdash (\amalg_{\Sigma,\Gamma}\mathcal{A})[\pi \circ (\beta \times \mathrm{Id}_\Gamma)] \\ & = & \Delta \times \Gamma \vdash (\amalg_{\Sigma,\Gamma}\mathcal{A})[\beta \circ \pi'] \\ & = & \Delta \times \Gamma \vdash (\amalg_{\Sigma,\Gamma}\mathcal{A})[\beta][\pi'] \end{array}$$

Then, the Beck-Chevalley condition is that the natural maps

$$\phi^{-1}((\beta \times \mathrm{Id}_\Gamma)^*(\eta_{\mathcal{A}})) \quad : \quad \Delta \vdash \amalg_{\Delta,\Gamma}(\mathcal{A}[\beta \times \mathrm{Id}_\Gamma]) \quad \longrightarrow \quad \Delta \vdash (\amalg_{\Sigma,\Gamma}\mathcal{A})[\beta]$$

are the identity.

But by definition of ϕ^{-1}, this means that the unique

$$\tau \quad : \quad \Delta \vdash \amalg_{\Delta,\Gamma}\mathcal{A}[\beta \times \mathrm{Id}_\Gamma] \quad \longrightarrow \quad \Delta \vdash (\amalg_{\Sigma,\Gamma}\mathcal{A})[\beta]$$

such that

$$\begin{tikzcd} \mathcal{A}[\beta \times \mathrm{Id}_\Gamma] \arrow[r, "\eta_{\mathcal{A}[\beta\times\mathrm{Id}_\Gamma]}"] \arrow[dr, "(\beta\times\mathrm{Id}_\Gamma)^*(\eta_{\mathcal{A}})"'] & (\amalg_{\Delta,\Gamma}\mathcal{A}[\beta \times \mathrm{Id}_\Gamma])[\pi'] \arrow[d, "\pi^*(\tau)"] \\ & (\amalg_{\Sigma,\Gamma}\mathcal{A})[\beta][\pi'] \end{tikzcd}$$

is the identity.
This amounts to showing

$$\Delta \vdash (\amalg_{\Sigma,\Gamma}\mathcal{A})[\beta] \quad = \quad \Delta \vdash \amalg_{\Delta,\Gamma}(\mathcal{A}[\beta \times \mathrm{Id}_\Gamma]) \tag{24}$$

and

$$\begin{array}{lll} \eta_{\mathcal{A}[\beta\times\mathrm{Id}_\Gamma]} & = & (\beta \times \mathrm{Id}_\Gamma)^*(\eta_{\mathcal{A}}) \\ & & : \quad \Delta \times \Gamma \vdash \mathcal{A}[\beta \times \mathrm{Id}_\Gamma] \quad \longrightarrow \quad \Delta \times \Gamma \vdash (\amalg_{\Delta,\Gamma}\mathcal{A}[\beta \times \mathrm{Id}_\Gamma])[\pi'] \end{array} \tag{25}$$

14.1 The Lifted Projection $\amalg$

Consider $\Sigma \times \Gamma \vdash \mathcal{A}$ with

$$\mathcal{A} \quad = \quad (Q, q^\imath, \delta, \Omega)$$

Define $\Sigma \vdash \amalg_{\Sigma,\Gamma}\mathcal{A}$ as

$$\amalg_{\Sigma,\Gamma}\mathcal{A} \quad := \quad (Q \times \Gamma + \{q^\imath\}, q^\imath, \delta_{\amalg\mathcal{A}}, \Omega_{\amalg\mathcal{A}})$$

where

$$\begin{array}{rcl} \delta_{\amalg\mathcal{A}}(q^\imath, a) & := & \bigcup_{b\in\Gamma}\{\gamma^{+b} \mid \gamma \in \delta(q^\imath, (a,b))\} \\ \delta_{\amalg\mathcal{A}}((q, \_), a) & := & \bigcup_{b\in\Gamma}\{\gamma^{+b} \mid \gamma \in \delta(q, (a,b))\} \end{array}$$

and, given $\gamma \in \mathcal{P}(Q \times D)$ and $b \in \Gamma$,

$$\begin{array}{rcl} \gamma^{+b} & := & \{((q^{+b}, d) \mid (q,d) \in \gamma\} \\ q^{+b} & := & (q, b) \end{array}$$

and

$$q^\imath \cdot (q_0, b_0) \cdot \ldots \cdot (q_n, b_n) \cdot \ldots \;\in\; \Omega_{\amalg\mathcal{A}}$$

iff

$$q^\imath \cdot q_0 \cdot \ldots \cdot q_n \cdot \ldots \;\in\; \Omega$$

14.2 Action on Acceptance Games of the Lifted Projection

Let $\pi := \pi_{\Sigma,\Gamma} \in \mathbf{Alph}[\Sigma\times\Gamma, \Sigma]$, and $\Sigma\times\Gamma \vdash \mathcal{A}$. Write $\Sigma \vdash \amalg(\mathcal{A})$ for $\Sigma \vdash \amalg_{\Sigma,\Gamma}\mathcal{A}$.

We now define a map

$$\wp(\amalg) \quad : \quad \wp_{\Sigma\times\Gamma}(\mathcal{A}) \quad\longrightarrow\quad \wp_{\Sigma}(\amalg_{\Sigma,\Gamma}\mathcal{A})$$

such that if

$$s \quad = \quad (\varepsilon, q^\imath) \to^* (p, q)$$

then

$$\wp(\amalg)(s) \quad = \quad (\varepsilon, q^\imath) \to^* (p, q^{++})$$

where

$$\text{either} \qquad q^{++} = q = q^\imath \qquad \text{or} \qquad q^{++} = q^{+b} = (q,b) \quad \text{for some } b \in \Gamma$$

Consider the map

$$\wp(\amalg) \quad : \quad \wp_{\Sigma\times\Gamma}(\mathcal{A}) \quad\longrightarrow\quad \wp_{\Sigma}(\amalg_{\Sigma,\Gamma}\mathcal{A})$$

defined as

$$\begin{array}{rcl} \wp(\amalg)(\varepsilon : (\varepsilon, q^\imath)) & := & \varepsilon : (\varepsilon, q^\imath) \\ \wp(\amalg)((\varepsilon, q^\imath) \to^* (p,q) \to (p, (a,b), \gamma)) & := & \wp(\amalg)((\varepsilon, q^\imath) \to^* (p,q)) \to (p, a, \gamma^{+b}) \\ \wp(\amalg)((\varepsilon, q^\imath) \to^* (p, (a,b), \gamma) \to (p.d, q)) & := & \wp(\amalg)((\varepsilon, q^\imath) \to^* (p, (a,b), \gamma)) \to (p.d, q^{+b}) \end{array}$$

▶ **Lemma 14.1.** *(i) If*

$$s \quad = \quad (\varepsilon, q^\imath) \to^* (p, q)$$

then

$$\wp(\amalg)(s) \quad = \quad (\varepsilon, q^\imath) \to^* (p, q^{++})$$

where

$$\textit{either} \qquad q^{++} = q = q^{\imath} \qquad \textit{or} \qquad q^{++} = q^{+b} = (q,b) \quad \textit{for some } b \in \Gamma$$

and if

$$s \quad = \quad (\varepsilon, q^{\imath}) \to^{*} (p, (a,b), \gamma)$$

then

$$\wp(\mathrm{II})(s) \quad = \quad (\varepsilon, q^{\imath}) \to^{*} (p, a, \gamma^{+b})$$

(ii) $\wp(\mathrm{II}) : \wp_{\Sigma\times\Gamma}(\mathcal{A}) \longrightarrow \wp_{\Sigma}(\mathrm{II}_{\Sigma,\Gamma}\mathcal{A})$

Proof. (i) By induction on s.

(ii) By induction on s. In the base case $s = \varepsilon : (\varepsilon, q^{\imath})$ and we are done since $\wp(\mathrm{II})(s) = \varepsilon : (\varepsilon, q^{\imath})$. For the induction step we consider two cases:

- If $s = (\varepsilon, q^{\imath}) \to^{*} (p,q) \to (p,(a,b),\gamma)$, then

  $$\wp(\mathrm{II})(s) \quad = \quad \wp(\mathrm{II})((\varepsilon, q^{\imath}) \to^{*} (p,q)) \to (p, a, \gamma^{+b})$$

  By induction hypothesis, we have

  $$\wp(\mathrm{II})((\varepsilon, q^{\imath}) \to^{*} (p,q)) \in \wp_{\Sigma}(\mathrm{II}_{\Sigma,\Gamma}\mathcal{A})$$

  Moreover, by (i)

  $$\wp(\mathrm{II})((\varepsilon, q^{\imath}) \to^{*} (p,q)) \quad = \quad (\varepsilon, q^{\imath}) \to^{*} (p, q^{++})$$

  where

  $$\text{either} \qquad q^{++} = q = q^{\imath} \qquad \text{or} \qquad q^{++} = q^{+c} = (q,c) \quad \text{for some } c \in \Gamma$$

  Since $\gamma \in \delta(q,(a,b))$, we have $\gamma^{+b} \in \delta_{\mathrm{II}\mathcal{A}}(q,a)$ in both cases, hence $\wp(\mathrm{II})(s) \in \wp_{\Sigma}(\mathrm{II}_{\Sigma,\Gamma}\mathcal{A})$.
- Otherwise, $s = (\varepsilon, q^{\imath}) \to^{*} (p,(a,b),\gamma) \to (p.d, q)$. Hence

  $$\wp(\mathrm{II})(s) \quad = \quad \wp(\mathrm{II})((\varepsilon, q^{\imath}) \to^{*} (p,(a,b),\gamma)) \to (p.d, q^{+b})$$

  By induction hypothesis, we have

  $$\wp(\mathrm{II})((\varepsilon, q^{\imath}) \to^{*} (p,(a,b),\gamma)) \in \wp_{\Sigma}(\mathrm{II}_{\Sigma,\Gamma}\mathcal{A})$$

  and moreover, by (i)

  $$\wp(\mathrm{II})((\varepsilon, q^{\imath}) \to^{*} (p,(a,b),\gamma)) \quad = \quad (\varepsilon, q^{\imath}) \to^{*} (p, a, \gamma^{+b})$$

  hence $(q^{+b}, d) \in \gamma^{+b}$ and we are done. ◀

▶ **Lemma 14.2.**

$$\begin{tikzcd}
\wp_{\Sigma\times\Gamma}(\mathcal{A}) \arrow[r, "\wp(\mathrm{II})"] \arrow[d, "\mathrm{tr}"'] & \wp_{\Sigma}(\mathrm{II}_{\Sigma,\Gamma}\mathcal{A}) \arrow[d, "\mathrm{tr}"] \\
\mathrm{Tr}_{\Sigma\times\Gamma} \arrow[r, "\mathrm{Tr}(\pi)"'] & \mathrm{Tr}_{\Sigma}
\end{tikzcd}$$

Proof. We show that for all $s \in \wp_{\Sigma\times\Gamma}(\mathcal{A})$, we have

$$\mathrm{Tr}(\pi) \circ \mathrm{tr}(s) \quad = \quad \mathrm{tr} \circ \wp(\amalg)(s)$$

We reason by induction on s. In the base case, $s = \varepsilon : (\varepsilon, q^\imath)$ and we are done since:

$$\mathrm{Tr}(\pi)(\mathrm{tr}(s)) \quad = \quad \varepsilon \quad = \quad \mathrm{tr}(\wp(\amalg)(s))$$

For the induction step, we consider two cases:

- If $s = (\varepsilon, q^\imath) \to^* (p, q) \to (p, (a, b), \gamma)$, then

  $$\mathrm{tr}(s) \quad = \quad \mathrm{tr}((\varepsilon, q^\imath) \to^* (p, q)) \cdot (a, b)$$

  hence, by definition of $\pi \in \mathbf{Alph}[\Sigma \times \Gamma, \Sigma]$, we have

  $$\mathrm{Tr}(\pi)(\mathrm{tr}(s)) \quad = \quad \mathrm{Tr}(\pi)(\mathrm{tr}((\varepsilon, q^\imath) \to^* (p, q))) \cdot a$$

  On the other hand,

  $$\wp(\amalg)(s) \quad = \quad \wp(\amalg)((\varepsilon, q^\imath) \to^* (p, q)) \to (p, a, \gamma^{+b})$$

  hence

  $$\mathrm{tr}(\wp(\amalg)(s)) \quad = \quad \mathrm{tr}(\wp(\amalg)((\varepsilon, q^\imath) \to^* (p, q))) \cdot a$$

  and we are donce since by induction hypothesis we have

  $$\mathrm{Tr}(\pi)(\mathrm{tr}((\varepsilon, q^\imath) \to^* (p, q))) \quad = \quad \mathrm{tr}(\wp(\amalg)((\varepsilon, q^\imath) \to^* (p, q)))$$

- Otherwise, $s = (\varepsilon, q^\imath) \to^* (p, (a, b), \gamma) \to (p.d, q)$. Hence

  $$\mathrm{tr}(s) \quad = \quad \mathrm{tr}((\varepsilon, q^\imath) \to^* (p, (a, b), \gamma)) \cdot d$$

  and thus

  $$\mathrm{Tr}(\pi)(\mathrm{tr}(s)) \quad = \quad \mathrm{Tr}(\pi)(\mathrm{tr}((\varepsilon, q^\imath) \to^* (p, (a, b), \gamma))) \cdot d$$

  On the other hand,

  $$\wp(\amalg)(s) \quad = \quad \wp(\amalg)((\varepsilon, q^\imath) \to^* (p, (a, b), \gamma)) \to (p.d, q^{+b})$$

  hence

  $$\mathrm{tr}(\wp(\amalg)(s)) \quad = \quad \mathrm{tr}(\wp(\amalg)((\varepsilon, q^\imath) \to^* (p, (a, b), \gamma))) \cdot d$$

  and we are done, since by induction hypothesis

  $$\mathrm{Tr}(\pi)(\mathrm{tr}((\varepsilon, q^\imath) \to^* (p, (a, b), \gamma))) \cdot d \quad = \quad \mathrm{tr}(\wp(\amalg)((\varepsilon, q^\imath) \to^* (p, (a, b), \gamma)))$$

◀

▶ **Lemma 14.3.** *If $\mathcal{A}$ is a complete automaton, then the map*

$$\wp(\amalg) \quad : \quad \wp_{\Sigma\times\Gamma}(\mathcal{A}) \quad \longrightarrow \quad \wp_\Sigma(\amalg_{\Sigma,\Gamma}\mathcal{A})$$

is an injection.

Proof. For injectivity, we have to check that given $s, t \in \wp_{\Sigma\times\Gamma}(\mathcal{A})$, if $\wp(\amalg)(s) = \wp(\amalg)(t)$ then $s = t$. Since $\wp(\amalg)$ is length-preserving, $\wp(\amalg)(s) = \wp(\amalg)(t)$ implies $|s| = |t|$. We reason by induction on $|s| = |t|$. In the base case $|s| = |t| = 0$, and we must have $s = t = \varepsilon : (\varepsilon, q^{\imath})$. For the induction step, there are two cases:

- In the first case, we have

  $$\begin{array}{lll} s & = & (\varepsilon, q^{\imath}) \to^* (p, q) \to (p, (a, b), \gamma) \\ t & = & (\varepsilon, q^{\imath}) \to^* (p', q') \to (p', (a', b'), \gamma') \end{array}$$

  Hence

  $$\begin{array}{lll} \wp(\amalg)(s) & = & (\varepsilon, q^{\imath++}) \to^* (p, q^{++}) \to (p, a, \gamma^{+b}) \\ \wp(\amalg)(t) & = & (\varepsilon, q^{\imath++}) \to^* (p', q'^{++}) \to (p', a', \gamma'^{+b'}) \end{array}$$

  Now, $\wp(\amalg)(s) = \wp(\amalg)(t)$ implies

  $$p = p' \quad q^{++} = q'^{++} \quad a = a' \quad \gamma^{+b} = \gamma'^{+b'}$$

  It follows that

  $$q = q' \quad \gamma = \gamma'$$

  Moreover, since $\mathcal{A}$ is complete, $\gamma = \gamma'$ is non-empty, hence $b = b'$.
  Now we are done since moreover

  $$(\varepsilon, q^{\imath++}) \to^* (p, q^{++}) \quad = \quad (\varepsilon, q^{\imath++}) \to^* (p', q'^{++})$$

  by induction hypothesis.
- In the second case we have

  $$\begin{array}{lll} s & = & (\varepsilon, q^{\imath}) \to^* (p, (a, b), \gamma) \to (p.d, q) \\ t & = & (\varepsilon, q^{\imath}) \to^* (p', (a', b'), \gamma') \to (p'.d', q') \end{array}$$

  Hence

  $$\begin{array}{lll} \wp(\amalg)(s) & = & (\varepsilon, q^{\imath++}) \to^* (p, a, \gamma^{+b}) \to (p.d, q^{+b}) \\ \wp(\amalg)(t) & = & (\varepsilon, q^{\imath++}) \to^* (p', a', \gamma'^{+b'}) \to (p'.d', q'^{+b'}) \end{array}$$

  Now, $\wp(\amalg)(s) = \wp(\amalg)(t)$ implies

  $$p = p' \quad a = a' \quad \gamma^{+b} = \gamma'^{+b'} \quad d = d' \quad q^{+b} = q'^{+b'}$$

  It follows that

  $$b = b' \quad \gamma = \gamma' \quad q = q'$$

  and we are done since moreover

  $$(\varepsilon, q^{\imath++}) \to^* (p, a, \gamma^{+b}) \quad = \quad (\varepsilon, q^{\imath++}) \to^* (p', a', \gamma'^{+b'})$$

  by induction hypothesis.

◀

▶ **Lemma 14.4.** *The map*

$$\wp(\amalg) \quad : \quad \wp_{\Sigma\times\Gamma}(\mathcal{A}) \quad \longrightarrow \quad \wp_{\Sigma}(\amalg_{\Sigma,\Gamma}\mathcal{A})$$

is a surjection.

Proof. We show by induction on $t \in \wp_\Sigma(\amalg_{\Sigma\times\Gamma}\mathcal{A})$ that there is $s \in \wp_{\Sigma\times\Gamma}(\mathcal{A})$ such that $\wp(\amalg)(s) = t$.

In the base case $t = \varepsilon : (\varepsilon, q^\imath)$ we are done by taking $s := t$. For the induction step, we consider two cases:

- If $t = (\varepsilon, q^\imath) \to^* (p, q^{++}) \to (p, a, \gamma^{++})$, then by induction hypothesis, there is $s' \in \wp_{\Sigma\times\Gamma}(\mathcal{A})$ such that

$$\wp(\amalg)(s') \quad = \quad (\varepsilon, q^\imath) \to^* (p, q^{++})$$

By definition of $\wp(\amalg)$, we have either $q = q^\imath = q^{++}$ or $q^{++} = (q, c)$ for some $c \in \Gamma$. By defintion of the transition function $\delta_{\amalg\mathcal{A}}$ of $\amalg\mathcal{A}$, there is $b \in \Gamma$ and $\gamma \in \delta(q, (a, b))$ such that

$$\gamma^{++} \quad = \quad \gamma^{+b} \quad = \quad \{((q, b), d) \mid (q, d) \in \gamma\}$$

It follows that by taking

$$s \quad := \quad s' \to (p, (a, b), \gamma)$$

we have $s \in \wp_{\Sigma\times\Gamma}(\mathcal{A})$ and $\wp(\amalg)(s) = t$.

- Otherwise, $t = (\varepsilon, q^\imath) \to^* (p, a, \gamma^{++}) \to (p.d, q^{++})$. Then by induction hypothesis, there is $s' \in \wp_{\Sigma\times\Gamma}(\mathcal{A})$ such that

$$\wp(\amalg)(s') \quad = \quad (\varepsilon, q^\imath) \to^* (p, a, \gamma^{++})$$

By definition of $\wp(\amalg)$, there is $b \in \Gamma$ such that

$$s' \quad = \quad (\varepsilon, q^\imath) \to^* (p, (a, b), \gamma)$$

and

$$\gamma^{++} \quad = \quad \gamma^{+b} \quad = \quad \{((q, b), d) \mid (q, d) \in \gamma\}$$

Moreover $q^{++} = q^{+b} = (q, b)$ with $(q, d) \in \gamma$, and it follows that by taking

$$s \quad := \quad s' \to (p.d, q)$$

we have $s \in \wp_{\Sigma\times\Gamma}(\mathcal{A})$ and $\wp(\amalg)(s) = t$. ◀

We thus obtain:

▶ **Corollary 14.5** (Prop. 7.1). *If $\mathcal{A}$ is a complete automaton, then the map*

$$\wp(\amalg) \quad : \quad \wp_{\Sigma\times\Gamma}(\mathcal{A}) \quad \longrightarrow \quad \wp_\Sigma(\amalg_{\Sigma,\Gamma}\mathcal{A})$$

is a bijection.

Proof. Injectivity is given by Lem. 14.3 and surjectivitty by Lem. 14.4. ◀

14.3 The Universal Lifting Property

In this Section, we define the unit maps

$$\eta_\mathcal{A} \quad : \quad (\Sigma \times \Gamma \vdash \mathcal{A}) \quad \longrightarrow \quad (\Sigma \times \Gamma \vdash (\amalg_{\Sigma,\Gamma}\mathcal{A})[\pi])$$

and show that they satisfy the unique lifting property (23).

14.3.1 The Units $\eta_{(-)}$

Recall from Cor. 14.5 that we have a bijection:

$$\wp(\amalg) \quad : \quad \wp_{\Sigma\times\Gamma}(\mathcal{A}) \quad \longrightarrow \quad \wp_{\Sigma}(\amalg_{\Sigma,\Gamma}\mathcal{A})$$

We show that there is moreover an injection:

$$\iota_{\Sigma,\Gamma} \quad : \quad \wp_{\Sigma}(\amalg_{\Sigma,\Gamma}\mathcal{A}) \quad \longrightarrow \quad \wp_{\Sigma\times\Gamma}((\amalg_{\Sigma,\Gamma}\mathcal{A})[\pi])$$

We define $\iota_{\Sigma,\Gamma}$ by induction on plays:

$$\begin{array}{rcl} \iota_{\Sigma,\Gamma}((\varepsilon, q^{\imath}_{\mathcal{A}})) & := & (\varepsilon, q^{\imath}_{\mathcal{A}}) \\ \iota_{\Sigma,\Gamma}(s \to (p, a, \gamma^{+b})) & := & \iota_{\Sigma,\Gamma}(s) \to (p, (a,b), \gamma^{+b}) \\ \iota_{\Sigma,\Gamma}(s \to (p, q^{+b})) & := & \iota_{\Sigma,\Gamma}(s) \to (p, q^{+b}) \end{array}$$

Note that for all $t \in \wp_{\Sigma}(\amalg_{\Sigma\times\Gamma}\mathcal{A})$, we have

$$\wp(\pi) \circ \iota_{\Sigma,\Gamma}(t) \quad = \quad t$$

and for all $t' \in \wp_{\Sigma\times\Gamma}(\mathcal{A})$,

$$\mathrm{tr}(\iota_{\Sigma,\Gamma} \circ \wp(\amalg)(t')) \quad = \quad \mathrm{tr}(t')$$

We now define $\eta_{\mathcal{A}}$ as the (necessarily unique) strategy such that

$$\mathrm{HS}(\eta_{\mathcal{A}}) \quad = \quad \{(t, \iota_{\Sigma,\Gamma} \circ \wp(\amalg)(t)) \mid t \in \wp_{\Sigma\times\Gamma}(\mathcal{A})\} \tag{26}$$

We define it by induction on plays

$$s \in \wp^{\mathsf{P}}_{\Sigma\times\Gamma}(\mathcal{A} \multimap\!\!\circledast (\amalg\mathcal{A})[\pi])$$

We let

$$((\varepsilon, q^{\imath}_{\mathcal{A}}) \ , \ (\varepsilon, q^{\imath}_{\mathcal{A}})) \quad \in \quad \eta_{\mathcal{A}}$$

and the property (26) is satisfied.

Assume now

$$s \quad : \quad * \quad \to^* \quad \big((p, q_{\mathcal{A}}) \ , \ (p, {q_{\mathcal{A}}}^{++})\big) \quad \in \quad \eta_{\mathcal{A}}$$

Then for all $(a,b) \in \Sigma \times \Gamma$ and all $\gamma_{\mathcal{A}} \in \delta_{\mathcal{A}}(q_{\mathcal{A}}, (a,b))$, we let

$$\begin{array}{rl} s' \quad : \quad * \quad \to^* \quad \big((p, q_{\mathcal{A}}) \ , \ (p, {q_{\mathcal{A}}}^{++})\big) & \stackrel{\mathsf{O}}{\to} \quad \big((p, (a,b), \gamma_{\mathcal{A}}) \ , \ (p, {q_{\mathcal{A}}}^{++})\big) \\ & \stackrel{\mathsf{P}}{\to} \quad \big((p, (a,b), \gamma_{\mathcal{A}}) \ , \ (p, (a,b), {\gamma_{\mathcal{A}}}^{+b})\big) \quad \in \quad \eta_{\mathcal{A}} \end{array}$$

Furthermore, for all $(q'_{\mathcal{A}}, d) \in \gamma_{\mathcal{A}}$, we let

$$\begin{array}{rl} s'' \quad : \quad * \quad \to^* \quad \big((p, q_{\mathcal{A}}) \ , \ (p, {q_{\mathcal{A}}}^{++})\big) & \stackrel{\mathsf{O}}{\to} \quad \big((p, (a,b), \gamma_{\mathcal{A}}) \ , \ (p, {q_{\mathcal{A}}}^{++})\big) \\ \stackrel{\mathsf{P}}{\to} \quad \big((p, (a,b), \gamma_{\mathcal{A}}) \ , \ (p, (a,b), {\gamma_{\mathcal{A}}}^{+b})\big) & \stackrel{\mathsf{O}}{\to} \quad \Big((p, (a,b), \gamma_{\mathcal{A}}) \ , \ (p \cdot d, {q'_{\mathcal{A}}}^{+b})\Big) \\ & \stackrel{\mathsf{P}}{\to} \quad \Big((p \cdot d, q'_{\mathcal{A}}) \ , \ (p \cdot d, {q'_{\mathcal{A}}}^{+b})\Big) \quad \in \quad \eta_{\mathcal{A}} \end{array}$$

▶ **Lemma 14.6.**

$$\eta_{\mathcal{A}} \quad : \quad \mathcal{A} \quad \longrightarrow_{\widehat{\mathbf{Aut}}^{(\mathrm{W/R})}_{\Sigma\times\Gamma}} \quad (\amalg\mathcal{A})[\pi]$$

14.3.2 The Unique Lifting Property (23)

We now discuss the lifting property (23). Consider some

$$\sigma \quad : \quad \Sigma \times \Gamma \vdash \mathcal{A} \quad \longrightarrow \quad \Sigma \times \Gamma \vdash \mathcal{B}[\pi]$$

We will define

$$\tau \quad : \quad \Sigma \vdash \amalg \mathcal{A} \quad \longrightarrow \quad \Sigma \vdash \mathcal{B}$$

such that

$$\mathrm{HS}(\tau) \quad = \quad \{(\wp(\amalg)(s), \wp(\pi)(t)) \mid (s,t) \in \mathrm{HS}(\sigma)\}$$

We define τ by induction on plays

$$s \in \sigma \subseteq \wp^{\mathsf{P}}_{\Sigma \times \Gamma}(\mathcal{A} \multimap\!\circledast\, \mathcal{B}[\pi])$$

For the base case, we let

$$((\varepsilon, q^{\imath}_{\mathcal{A}}) \ , \ (\varepsilon, q^{\imath}_{\mathcal{B}})) \quad \in \quad \tau$$

Consider now

$$\begin{array}{ccccccc} s & : & * & \to^* & ((p, q_{\mathcal{A}}) \ , \ (p, q_{\mathcal{B}})) & \in & \sigma \\ \tilde{s} & : & * & \to^* & ((p, {q_{\mathcal{A}}}^{++}) \ , \ (p, q_{\mathcal{B}})) & \in & \tau \end{array}$$

with $\mathrm{HS}(\tilde{s}) = (\wp(\amalg)(u), \wp(\pi)(v))$ for $\mathrm{HS}(s) = (u, v)$.
Assume

$$(u \to (p, (a,b), \gamma_{\mathcal{A}}) \ , \ v \to (p, (a,b), \gamma_{\mathcal{B}})) \quad \in \quad \mathrm{HS}(\sigma)$$

so that

$$\begin{array}{l} s' \quad : \quad * \quad \to^* \quad ((p, q_{\mathcal{A}}) \ , \ (p, q_{\mathcal{B}})) \quad \xrightarrow{\mathsf{O}} \quad ((p, (a,b), \gamma_{\mathcal{A}}) \ , \ (p, q_{\mathcal{B}})) \\ \qquad\qquad \xrightarrow{\mathsf{P}} \quad ((p, (a,b), \gamma_{\mathcal{A}}) \ , \ (p, (a,b), \gamma_{\mathcal{B}})) \quad \in \quad \sigma \end{array}$$

We put

$$\begin{array}{l} \tilde{s}' \quad : \quad * \quad \to^* \quad ((p, {q_{\mathcal{A}}}^{++}) \ , \ (p, q_{\mathcal{B}})) \quad \xrightarrow{\mathsf{O}} \quad ((p, a, {\gamma_{\mathcal{A}}}^{+b}) \ , \ (p, q_{\mathcal{B}})) \\ \qquad\qquad \xrightarrow{\mathsf{P}} \quad ((p, a, {\gamma_{\mathcal{A}}}^{+b}) \ , \ (p, a, \gamma_{\mathcal{B}})) \quad \in \quad \tau \end{array}$$

Assume moreover

$$\Big(u \to (p, (a,b), \gamma_{\mathcal{A}}) \to (p \cdot d, {q'_{\mathcal{A}}}^{+b}) \ , \ v \to (p, (a,b), \gamma_{\mathcal{B}}) \to (p \cdot d, q'_{\mathcal{B}})\Big) \quad \in \quad \mathrm{HS}(\sigma)$$

so that

$$\begin{array}{l} s'' \quad : \quad * \quad \to^* \quad ((p, q_{\mathcal{A}}) \ , \ (p, q_{\mathcal{B}})) \quad \xrightarrow{\mathsf{O}} \quad ((p, (a,b), \gamma_{\mathcal{A}}) \ , \ (p, q_{\mathcal{B}})) \\ \quad \xrightarrow{\mathsf{P}} \quad ((p, (a,b), \gamma_{\mathcal{A}}) \ , \ (p, (a,b), \gamma_{\mathcal{B}})) \quad \xrightarrow{\mathsf{O}} \quad ((p, (a,b), \gamma_{\mathcal{A}}) \ , \ (p \cdot d, q'_{\mathcal{B}})) \\ \qquad\qquad \xrightarrow{\mathsf{P}} \quad ((p \cdot d, q'_{\mathcal{A}}) \ , \ (p \cdot d, q'_{\mathcal{B}})) \quad \in \quad \sigma \end{array}$$

We then put

$$\begin{array}{lllllll} \tilde{s}'' & : & * & \to^* & ((p, q_{\mathcal{A}}{}^{++})\ ,\ (p, q_{\mathcal{B}})) & \xrightarrow{\mathsf{O}} & ((p, a, \gamma_{\mathcal{A}}{}^{+b})\ ,\ (p, q_{\mathcal{B}})) \\ \multicolumn{7}{l}{\quad \xrightarrow{\mathsf{P}} \quad ((p, a, \gamma_{\mathcal{A}}{}^{+b})\ ,\ (p, a, \gamma_{\mathcal{B}})) \quad \xrightarrow{\mathsf{O}} \quad ((p, a, \gamma_{\mathcal{A}}{}^{+b})\ ,\ (p \cdot d, q'_{\mathcal{B}}))} \\ \multicolumn{7}{r}{\xrightarrow{\mathsf{P}} \quad ((p \cdot d, q'_{\mathcal{A}}{}^{+b})\ ,\ (p \cdot d, q'_{\mathcal{B}})) \quad \in \quad \tau} \end{array}$$

This completes the definition of τ. It easy to see that τ is indeed a strategy. For P-determinism, note that if τ contains

$$\begin{array}{lllllll} s & : & * & \to^* & ((p, a, \gamma_{\mathcal{A}}{}^{+b})\ ,\ u) & \xrightarrow{\mathsf{P}} & v \\ s' & : & * & \to^* & ((p, a, \gamma'_{\mathcal{A}}{}^{+b'})\ ,\ u') & \xrightarrow{\mathsf{P}} & v' \end{array}$$

then since $\mathcal{A}$ is complete, we have

$$\gamma_{\mathcal{A}}{}^{+b} \quad = \quad \gamma'_{\mathcal{A}}{}^{+b'} \qquad \Longrightarrow \qquad (\gamma_{\mathcal{A}} = \gamma'_{\mathcal{B}} \quad \text{and} \quad b = b')$$

Moreover,

▶ **Lemma 14.7.** *If*

$$\sigma \quad : \quad \mathcal{A} \quad \longrightarrow_{\widehat{\mathbf{Aut}}_{\Sigma\times\Gamma}^{(\mathrm{W/R})}} \quad \mathcal{B}[\pi]$$

then

$$\tau \quad : \quad \amalg(\mathcal{A}) \quad \longrightarrow_{\widehat{\mathbf{Aut}}_{\Sigma}^{(\mathrm{W/R})}} \quad \mathcal{B}$$

We now check that τ satisfies the lifting property (23):

▶ **Lemma 14.8.** $\sigma = \pi^*(\tau) \circ \eta_{\mathcal{A}}$

Proof. Thanks to Lem.4.6.(ii) (or Cor. 6.7), Prop. 4.10 and Lem. 8.3 we just have to check

$$\mathrm{HS}(\sigma) \quad = \quad ((\wp(\pi) \times \wp(\pi))^{-1} \circ \mathrm{HS}(\tau)) \circ \mathrm{HS}(\eta_{\mathcal{A}})$$

where $(\wp(\pi) \times \wp(\pi))^{-1}$ is defined as in Lem. 8.3.

If $(t, u) \in \mathrm{HS}(\sigma)$, then by construction $(\wp(\amalg)(t), \wp(\pi)(u)) \in \mathrm{HS}(\tau)$ hence for all t' such that $\wp(\pi)(t') = \wp(\amalg)(t)$ and $\mathrm{tr}(t') = \mathrm{tr}(u)$, we have

$$(t', u) \in (\wp(\pi) \times \wp(\pi))^{-1} \circ \mathrm{HS}(\tau)$$

On the other hand,

$$(t, \iota_{\Sigma,\Gamma} \circ \wp(\amalg)(t)) \in \mathrm{HS}(\eta_{\mathcal{A}})$$

And we are done by taking $t' := \iota_{\Sigma,\Gamma} \circ \wp(\amalg)(t)$ since

$$\wp(\pi) \circ \iota_{\Sigma,\Gamma} \circ \wp(\amalg)(t) \quad = \quad \wp(\amalg)(t) \qquad \text{and} \qquad \mathrm{tr}(\iota_{\Sigma,\Gamma} \circ \wp(\amalg)(t)) = \mathrm{tr}(t) = \mathrm{tr}(u)$$

Conversely, assume

$$(t, u) \in ((\wp(\pi) \times \wp(\pi))^{-1} \circ \mathrm{HS}(\tau)) \circ \mathrm{HS}(\eta_{\mathcal{A}})$$

so that

$$(t, \iota_{\Sigma,\Gamma} \circ \wp(\amalg)(t)) \in \mathrm{HS}(\eta_{\mathcal{A}})$$

and

$$(\iota_{\Sigma,\Gamma} \circ \wp(\amalg)(t), u) \in ((\wp(\pi) \times \wp(\pi))^{-1} \circ \mathrm{HS}(\tau))$$

It follows that

$$(\wp(\amalg)(t), \wp(\pi)(u)) \in \mathrm{HS}(\tau)$$

Hence there are $(t', u') \in \mathrm{HS}(\sigma)$ such that

$$\wp(\amalg)(t) = \wp(\amalg)(t') \qquad \wp(\pi)(u) = \wp(\pi)(u')$$

But by Lem. 14.3 this implies $t = t'$ and since $\mathrm{tr}(t) = \mathrm{tr}(u)$ and $\mathrm{tr}(t') = \mathrm{tr}(u')$, we also get $\mathrm{tr}(u) = \mathrm{tr}(u')$, hence $u = u'$ by Lem. 7.6 and we are done. ◀

For the unicity part of the lifting property of $\eta_{\mathcal{A}}$, it is sufficient to check:

▶ **Lemma 14.9.** *If $\pi^*(\theta) \circ \eta_{\mathcal{A}} = \pi^*(\theta') \circ \eta_{\mathcal{A}}$ then $\theta = \theta'$.*

Proof. Reasonning as in the proof of Lem. 14.8, thanks to Lem.ii.(ii) (or Cor. 6.7), Prop. 4.10 and Lem. 8.3, we just have to check

$$\begin{array}{rcl} ((\wp(\pi) \times \wp(\pi))^{-1} \circ \mathrm{HS}(\theta)) \circ \mathrm{HS}(\eta_{\mathcal{A}}) & = & ((\wp(\pi) \times \wp(\pi))^{-1} \circ \mathrm{HS}(\theta')) \circ \mathrm{HS}(\eta_{\mathcal{A}}) \\ & & \Longrightarrow \quad \mathrm{HS}(\theta) \quad = \quad \mathrm{HS}(\theta') \end{array}$$

where $(\wp(\pi) \times \wp(\pi))^{-1}$ is defined as in Lem. 8.3.

Let $(t, u) \in \mathrm{HS}(\theta)$, so that, for all $t', u' \in \wp_{\Sigma\times\Gamma}(\amalg(\mathcal{A})[\pi] \multimap\!\circledast\, \mathcal{B}[\pi])$ with

$$\wp(\pi)(t') = t \qquad \wp(\pi)(u') = u \qquad \mathrm{tr}(t') = \mathrm{tr}(u')$$

we have

$$(t', u') \in ((\wp(\pi) \times \wp(\pi))^{-1} \circ \mathrm{HS}(\theta))$$

On the other hand, for all $t'' \in \wp_{\Sigma\times\Gamma}(\mathcal{A})$, we have

$$(t'', \iota_{\Sigma,\Gamma} \circ \wp(\amalg)(t'')) \in \mathrm{HS}(\eta_{\mathcal{A}})$$

Taking $t'' := \wp(\amalg)^{-1}(t)$ we get

$$(\wp(\amalg)^{-1}(t), \iota_{\Sigma,\Gamma}(t)) \in \mathrm{HS}(\eta_{\mathcal{A}})$$

Let now $t' := \iota_{\Sigma,\Gamma}(t)$. Note that $\mathrm{tr}(\wp(\pi)(t')) = \mathrm{tr}(t) = \mathrm{tr}(u)$ since $\wp(\pi)(t') = t$, and then $\mathrm{Tr}(\pi) \circ \mathrm{tr}(t') = \mathrm{tr}(u)$ by Lem. 7.5. Hence, thanks to Lem. 7.6, there is $u' \in \wp(\pi)^{-1}(u)$ with $\mathrm{tr}(u') = \mathrm{tr}(t')$. Since $\wp(\pi) \circ \iota_{\Sigma,\Gamma}(t) = t$ we obtain

$$(\wp(\amalg)^{-1}(t), u') \in ((\wp(\pi) \times \wp(\pi))^{-1} \circ \mathrm{HS}(\theta)) \circ \mathrm{HS}(\eta_{\mathcal{A}})$$

Hence

$$(\wp(\amalg)^{-1}(t), u') \in ((\wp(\pi) \times \wp(\pi))^{-1} \circ \mathrm{HS}(\theta')) \circ \mathrm{HS}(\eta_{\mathcal{A}})$$

It follows that

$$(t''', u') \in ((\wp(\pi) \times \wp(\pi))^{-1} \circ \mathrm{HS}(\theta'))$$

for some t''' such that

$$t''' = \iota_{\Sigma,\Gamma} \circ \wp(\amalg) \circ \wp(\amalg)^{-1}(t) = \iota_{\Sigma,\Gamma}(t)$$

It follows that

$$(\wp(\pi) \circ \iota_{\Sigma,\Gamma}(t), \wp(\pi)(u')) \in \mathrm{HS}(\theta')$$

and thus $(t, u) \in \mathrm{HS}(\theta')$. ◀

Thanks to [13, Thm. IV.1.2.(ii)], we thus get:

▶ **Proposition 14.10.** *For each projection $\pi \in \mathbf{Alph}[\Sigma \times \Gamma, \Sigma]$, we have in $\widehat{\mathbf{Aut}}^{(\mathrm{W/R})}$ an adjunction*

$$\amalg_{\Sigma,\Gamma} \dashv \pi$$

14.4 The Beck-Chevalley Condition

We now check that the adjunction $\amalg \dashv \pi^*$ is preserved by substitution, in the sense of the Beck-Chevalley condion. We therefore have to check (24) and (25),

$$\Delta \vdash (\amalg_{\Sigma,\Gamma}\mathcal{A})[\beta] \quad = \quad \Delta \vdash \amalg_{\Delta,\Gamma}(\mathcal{A}[\beta \times \mathrm{Id}_\Gamma])$$

and

$$\begin{array}{rcl} \eta_{\mathcal{A}[\beta \times \mathrm{Id}_\Gamma]} & = & (\beta \times \mathrm{Id}_\Gamma)^*(\eta_{\mathcal{A}}) \\ & & : \quad \Delta \times \Gamma \vdash \mathcal{A}[\beta \times \mathrm{Id}_\Gamma] \quad \longrightarrow \quad \Delta \times \Gamma \vdash (\amalg_{\Delta,\Gamma}\mathcal{A}[\beta \times \mathrm{Id}_\Gamma])[\pi'] \end{array}$$

given $\beta \in \mathbf{Alph}[\Delta, \Sigma]$ and $\Sigma \times \Gamma \vdash \mathcal{A}$.

▶ **Lemma 14.11.**

$$\Delta \vdash (\amalg_{\Sigma,\Gamma}\mathcal{A})[\beta] \quad = \quad \Delta \vdash \amalg_{\Delta,\Gamma}(\mathcal{A}[\beta \times \mathrm{Id}_\Gamma])$$

Proof. It is sufficient to check the equality of the corresponding transition functions. We have

$$\gamma \in \delta_{(\amalg_{\Sigma,\Gamma}\mathcal{A})[\beta]}(q, c)$$

iff

$$\gamma \in \delta_{(\amalg_{\Sigma,\Gamma}\mathcal{A})}(q, \beta(c))$$

iff

$$\gamma' \in \delta_{\mathcal{A}}(q, (\beta(c), b')) \qquad \text{for some } b' \in \Gamma \text{ and } \gamma' \text{ s.t. } \gamma = \gamma'^{+b'}$$

iff

$$\gamma' \in \delta_{\mathcal{A}[\beta \times \mathrm{Id}_\Gamma]}(q, (c, b')) \qquad \text{for some } b' \in \Gamma \text{ and } \gamma' \text{ s.t. } \gamma = \gamma'^{+b'}$$

iff

$$\gamma \in \delta_{\amalg_{\Sigma,\Gamma}(\mathcal{A}[\beta \times \mathrm{Id}_\Gamma])}(q, c)$$

◀

For the equation (25) (preservation of η by substitution), we use the following preliminary lemma:

▶ **Lemma 14.12.**

$$\iota_{\Sigma\times\Gamma} \circ \wp(\text{II}) \circ \wp(\beta \times \text{Id}_\Gamma) \quad = \quad \wp(\beta \times \text{Id}_\Gamma) \circ \iota_{\Delta\times\Gamma} \circ \wp(\text{II})$$

Proof. By induction on plays $s \in \wp_{\Delta\times\Gamma}(\mathcal{A}[\beta \times \text{Id}_\Gamma])$. The only non-trivial case is that of

$$s \quad : \quad * \quad \to^* \quad (p, q) \quad \to \quad (p, (c, b), \gamma)$$

with

$$\gamma \in \delta(q, (\beta(c), b))$$

But then we have

$$\begin{array}{lll} & \iota_{\Sigma\times\Gamma} \circ \wp(\text{II}) \circ \wp(\beta \times \text{Id}_\Gamma)(s) & \\ = & \iota_{\Sigma\times\Gamma} \circ \wp(\text{II}) \circ \wp(\beta \times \text{Id}_\Gamma)(* \to (p, q)) \quad \to & \iota_{\Sigma\times\Gamma} \circ \wp(\text{II})(p, (\beta(c), b), \gamma) \\ = & \iota_{\Sigma\times\Gamma} \circ \wp(\text{II}) \circ \wp(\beta \times \text{Id}_\Gamma)(* \to (p, q)) \quad \to & \iota_{\Sigma\times\Gamma}(p, \beta(c), \gamma^{+b}) \\ = & \iota_{\Sigma\times\Gamma} \circ \wp(\text{II}) \circ \wp(\beta \times \text{Id}_\Gamma)(* \to (p, q)) \quad \to & (p, (\beta(c), b), \gamma^{+b}) \end{array}$$

and

$$\begin{array}{lll} & \wp(\beta \times \text{Id}_\Gamma) \circ \iota_{\Delta\times\Gamma} \circ \wp(\text{II})(s) & \\ = & \wp(\beta \times \text{Id}_\Gamma) \circ \iota_{\Delta\times\Gamma} \circ \wp(\text{II})(* \to (p, q)) \quad \to & \wp(\beta \times \text{Id}_\Gamma) \circ \iota_{\Delta\times\Gamma}(p, c, \gamma^{+b}) \\ = & \wp(\beta \times \text{Id}_\Gamma) \circ \iota_{\Delta\times\Gamma} \circ \wp(\text{II})(* \to (p, q)) \quad \to & \wp(\beta \times \text{Id}_\Gamma)(p, (c, b), \gamma^{+b}) \\ = & \wp(\beta \times \text{Id}_\Gamma) \circ \iota_{\Delta\times\Gamma} \circ \wp(\text{II})(* \to (p, q)) \quad \to & (p, (\beta(c), b), \gamma^{+b}) \end{array}$$

and we are done by induction hypothesis. ◀

▶ **Lemma 14.13.**

$$\eta_{\mathcal{A}[\beta\times\text{Id}_\Gamma]} \quad = \quad (\beta \times \text{Id}_\Gamma)^*(\eta_\mathcal{A})$$

Proof. Thanks to Lem.4.6.(ii) (or Cor. 6.7) and Lem. 8.3 we just have to check

$$\text{HS}(\eta_{\mathcal{A}[\beta\times\text{Id}_\Gamma]}) \quad = \quad ((\wp(\beta \times \text{Id}_\Gamma) \times \wp(\beta \times \text{Id}_\Gamma))^{-1} \circ \text{HS}(\eta_\mathcal{A}))$$

where $(\wp(\beta \times \text{Id}_\Gamma) \times \wp(\beta \times \text{Id}_\Gamma))^{-1}$ is defined as in Lem. 8.3.

Assume $(s, u) \in \text{HS}(\eta_{\mathcal{A}[\beta\times\text{Id}_\Gamma]})$, so that

$$u \quad = \quad \iota_{\Delta,\Gamma} \circ \wp(\text{II})(s)$$

We must show

$$(\wp(\beta \times \text{Id}_\Gamma)(s) \ , \ \wp(\beta \times \text{Id}_\Gamma)(u)) \in \text{HS}(\eta_\mathcal{A})$$

that is

$$\wp(\beta \times \text{Id}_\Gamma)(u) \quad = \quad \iota_{\Sigma,\Gamma} \circ \wp(\text{II}) \circ \wp(\beta \times \text{Id}_\Gamma)(s)$$

and we conclude by Lem. 14.12.

Conversely, consider

$$(s, u) \in ((\wp(\beta \times \text{Id}_\Gamma) \times \wp(\beta \times \text{Id}_\Gamma))^{-1} \circ \text{HS}(\eta_\mathcal{A}))$$

that is

$$\wp(\beta \times \mathrm{Id}_\Gamma)(u) \quad = \quad \iota_{\Sigma,\Gamma} \circ \wp(\coprod) \circ \wp(\beta \times \mathrm{Id}_\Gamma)(s)$$

We must show

$$u \quad = \quad \iota_{\Delta,\Gamma} \circ \wp(\coprod)(s)$$

By Lem. 14.12 we have

$$\wp(\beta \times \mathrm{Id}_\Gamma)(u) \quad = \quad \wp(\beta \times \mathrm{Id}_\Gamma) \circ \iota_{\Delta,\Gamma} \circ \wp(\coprod)(s)$$

Moreover, we have $\mathrm{tr}(u) = \mathrm{tr}(s)$, and since

$$\mathrm{tr}(s) \quad = \quad \mathrm{tr}(\iota_{\Delta,\Gamma} \circ \wp(\coprod)(s))$$

by Lem. 7.6 we get

$$u \quad = \quad \iota_{\Delta,\Gamma} \circ \wp(\coprod)(s)$$

◀

14.5 Relation with Existential Quantification in $\mathbf{Set}^{\to}$

Let $\pi := \pi_{\Sigma,\Gamma} \in \mathbf{Alph}[\Sigma \times \Gamma, \Sigma]$. Following Sect. 7.4, the map

$$\mathrm{Tr}(\pi) \quad : \quad \mathrm{Tr}_{\Sigma\times\Gamma} \quad \longrightarrow \quad \mathrm{Tr}_\Sigma$$

induces a change-of-base functor

$$\pi^\bullet \quad : \quad \mathbf{Set}/\mathrm{Tr}_\Sigma \quad \longrightarrow \quad \mathbf{Set}/\mathrm{Tr}_{\Sigma\times\Gamma}$$

Recall from Cor. 7.14 that $\pi^\bullet$ is actually isomorphic to the action of the substitution functor π^*:

$$\pi^\bullet(\wp_\Sigma(\mathcal{A}) \xrightarrow{\mathrm{tr}} \mathrm{Tr}_\Sigma) \quad \simeq \quad \pi^*(\wp_\Sigma\mathcal{A}) \xrightarrow{\mathrm{tr}} \mathrm{Tr}_{\Sigma\times\Gamma} \qquad \text{where} \quad \pi^*(\wp_\Sigma\mathcal{A}) = \wp_{\Sigma\times\Gamma}(\mathcal{A}[\pi])$$

Consider, in $\mathbf{Set}^{\to}$, existential quantification along

$$\pi^\bullet \quad : \quad \mathbf{Set}/\mathrm{Tr}_\Sigma \quad \longrightarrow \quad \mathbf{Set}/\mathrm{Tr}_{\Sigma\times\Gamma}$$

It is given by the functor

$$\coprod_\pi \quad : \quad \mathbf{Set}/\mathrm{Tr}_{\Sigma\times\Gamma} \quad \longrightarrow \quad \mathbf{Set}/\mathrm{Tr}_\Sigma$$

whose action on objects is

$$(A \xrightarrow{f} \mathrm{Tr}_{\Sigma\times\Gamma}) \quad \mapsto \quad (A \xrightarrow{\mathrm{Tr}(\pi)\circ f} \mathrm{Tr}_\Sigma)$$

and on morphisms

$$\begin{tikzcd}
A \arrow[rr, "h"] & & B & \mapsto & A \arrow[rr, "h"] & & B \\
A \arrow[rr, "h"] \arrow[dr, "f"'] & & B \arrow[dl, "g"] & & A \arrow[rr, "h"] \arrow[dr, "\mathrm{Tr}(\pi)\circ f"'] & & B \arrow[dl, "\mathrm{Tr}(\pi)\circ g"] \\
& \mathrm{Tr}_{\Sigma\times\Gamma} & & & & \mathrm{Tr}_\Sigma &
\end{tikzcd}$$

It is well known (see e.g. [12, Prop. 1.9.8, p. 99]) that $\amalg_\pi$ is an existential quantification for the codomain fibration $\mathsf{cod} : \mathbf{Set}^{\to} \to \mathbf{Set}$. In particular, $\amalg_\pi$ is left adjoint to $\pi^\bullet$ (and hence to π^*):

$$\amalg_\pi \dashv \pi^*$$

and moreover the Beck-Chevalley condition is satisfied (we come back on this point in Sect. 14.4).

The action *on plays* of the lifted projection of automata $\amalg_{\Sigma,\Gamma}$ is very close to that of $\amalg_\pi$. First,

$$\wp(\amalg) \quad : \quad \wp_{\Sigma\times\Gamma}(\mathcal{A}) \quad \longrightarrow \quad \wp_\Sigma(\amalg_{\Sigma,\Gamma}\mathcal{A})$$

is a bijection by Cor. 14.5. Thanks to Lem. 14.2, we thus have, in $\mathbf{Set}/\mathrm{Tr}_\Sigma$,

$$\begin{tikzcd}
\wp_{\Sigma\times\Gamma}(\mathcal{A}) \arrow[rr, "\simeq", "\wp(\amalg)"'] \arrow[dr, "\mathrm{Tr}(\pi)\mathrm{o}\mathrm{tr}"'] & & \wp_\Sigma(\amalg_{\Sigma,\Gamma}\mathcal{A}) \arrow[dl, "\mathrm{tr}"] \\
& \mathrm{Tr}_\Sigma &
\end{tikzcd}$$

Hence:

▶ **Corollary 14.14.** *In* $\mathbf{Set}/\mathrm{Tr}_\Sigma$*:*

$$\amalg_\pi(\wp_{\Sigma\times\Gamma}(\mathcal{A}) \xrightarrow{\mathrm{tr}} \mathrm{Tr}_{\Sigma\times\Gamma}) \quad \simeq \quad \wp_\Sigma(\amalg_{\Sigma,\Gamma}(\mathcal{A})) \xrightarrow{\mathrm{tr}} \mathrm{Tr}_\Sigma$$

14.6 Non-Functoriality of Usual Projection

Given $\Sigma \times \Gamma \vdash \mathcal{A}$, the usual projection (see e.g. [23]) $\Sigma \vdash \widetilde{\amalg}_{\Sigma,\Gamma}\mathcal{A}$ is defined as follows (we leave the subscript implicit):

- $\widetilde{\amalg}\mathcal{A}$ has the same states, initial state and acceptance condition as $\mathcal{A}$.
- Given a state q and $a \in \Sigma$,

$$\delta_{\widetilde{\amalg}\mathcal{A}}(q, a) \quad := \quad \bigcup_{b\in\Gamma} \delta_{\mathcal{A}}(q, (a, b))$$

Let us now discuss the possible action of $\widetilde{\amalg}$ on morphisms. Given

$$\Sigma \times \Gamma \vdash \sigma : \mathcal{A} -\!\circledast\ \mathcal{B}[\pi] \qquad \text{where } \pi \in \mathbf{Alph}[\Sigma \times \Gamma, \Sigma]$$

with $\Sigma \times \Gamma \vdash \mathcal{A}$ and $\Sigma \vdash \mathcal{B}$ we would like to define

$$\Sigma \vdash \widetilde{\amalg}\sigma : \widetilde{\amalg}\mathcal{A} -\!\circledast\ \widetilde{\amalg}(\mathcal{B}[\pi])$$

Consider the two following plays:

$$((p, q_{\mathcal{A}}),\ (p, q_{\mathcal{B}})) \quad \stackrel{\mathsf{O}}{\to} \quad ((p, (a, b), \gamma_{\mathcal{A}}),\ (p, q_{\mathcal{B}})) \quad \stackrel{\mathsf{P}}{\to} \quad ((p, (a, b), \gamma_{\mathcal{A}}),\ (p, (a, b), \gamma_{\mathcal{B}}))$$

$$((p, q_{\mathcal{A}}),\ (p, q_{\mathcal{B}})) \quad \stackrel{\mathsf{O}}{\to} \quad ((p, (a, b'), \gamma_{\mathcal{A}}),\ (p, q_{\mathcal{B}})) \quad \stackrel{\mathsf{P}}{\to} \quad ((p, (a, b'), \gamma_{\mathcal{A}}),\ (p, (a, b'), \gamma'_{\mathcal{B}}))$$

When projecting these two plays on the alphabet Σ, one obtains

$$((p, q_{\mathcal{A}}),\ (p, q_{\mathcal{B}})) \quad \stackrel{\mathsf{O}}{\to} \quad ((p, a, \gamma_{\mathcal{A}}),\ (p, q_{\mathcal{B}})) \quad \stackrel{\mathsf{P}}{\to} \quad ((p, a, \gamma_{\mathcal{A}}),\ (p, a, \gamma_{\mathcal{B}}))$$

$$((p, q_{\mathcal{A}}),\ (p, q_{\mathcal{B}})) \quad \stackrel{\mathsf{O}}{\to} \quad ((p, a, \gamma_{\mathcal{A}}),\ (p, q_{\mathcal{B}})) \quad \stackrel{\mathsf{P}}{\to} \quad ((p, a, \gamma_{\mathcal{A}}),\ (p, a, \gamma'_{\mathcal{B}}))$$

But there two plays are no longer part of a strategy.

A (bad) idea to remedy to this would be to fix a total order on states and $\mathcal{P}(Q \times D)$, and force projection to always take the least available choice. This this is not functorial, since one can compose σ with a strategy $\Sigma \vdash \tau \Vdash \mathcal{B} \multimap\!\circledast\, \mathcal{C}$ which is insensitive to Γ but swaps priorities.

▶ **Example 14.15.** Consider

$$\Sigma \times \Gamma \vdash \sigma : \mathcal{A} \multimap\!\circledast\, \mathcal{B}[\pi]$$

as above, and assume that $\gamma_{\mathcal{B}}$ has priority over $\gamma'_{\mathcal{B}}$, and that

$$\Sigma \vdash \widetilde{\amalg}\sigma : \widetilde{\amalg}\mathcal{A} \multimap\!\circledast\, \widetilde{\amalg}(\mathcal{B}[\pi])$$

contains the play

$$((p, q_{\mathcal{A}}),\ (p, q_{\mathcal{B}})) \quad \xrightarrow{\mathsf{O}} \quad ((p, a, \gamma_{\mathcal{A}}),\ (p, q_{\mathcal{B}})) \quad \xrightarrow{\mathsf{P}} \quad ((p, a, \gamma_{\mathcal{A}}),\ (p, a, \gamma_{\mathcal{B}}))$$

Consider now

$$\Sigma \vdash \tau : \mathcal{B} \multimap\!\circledast\, \mathcal{B}$$

with plays

$$\begin{array}{ccccc} ((p, q_{\mathcal{B}}),\ (p, q_{\mathcal{B}})) & \xrightarrow{\mathsf{O}} & ((p, a, \gamma_{\mathcal{B}}),\ (p, q_{\mathcal{B}})) & \xrightarrow{\mathsf{P}} & ((p, a, \gamma_{\mathcal{B}}),\ (p, a, \gamma'_{\mathcal{B}})) \\ ((p, q_{\mathcal{B}}),\ (p, q_{\mathcal{B}})) & \xrightarrow{\mathsf{O}} & ((p, a, \gamma'_{\mathcal{B}}),\ (p, q_{\mathcal{B}})) & \xrightarrow{\mathsf{P}} & ((p, a, \gamma'_{\mathcal{B}}),\ (p, a, \gamma_{\mathcal{B}})) \end{array}$$

so that

$$\Sigma \times \Gamma \vdash \pi^*(\tau) : \mathcal{B}[\pi] \multimap\!\circledast\, \mathcal{B}[\pi] \qquad \text{where } \pi \in \mathbf{Alph}[\Sigma \times \Gamma, \Sigma]$$

contains

$$\begin{array}{ccccc} ((p, q_{\mathcal{B}}),\ (p, q_{\mathcal{B}})) & \xrightarrow{\mathsf{O}} & ((p, (a,b), \gamma_{\mathcal{B}}),\ (p, q_{\mathcal{B}})) & \xrightarrow{\mathsf{P}} & ((p, (a,b), \gamma_{\mathcal{B}}),\ (p, (a,b), \gamma'_{\mathcal{B}})) \\ ((p, q_{\mathcal{B}}),\ (p, q_{\mathcal{B}})) & \xrightarrow{\mathsf{O}} & ((p, (a,b), \gamma'_{\mathcal{B}}),\ (p, q_{\mathcal{B}})) & \xrightarrow{\mathsf{P}} & ((p, (a,b), \gamma'_{\mathcal{B}}),\ (p, (a,b), \gamma_{\mathcal{B}})) \end{array}$$

as well as

$$\begin{array}{ccccc} ((p, q_{\mathcal{B}}),\ (p, q_{\mathcal{B}})) & \xrightarrow{\mathsf{O}} & ((p, (a,b'), \gamma_{\mathcal{B}}),\ (p, q_{\mathcal{B}})) & \xrightarrow{\mathsf{P}} & ((p, (a,b'), \gamma_{\mathcal{B}}),\ (p, (a,b'), \gamma'_{\mathcal{B}})) \\ ((p, q_{\mathcal{B}}),\ (p, q_{\mathcal{B}})) & \xrightarrow{\mathsf{O}} & ((p, (a,b'), \gamma'_{\mathcal{B}}),\ (p, q_{\mathcal{B}})) & \xrightarrow{\mathsf{P}} & ((p, (a,b'), \gamma'_{\mathcal{B}}),\ (p, (a,b'), \gamma_{\mathcal{B}})) \end{array}$$

We thus have, in

$$\Sigma \times \Gamma \vdash \pi^*(\tau) \circ \sigma \quad : \quad \mathcal{A} \multimap\!\circledast\, \mathcal{B}[\pi]$$

the plays

$$\begin{array}{ccccc} ((p, q_{\mathcal{A}}),\ (p, q_{\mathcal{B}})) & \xrightarrow{\mathsf{O}} & ((p, (a,b), \gamma_{\mathcal{A}}),\ (p, q_{\mathcal{B}})) & \xrightarrow{\mathsf{P}} & ((p, (a,b), \gamma_{\mathcal{A}}),\ (p, (a,b), \gamma'_{\mathcal{B}})) \\ ((p, q_{\mathcal{B}}),\ (p, q_{\mathcal{B}})) & \xrightarrow{\mathsf{O}} & ((p, (a,b'), \gamma_{\mathcal{A}}),\ (p, q_{\mathcal{B}})) & \xrightarrow{\mathsf{P}} & ((p, (a,b'), \gamma_{\mathcal{A}}),\ (p, (a,b'), \gamma_{\mathcal{B}})) \end{array}$$

which project to

$$((p, q_{\mathcal{A}}),\ (p, q_{\mathcal{B}})) \quad \overset{\mathsf{O}}{\to} \quad ((p, a, \gamma_{\mathcal{A}}),\ (p, q_{\mathcal{B}})) \quad \overset{\mathsf{P}}{\to} \quad ((p, a, \gamma_{\mathcal{A}}),\ (p, a, \gamma_{\mathcal{B}}))$$

$$((p, q_{\mathcal{A}}),\ (p, q_{\mathcal{B}})) \quad \overset{\mathsf{O}}{\to} \quad ((p, a, \gamma_{\mathcal{A}}),\ (p, q_{\mathcal{B}})) \quad \overset{\mathsf{P}}{\to} \quad ((p, a, \gamma_{\mathcal{A}}),\ (p, a, \gamma'_{\mathcal{B}}))$$

so that, since $\gamma_{\mathcal{B}}$ has priority over $\gamma'_{\mathcal{B}}$, the strategy

$$\Sigma \vdash \widetilde{\amalg}(\pi^*(\tau) \circ \sigma) \quad : \quad \widetilde{\amalg}(\mathcal{A}) \multimap\!\!\circledast\ \widetilde{\amalg}(\mathcal{B}[\pi])$$

should only contain

$$((p, q_{\mathcal{A}}),\ (p, q_{\mathcal{B}})) \quad \overset{\mathsf{O}}{\to} \quad ((p, a, \gamma_{\mathcal{A}}),\ (p, q_{\mathcal{B}})) \quad \overset{\mathsf{P}}{\to} \quad ((p, a, \gamma_{\mathcal{A}}),\ (p, a, \gamma_{\mathcal{B}}))$$

On the other hand, again since $\gamma_{\mathcal{B}}$ has priority over $\gamma'_{\mathcal{B}}$, the strategy

$$\Sigma \vdash \widetilde{\amalg}\sigma \quad : \quad \widetilde{\amalg}(\mathcal{A}) \multimap\!\!\circledast\ \widetilde{\amalg}(\mathcal{B}[\pi])$$

also contains

$$((p, q_{\mathcal{A}}),\ (p, q_{\mathcal{B}})) \quad \overset{\mathsf{O}}{\to} \quad ((p, a, \gamma_{\mathcal{A}}),\ (p, q_{\mathcal{B}})) \quad \overset{\mathsf{P}}{\to} \quad ((p, a, \gamma_{\mathcal{A}}),\ (p, a, \gamma_{\mathcal{B}}))$$

so that

$$\Sigma \vdash \widetilde{\amalg}(\pi^*(\tau)) \circ \widetilde{\amalg}\sigma \quad : \quad \widetilde{\amalg}(\mathcal{A}) \multimap\!\!\circledast\ \widetilde{\amalg}(\mathcal{B}[\pi])$$

contains

$$((p, q_{\mathcal{A}}),\ (p, q_{\mathcal{B}})) \quad \overset{\mathsf{O}}{\to} \quad ((p, a, \gamma_{\mathcal{A}}),\ (p, q_{\mathcal{B}})) \quad \overset{\mathsf{P}}{\to} \quad ((p, a, \gamma_{\mathcal{A}}),\ (p, a, \gamma'_{\mathcal{B}}))$$

Hence

$$\widetilde{\amalg}(\pi^*(\tau) \circ \sigma) \quad \neq \quad \widetilde{\amalg}(\pi^*(\tau)) \circ \widetilde{\amalg}\sigma$$

14.6.1 On (23) w.r.t. $\widetilde{\amalg}$.

Also when looking at unique lifting property (23), it seems that $\widetilde{\amalg}$ would not have been sufficient. In particular, the counter-example Ex. 14.15 might be adapted to the present situation.

▶ **Example 14.16** (Adapted from Ex. 14.15)**.** Assume that

$$\Sigma \times \Gamma \vdash \sigma : \mathcal{A} \multimap\!\!\circledast\ \mathcal{B}[\pi]$$

contains the plays

$$((p, q_{\mathcal{A}}),\ (p, q_{\mathcal{B}})) \quad \overset{\mathsf{O}}{\to} \quad ((p, (a, b), \gamma_{\mathcal{A}}),\ (p, q_{\mathcal{B}})) \quad \overset{\mathsf{P}}{\to} \quad ((p, (a, b), \gamma_{\mathcal{A}}),\ (p, (a, b), \gamma_{\mathcal{B}}))$$

$$((p, q_{\mathcal{A}}),\ (p, q_{\mathcal{B}})) \quad \overset{\mathsf{O}}{\to} \quad ((p, (a, b'), \gamma_{\mathcal{A}}),\ (p, q_{\mathcal{B}})) \quad \overset{\mathsf{P}}{\to} \quad ((p, (a, b'), \gamma_{\mathcal{A}}),\ (p, (a, b'), \gamma'_{\mathcal{B}}))$$

Then, *via* $\eta_{\mathcal{A}}$, τ would only see

$$((p, q_{\mathcal{A}}),\ (p, q_{\mathcal{B}})) \quad \overset{\mathsf{O}}{\to} \quad ((p, a, \gamma_{\mathcal{A}}),\ (p, q_{\mathcal{B}}))$$

$$((p, q_{\mathcal{A}}),\ (p, q_{\mathcal{B}})) \quad \overset{\mathsf{O}}{\to} \quad ((p, a, \gamma_{\mathcal{A}}),\ (p, q_{\mathcal{B}}))$$

hence, while they should be played by τ, the following plays can not be part of a strategy:

$$((p, q_{\mathcal{A}}),\ (p, q_{\mathcal{B}})) \quad \overset{\mathsf{O}}{\to} \quad ((p, a, \gamma_{\mathcal{A}}),\ (p, q_{\mathcal{B}})) \quad \overset{\mathsf{P}}{\to} \quad ((p, a, \gamma_{\mathcal{A}}),\ (p, a, \gamma_{\mathcal{B}}))$$

$$((p, q_{\mathcal{A}}),\ (p, q_{\mathcal{B}})) \quad \overset{\mathsf{O}}{\to} \quad ((p, a, \gamma_{\mathcal{A}}),\ (p, q_{\mathcal{B}})) \quad \overset{\mathsf{P}}{\to} \quad ((p, a, \gamma_{\mathcal{A}}),\ (p, a, \gamma'_{\mathcal{B}}))$$

15 The Synchronous Arrow and Language Inclusion

In this section, we discuss the *completeness* of the synchronous arrow w.r.t. language inclusion.

15.1 Correctness of Projection on Non-Deterministic Automata

We now check that, on non-deterministic automata, the projection defined in Sect. 14

$$\amalg_{\Sigma,\Gamma} \quad : \quad \mathbf{Aut}_{\Sigma\times\Gamma} \quad \to \quad \mathbf{Aut}_{\Sigma}$$

implements the operation of projection on languages. Recal that the first projection $\pi_{\Sigma,\Gamma} \in \mathbf{Alph}[\Sigma\times\Gamma,\Sigma]$ induces a tree map

$$\pi_{\Sigma,\Gamma} \in \mathbf{Tree}[\Sigma\times\Gamma,\Sigma]$$

Given an automaton $\Sigma\times\Gamma \vdash \mathcal{A}$, we write

$$\pi_{\Sigma,\Gamma}(\mathcal{L}(\mathcal{A}))$$

for the action of $\pi_{\Sigma,\Gamma}$ on $\mathcal{L}(\mathcal{A})$

We now check that the projection operation $\amalg_{\Sigma,\Gamma}$ implements existential quatification on non-deterministic automaton, *i.e.* that for a complete non-deterministic automaton $\Sigma\times\Gamma \vdash \mathcal{N}$, we have

$$\mathcal{L}(\amalg_{\Sigma,\Gamma}\mathcal{N}) \quad = \quad \pi_{\Sigma,\Gamma}(\mathcal{L}(\mathcal{N}))$$

The inclusion

$$\pi_{\Sigma,\Gamma}(\mathcal{L}(\mathcal{N})) \quad \subseteq \quad \mathcal{L}(\amalg_{\Sigma,\Gamma}\mathcal{N})$$

directly follows from the categorical properties of $\amalg$ established in Sect. 14 and doest not require $\mathcal{N}$ to be non-deterministic. Consider a complete automaton $\Sigma\times\Gamma \vdash \mathcal{A}$. The unit $\eta_{\mathcal{A}}$ of the adjunction of Prop. 14.10 gives

$$\Sigma\times\Gamma \Vdash \mathcal{A} \multimap\!\circledast (\amalg_{\Sigma,\Gamma})[\pi]$$

By Prop. 12.7 (a.k.a. Prop. 6.1), it follows that

$$\mathcal{L}(\mathcal{A}) \quad \subseteq \quad \mathcal{L}(\amalg_{\Sigma,\Gamma})[\pi]$$

Hence

$$\pi_{\Sigma,\Gamma}(\mathcal{L}(\mathcal{A})) \quad \subseteq \quad \mathcal{L}(\amalg_{\Sigma,\Gamma}\mathcal{A})$$

Conversely, consider a complete non-deterministic automaton $\Sigma\times\Gamma \vdash \mathcal{N}$, a tree t and a strategy σ such that

$$\mathbf{1} \vdash \sigma \Vdash \mathcal{G}(\amalg_{\Sigma,\Gamma}\mathcal{N}, \dot{t})$$

Then, since $\amalg_{\Sigma,\Gamma}\mathcal{N}$ is non-deterministic and complete, and since σ is total, for each tree position $p \in D^*$, there is exactly one non-empty play s of σ such that $\mathrm{tr}_D(s) = (p)$. Note that s is of the form

$$s \quad : \quad * \quad \to^* \quad (p,q) \quad \xrightarrow{\mathsf{P}} \quad (p,\bullet,\gamma^{+b}) \qquad \text{where} \quad \gamma \in \delta_{\mathcal{N}}(q,(t(p),b)) \tag{27}$$

We therefore define the tree

$$u \quad : \quad p \in D \quad \mapsto \quad (t(p), b)$$

It is then easy, following the usual pattern for projection (see e.g. [23]) to build a strategy τ such that

$$\mathbf{1} \vdash \tau \Vdash \mathcal{G}(\mathcal{N}, \dot{u})$$

We inductivelly associate to each $s \in \sigma$ as in (27) above a play

$$\tilde{s} \quad : \quad * \quad \to^* \quad (p, q) \quad \xrightarrow{\mathsf{P}} \quad (p, \bullet, \gamma) \quad \in \quad \tau$$

First, we let $(\varepsilon, q^\imath_{\mathcal{N}}) \in \tau$. Then given $s' \in \sigma$ of the form

$$s' \quad : \quad s \quad \xrightarrow{\mathsf{O}} \quad (p, q) \quad \xrightarrow{\mathsf{P}} \quad (p, \bullet, \gamma^{+b}) \qquad \text{where} \quad \gamma \in \delta_{\mathcal{N}}(q, (t(p), b))$$

with $\tilde{s} \in \tau$, we put

$$\tilde{s}' \quad : \quad \tilde{s} \quad \xrightarrow{\mathsf{O}} \quad (p, q) \quad \xrightarrow{\mathsf{P}} \quad (p, \bullet, \gamma) \quad \in \quad \tau$$

Note that we then have $\gamma \in \delta_{\mathcal{N}}(q, u(p))$ by definition of u.

We thus have shown:

▶ **Proposition 15.1** (Prop. 7.5)**.** *If $\Sigma \times \Gamma \vdash \mathcal{N}$ is a non-deterministic* complete *automaton, then*

$$\mathcal{L}(\amalg_{\Sigma,\Gamma}\mathcal{N}) \quad = \quad \pi_{\Sigma,\Gamma}(\mathcal{L}(\mathcal{N}))$$

15.2 Completeness w.r.t. Language Inclusion

We now give a result stating that for automata of a specific form, the synchronous arrow $\_ \multimap_{\circledast} \_$ is complete w.r.t. language inclusion. Specifically, we show that given automata $\Sigma \vdash \mathcal{A}$ and $\Sigma \vdash \mathcal{B}$,

$$\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B}) \qquad \Longrightarrow \qquad \Sigma \Vdash \mathrm{ND}(\mathcal{A}) \multimap_{\circledast} {\sim}\mathrm{ND}({\sim}\mathcal{B})$$

▶ **Proposition 15.2** (Prop. 7.6)**.** *Consider regular automata $\Sigma \vdash \mathcal{A}$ with $\Sigma \vdash \mathcal{B}$*
If $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$ then $\Sigma \Vdash \mathrm{ND}(\mathcal{A}) \multimap_{\circledast} \widehat{{\sim}\mathcal{C}}$ for $\mathcal{C} := \mathrm{ND}({\sim}\mathcal{B})$

Note that according to the definition of $\curlywedgedownarrow$ given in Sect. 13.2, given $\pi \in \mathbf{Alph}[\Sigma \times \Gamma, \Sigma]$ we have

$$\curlywedgedownarrow_{\Sigma \times \Gamma} \quad = \quad \curlywedgedownarrow_{\Sigma}[\pi]$$

Proof. Assume $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$. It follows from Prop. 6.4 that $\mathcal{L}(\mathcal{A}) \cap \mathcal{L}({\sim}\mathcal{B}) = \emptyset$, and we get from Prop. 7.4 that $\mathcal{L}(\mathrm{ND}(\mathcal{A})) \cap \mathcal{L}(\mathrm{ND}({\sim}\mathcal{B})) = \emptyset$. By Prop. 6.3, we get $\mathcal{L}(\mathrm{ND}(\mathcal{A}) \circledast \mathrm{ND}({\sim}\mathcal{B})) = \emptyset$.

Since $\mathrm{ND}(\mathcal{A}) \circledast \mathrm{ND}({\sim}\mathcal{B})$ is non-deterministic (Rem. 7), it follows from Prop. 15.1 that $\mathcal{L}(\amalg_{\mathbf{1},\Sigma}(\mathrm{ND}(\mathcal{A}) \circledast \mathrm{ND}({\sim}\mathcal{B}))) = \emptyset$. By Prop. 6.4, we obtain

$$\mathbf{1} \Vdash {\sim}(\amalg_{\mathbf{1},\Sigma}(\mathrm{ND}(\mathcal{A}) \circledast \mathrm{ND}({\sim}\mathcal{B})))$$

and by Cor. 13.4, since $\mathrm{ND}(\mathcal{A}) \circledast \mathrm{ND}({\sim}\mathcal{B})$ is complete, we get

$$\mathbf{1} \Vdash \amalg_{\mathbf{1},\Sigma}(\mathrm{ND}(\mathcal{A}) \circledast \mathrm{ND}({\sim}\mathcal{B})) \multimap_{\circledast} \curlywedgedownarrow_{\mathbf{1}}$$

It then follows from the adjunction $\amalg_{\mathbf{1},\Sigma} \dashv \pi^*$ (Prop. 14.10) that

$$\Sigma \Vdash (\mathrm{ND}(\mathcal{A}) \circledast \mathrm{ND}(\sim\mathcal{B})) \multimap_{\circledast} \curlywedge_{\mathbf{1}}[\pi]$$

and Prop. 13.2.(ii) gives

$$\Sigma \Vdash \mathrm{ND}(\mathcal{A}) \multimap_{\circledast} \widehat{\sim\mathcal{C}})$$

for $\mathcal{C} := \mathrm{ND}(\sim\mathcal{B})$ ◀

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