Semantics and Verification
Homework

This homework consists of two parts, which will be graded independently:

Part I: Questions of §1 and §2. The solutions should arrive latest by the **24th of February, midnight**.

Part II: Questions of §3. The solutions should arrive latest by the **10th of March, midnight**.

The two parts should be returned separetely. The solutions must be sent either in paper or on the "portail des études"

- at https://etudes.ens-lyon.fr/mod/assign/view.php?id=179133 for **Part I**,
- at https://etudes.ens-lyon.fr/mod/assign/view.php?id=179134 for **Part II**.

The questions marked with asterisks ($*$) are more advanced. We refer to the course notes for missing definitions and results from the course. The course notes are available at http://perso.ens-lyon.fr/colin.riba/teaching/sv/notes.pdf.

---

1 Toward Stone Duality

Warning. *In this homework, we always assume that* AP *is a **finite** non-empty set.*

In the course, we devised a topological notion of "observable property", which consists of the Boolean algebra of clopen sets of a topological space. For spaces $(\mathbf{2}^{\mathrm{AP}})^\omega$, this amounts to the Boolean algebra of sets of the form $\mathsf{ext}(W)$ for some **finite** $W \subseteq (\mathbf{2}^{\mathrm{AP}})^*$. We also devised LML, a base modal logic for linear-time properties, whose formulae define exactly the observable properties on $(\mathbf{2}^{\mathrm{AP}})^\omega$. We noted that LML is very weak w.r.t. linear-time properties, and will consider LTL, an extension of LML with a restricted form of least and greatest fixpoints.

We shall see in this homework that spaces $(\mathbf{2}^{\mathrm{AP}})^\omega$ enjoy further topological properties, allowing to recover the whole set $(\mathbf{2}^{\mathrm{AP}})^\omega$ from the Boolean algebra of clopen sets of its topology, *i.e.* from LML.

We use the following notation.

Notation 1.1. *Given a topological space (X, Ω), we let $\mathbf{K}\Omega$ be the set of clopen subsets of X (so $S \in \mathbf{K}\Omega$ iff $S \in \mathcal{P}(X)$ is both closed and open).*

Remark. *Notation 1.1 is not standard, as* $\mathbf{K}\Omega$ *usually designates the compact open subsets of a space* (X, Ω)*. It follows from results of the course that* $\mathbf{K}\Omega$ *is indeed the set of compact open subsets of* X *when* (X, Ω) *is compact Hausdorff.*

It follows from results of the course that in the case of $(\mathbf{2}^{\mathrm{AP}})^\omega$ we have

$$\mathbf{K}\Omega((\mathbf{2}^{\mathrm{AP}})^\omega) \quad = \quad \{\mathsf{ext}(W) \mid W \subseteq (\mathbf{2}^{\mathrm{AP}})^* \text{ is finite}\}$$

Given an ω-word $\sigma \in (\mathbf{2}^{\mathrm{AP}})^\omega$, let

$$\mathcal{F}_\sigma \quad := \quad \{\mathsf{ext}(W) \in \mathbf{K}\Omega((\mathbf{2}^{\mathrm{AP}})^\omega) \mid \sigma \in \mathsf{ext}(W)\}$$

Note that for $\sigma, \beta \in (\mathbf{2}^{\mathrm{AP}})^\omega$, we evidently have $\mathcal{F}_\sigma \neq \mathcal{F}_\beta$ whenever $\sigma \neq \beta$.

In this homework, we shall see that sets of the form $\mathcal{F}_\sigma$ for some $\sigma \in (\mathbf{2}^{\mathrm{AP}})^\omega$ can be completely characterized by purely order-theoretic properties, namely as **prime filters** on the Boolean algebra $\mathbf{K}\Omega((\mathbf{2}^{\mathrm{AP}})^\omega)$. This fact, which is part of **Stone's Representation Theorem**, holds for any **Stone space**.

Definition 1.2 (Stone Space)**.** *A **Stone space** is a topological space* (X, Ω) *which is compact and satisfies the two following conditions:*

(X, Ω) **is** T_0**:** *for any distinct points* $x, y \in X$*, there is an open containing one and not the other, i.e. either there is some* $U \in \Omega$ *such that* $x \in U$ *and* $y \notin U$*, or there is some* $V \in \Omega$ *such that* $x \notin V$ *and* $y \in V$*.*

(X, Ω) **is zero-dimensional:** *the clopen subsets of* X *form a base for the topology, i.e. every* $U \in \Omega$ *is a (possibly infinite or empty) union of clopens.*

Question 1. *Show that every Stone space* (X, Ω) *is Hausdorff (if* $x, y \in X$ *are distinct, then there are disjoint* $U, V \in \Omega$ *such that* $x \in U$ *and* $y \in V$*).* Quest. 1

Example 1.3. *It follows from results of the course that* $(\mathbf{2}^{\mathrm{AP}})^\omega$ *is a Stone space.*

We shall target the following two instances of Stone's Representation Theorem:

- Every Boolean algebra B is isomorphic to the Boolean algebra $\mathbf{K}\Omega(\mathbf{Sp}(B))$ for some Stone space $(\mathbf{Sp}(B), \Omega(\mathbf{Sp}(B)))$, called the **spectrum** of B.
- Every Stone space (X, Ω) is homeomorphic to the spectrum of the Boolean algebra $\mathbf{K}\Omega$.

Stone's Representation Theorem has a logical meaning, w.r.t. which the following notion is pertinent in our context.

Definition 1.4. *We let* $\mathfrak{L}(\mathsf{LML})$ *be the set of closed* LML*-formulae quotiented by logical equivalence* $\equiv$*.*

It follows from results of the course that we can identify the sets $\mathbf{K}\Omega((\mathbf{2}^{\mathrm{AP}})^\omega)$ and $\mathfrak{L}(\mathsf{LML})$.

Notation 1.5. *We shall always notationaly confuse a closed* LML*-formula* φ *with its equivalence class* $[\varphi]_\equiv \in \mathfrak{L}(\mathsf{LML})$*, where, as usual*

$$[\varphi]_\equiv \quad = \quad \{\psi \mid \psi \textit{ is a closed } \mathsf{LML}\textit{-formula such that } \varphi \equiv \psi\}$$

We equip $\mathfrak{L}(\mathsf{LML})$ with the relation

$$\varphi \leq \psi \quad := \quad (\varphi \to \psi) \equiv \top$$

Note that

$$\varphi \leq \psi \quad \text{iff} \quad \varphi \equiv (\varphi \wedge \psi) \quad \text{iff} \quad (\varphi \vee \psi) \equiv \psi$$

Question 2. *Show that* $\leq$ *is a partial order on* $\mathfrak{L}(\mathsf{LML})$ Quest. 2

Remark 1.6 (On Lindenbaum-Tarski Algebras)**.** *The set* $\mathfrak{L}(\mathsf{LML})$ *defined in Def. 1.4 is reminiscent from* ***Lindenbaum-Tarski algebras****. However, Lindenbaum-Tarski algebras are usually defined as the quotient of formulae w.r.t.* ***provable*** *logical equivalence.*

Remark 1.7 (On Filters)**.** *The notion of (prime) filter considered here is the lattice-theoretic one. General topology uses an instance on powersets of the lattice-theoretic notion of filter. However, one must be aware that the algebraic and the topological approaches differ concerning convergence. We follow here the algebraic approach, since it is the relevant one in the context of Stone Duality.*

2 Lattices and Boolean Algebras

In this Section, we discuss an algebraic presentation of (semi)lattices, distributive lattices and Boolean algebras.

2.1 Semilattices

A lattice is a partial order $(L, \leq)$, which, similarly to a complete lattice, has joins and meets. But in contrast with complete lattices, only **finite** joins and meets are required to exist in a lattice. As we shall see, this amounts to ask for binary joins and meets as well as for a least and a greatest element. It turns out that in contrast with complete lattices, we must assume joins and meets separately. This leads to the notions of meet and join semilattices.

Definition 2.1 (Semilattices)**.**

(1) A ***meet semilattice*** *is a partial order having all finite meets (i.e. greatest lower bounds* $\wedge, \top$*).*

(2) A ***join semilattice*** *is a partial order having all finite joins (i.e. least upper bounds* $\vee, \bot$*).*

A meet semilattice can equivalently be defined as a partial order $(L, \leq)$ equipped with binary meets $\wedge : L \times L \to L$ and a greatest element $\top \in L$. Similarly, a join semilattice is a partial order $(L, \leq)$ equipped with binary joins $\vee : L \times L \to L$ and a least element $\bot \in L$.

Question 3. *Let $(L, \leq)$ be a partial order.* Quest. 3

(1) Show that $(L, \leq)$ is a meet semilattice if and only if L has binary meets $\wedge : L \times L \to L$ and a greatest element $\top \in L$.

(2) Show that $(L, \leq)$ is a join semilattice if and only if L has binary joins $\vee : L \times L \to L$ and a least element $\bot \in L$.

We shall now see that the partial order $\leq$ can be recovered from equational axioms on $(L, \wedge, \top)$ and $(L, \vee, \bot)$.

Definition 2.2.

*(1) A **monoid** is a set A equipped with a binary operation $\circledast : A \times A \to A$ and a constant $\mathbf{I} \in A$ such that for all $a, b, c \in A$ we have*

$$a \circledast (b \circledast c) \;=\; (a \circledast b) \circledast c \qquad a \circledast \mathbf{I} \;=\; a \qquad \mathbf{I} \circledast a \;=\; a$$

*(2) A **commutative monoid** is a monoid $(A, \circledast, \mathbf{I})$ such that for all $a, b \in A$ we have*

$$a \circledast b \;=\; b \circledast a$$

*(3) An element $a \in A$ of a monoid $(A, \circledast, \mathbf{I})$ is **idempotent** if*

$$a \circledast a \;=\; a$$

Question 4. *Prove the following.* Quest. 4

(1) Let $(L, \leq)$ be a meet semilattice with binary meets $\wedge : L \times L \to L$ and greatest element $\top \in L$. Then $(L, \wedge, \top)$ is a commutative monoid in which every element is idempotent. Moreover, we have $a \leq b$ iff $a = a \wedge b$.

(2) Let $(L, \leq)$ be a join semilattice with binary joins $\vee : L \times L \to L$ and least element $\bot \in L$. Then $(L, \vee, \bot)$ is a commutative monoid in which every element is idempotent. Moreover, we have $a \leq b$ iff $a \vee b = b$.

Question 5. *Prove the following.* Quest. 5

(1) Given a commutative monoid $(L, \wedge, \top)$ in which every element is idempotent, let $a \leq_\wedge b$ iff $a = a \wedge b$. Then $(L, \leq_\wedge)$ is a meet semilattice with binary meets given by $\wedge$ and with greatest element $\top$.

(2) Given a commutative monoid $(L, \vee, \bot)$ in which every element is idempotent, let $a \leq_\vee b$ iff $a \vee b = b$. Then $(L, \leq_\vee)$ is a join semilattice with binary joins given by $\vee$ and with least element $\bot$.

Question 6. *Show the following, for the partial order $(\mathfrak{L}(\mathsf{LML}), \leq)$ (see Notation 1.5).* Quest. 6

(1) $(\mathfrak{L}(\mathsf{LML}), \leq)$ is a meet semilattice with greatest element $\top$ and with binary meets given by

$$\begin{array}{rclcl} (-) \wedge (-) & : & \mathfrak{L}(\mathsf{LML}) \times \mathfrak{L}(\mathsf{LML}) & \longrightarrow & \mathfrak{L}(\mathsf{LML}) \\ & & (\varphi, \psi) & \longrightarrow & \varphi \wedge \psi \end{array}$$

(2) $(\mathfrak{L}(\mathsf{LML}), \leq)$ is a join semilattice with least element $\bot$ and with binary joins given by

$$\begin{array}{rclcl} (-) \vee (-) & : & \mathfrak{L}(\mathsf{LML}) \times \mathfrak{L}(\mathsf{LML}) & \longrightarrow & \mathfrak{L}(\mathsf{LML}) \\ & & (\varphi, \psi) & \longrightarrow & \varphi \vee \psi \end{array}$$

Definition 2.3 (Semilattice Morphism)**.** *Let $(L, \leq)$ and $(L', \leq')$ be partial orders and let $f : L \to L'$ be a function.*

*(1) If $(L, \leq)$ and $(L', \leq')$ are meet semilattices, then f is a **map of meet semilattices** if it preserves finite meets, i.e. if for all finite $S \subseteq L$ we have*

$$f(\bigwedge S) \quad = \quad \bigwedge{}'\{f(s) \mid s \in S\}$$

*(2) If $(L, \leq)$ and $(L', \leq')$ are join semilattices, then f is a **map of join semilattices** if it preserves finite joins, i.e. if for all finite $S \subseteq L$ we have*

$$f(\bigvee S) \quad = \quad \bigvee{}'\{f(s) \mid s \in S\}$$

Note that $f : L \to L'$ is a map of meet (resp. join) semilattices iff $f(\top) = \top'$ and $f(a \wedge b) = f(a) \wedge' f(b)$ (resp. $f(\bot) = \bot'$ and $f(a \vee b) = f(a) \vee' f(b)$).

Question 7. *Show that a map of meet (resp. join) semilattices is monotone.* Quest. 7

2.2 Lattices

Definition 2.4 (Lattice)**.** *A **lattice** is a partial order having all finite joins and all finite meets.*

Of course, a **finite** join (resp. meet) semilattice has all joins (resp. all meets), and is thus a (complete) lattice. But this does not hold for **infinite** semilattices.

Question 8. *Consider the partial order $(L, \sqsubseteq)$ where* Quest. 8

$$L \quad := \quad \mathbb{N} \cup \{\alpha, \beta, \top\}$$

and where $\sqsubseteq$ is the reflexive-transitive closure of $\sqsubset$, where

$$a \sqsubset b \quad \textit{iff} \quad \begin{cases} a < b \text{ in } \mathbb{N}, \text{ or} \\ a \in \mathbb{N} \text{ and } b \in \{\alpha, \beta\}, \text{ or} \\ a \in \{\alpha, \beta\} \text{ and } b = \top \end{cases}$$

Show that $(L, \sqsubseteq)$ is a join-semilattice but is not a lattice.

The results of §2.1 give a purely algebraic characterization of lattices.

Question 9. *Consider a set L equipped with two binary operations $\wedge, \vee : L \times L \to L$ and two constants $\top, \bot \in L$. Assume that $(L, \wedge, \top)$ and $(L, \vee, \bot)$ are commutative monoids in which every element is idempotent. Show that the following are equivalent:* Quest. 9

(i) The partial order $\leq_\vee$ induced by $(L, \vee, \bot)$ coincides with the partial order $\leq_\wedge$ induced by $(L, \wedge, \top)$.

(ii) $(L, \vee, \wedge, \bot, \top)$ satisfies the two following ***absorptive laws****:*

$$\begin{array}{rcl} a \vee (a \wedge b) & = & a \\ a \wedge (a \vee b) & = & a \end{array}$$

As a consequence, if $(L, \vee, \wedge, \bot, \top)$ satisfies either of the equivalent conditions of Question 9, then, for $\leq = \leq_\wedge = \leq_\vee$, $(L, \leq)$ is a lattice with finite meets given by $(\wedge, \top)$ and with finite joins given by $(\vee, \bot)$.

Question 10. *Show that the partial order $(\mathfrak{L}(\mathsf{LML}), \leq)$ is a lattice.* Quest. 10

Definition 2.5 (Lattice Morphism)**.** *Given lattices $(L, \leq)$ and $(L', \leq')$, a function $f : L \to L'$ is a* ***morphisms of lattices*** *if f is both a map of meet and join semilattices from $(L, \leq)$ to $(L', \leq')$.*

Question 11. *Show that the function* Quest. 11

$$\begin{array}{rcccl} \bigcirc & : & \mathfrak{L}(\mathsf{LML}) & \longrightarrow & \mathfrak{L}(\mathsf{LML}) \\ & & \varphi & \longmapsto & \bigcirc\varphi \end{array}$$

is a morphism of lattices.

2.3 Distributive Lattices

Question 12. *Show that the following two* ***distributive laws*** *are equivalent in a lattice $(L, \vee, \wedge, \bot, \top)$:* Quest. 12

$$\begin{array}{rrcl} \forall a, b, c \in L, & a \wedge (b \vee c) & = & (a \wedge b) \vee (a \wedge c) \\ \forall a, b, c \in L, & a \vee (b \wedge c) & = & (a \vee b) \wedge (a \vee c) \end{array}$$

Definition 2.6 (Distributive Lattice)**.** *A lattice is* ***distributive*** *if it satisfies either of the distributive laws of Question 12.*

Question 13. *Show that the lattice $(\mathfrak{L}(\mathsf{LML}), \leq)$ is distributive.* Quest. 13

Question 14. *Consider the following lattice M_3:* Quest. 14

$$\begin{tikzcd} & \top & \\ a \arrow[ur, dash] & b \arrow[u, dash] & c \arrow[ul, dash] \\ & \bot \arrow[ul, dash] \arrow[u, dash] \arrow[ur, dash] & \end{tikzcd}$$

(i.e. $\bot \leq a, b, c \leq \top$, with a, b, c incomparable). Show that M_3 is not distributive.

2.4 Boolean Algebras

Definition 2.7. *Let $(L, \vee, \wedge, \bot, \top)$ be a lattice. We say that $c \in L$ is a **complement** of $a \in L$ if $a \vee c = \top$ and $a \wedge c = \bot$.*

Question 15. *Show that if $(L, \leq)$ is a distributive lattice, then $a \in L$ has at most one complement.* Quest. 15

Definition 2.8 (Boolean Algebra)**.** *A **Boolean algebra** is a distributive lattice in which every element b has a (necessarily unique) complement $\neg b$.*

As expected:

Example 2.9. *Given a topological space (X, Ω), the clopens $(\mathbf{K}\Omega, \subseteq)$ form a Boolean algebra.*

Question 16. *Show that $(\mathfrak{L}(\mathsf{LML}), \leq)$ is a Boolean algebra.* Quest. 16

Question 17. *Show that the following **De Morgan Laws** hold in every Boolean algebra $(B, \vee, \wedge, \bot, \top)$:* Quest. 17

$$a \wedge b \;=\; \neg(\neg a \vee \neg b) \qquad a \vee b \;=\; \neg(\neg a \wedge \neg b) \qquad a \;=\; \neg\neg a$$

It would be natural to ask morphisms of Boolean algebras to preserve all the structure (finite meets, joins and complements). It is actually sufficient to ask for preservation of meets and joins.

Definition 2.10 (Boolean Algebra Morphism)**.** *Given Boolean algebras $(B, \leq)$ and $(B', \leq')$, a function $f : B \to B'$ is a **map of Boolean algebras** if f is a map of lattices from $(B, \leq)$ to $(B', \leq')$.*

Question 18. *Show that if f is a map of Boolean algebras from $(B, \leq)$ to $(B', \leq')$, then f preserves complements.* Quest. 18

Example 2.11. *It follows from results of the course that*

$$\begin{array}{rccccc} [\![-]\!] & : & \mathfrak{L}(\mathsf{LML}) & \longrightarrow & \mathbf{K}\Omega((\mathbf{2}^{\mathrm{AP}})^{\omega}) \\ & & \varphi & \longmapsto & [\![\varphi]\!] \end{array}$$

is a bijection. It is easy to see that $[\![-]\!]$ is a morphism of Boolean algebras.

3 Representation of Boolean Algebras

As alluded to in §1, the space $(\mathbf{2}^{\mathrm{AP}})^{\omega}$ can be exactly described as a space of prime filters over $\mathfrak{L}(\mathsf{LML})$. This generalizes to any Stone space. We present some basic definitions and facts about filters in §3.1 and then turn to the representation of Boolean algebras as Stone spaces in §3.2.

3.1 Filters and Ultrafilters

Definition 3.1 (Filter on a Partial Order). *Let $(A, \leq)$ be a partial order. Then $\mathcal{F} \subseteq A$ is a **filter** if $\mathcal{F}$ is:*

upward-closed: *if $a \in \mathcal{F}$ and $a \leq b$ then $b \in \mathcal{F}$, and*

codirected: *$\mathcal{F}$ is non-empty and for all $a, b \in \mathcal{F}$ there is some $c \in \mathcal{F}$ such that $c \leq a$ and $c \leq b$.*

Question 19. *Let $(L, \wedge, \top)$ be a meet semilattice. Show that $\mathcal{F} \subseteq L$ is a filter iff* Quest. 19

(i) $\mathcal{F}$ is upward-closed, and

(ii) $\top \in \mathcal{F}$, and

(iii) $a \wedge b \in \mathcal{F}$ whenever $a \in \mathcal{F}$ and $b \in \mathcal{F}$.

Definition 3.2 (Prime Filter). *Let $(L, \vee, \bot)$ be a join semilattice. A filter $\mathcal{F}$ on $(L, \leq)$ is **prime** if*

(i) $\bot \notin \mathcal{F}$, and

(ii) if $a \vee b \in \mathcal{F}$ then either $a \in \mathcal{F}$ or $b \in \mathcal{F}$.

A filter $\mathcal{F}$ on a join semilattice $(L, \leq)$ is **proper** if $\bot \notin \mathcal{F}$.

Definition 3.3 (Finite Intersection Property). *Let $(L, \leq)$ be a lattice. A family of sets $F \subseteq L$ is said to have the **finite intersection property** if $\bigwedge S \neq \bot$ for all finite $S \subseteq F$.*

Question 20. *Let $(L, \leq)$ be a lattice. Show that if $F \subseteq L$ has the finite intersection property, then* Quest. 20

$$\mathrm{Filt}(F) \quad := \quad \{a \in L \mid a \geq \textstyle\bigwedge S \text{ for some finite } S \subseteq F\ \}$$

is a proper filter on $(L, \leq)$.

Definition 3.4 (Ultrafilter). *An **ultrafilter** $\mathcal{F}$ on a lattice L is a maximal proper filter: $\mathcal{F} \subseteq L$ is a proper filter and for any proper filter $\mathcal{H}$ on L such that $\mathcal{F} \subseteq \mathcal{H}$, we have $\mathcal{H} = \mathcal{F}$.*

Question* 21. *Let $\mathcal{F}$ be a filter on a distributive lattice. Show that if $\mathcal{F}$ is an ultrafilter, then $\mathcal{F}$ is prime.* *Quest. 21

In the case of Boolean algebras, we have the following neat characterization of ultrafilters.

Question 22. *Let $(B, \leq)$ be a Boolean algebra and let $\mathcal{F} \subseteq B$ be a filter. Show that the following are equivalent:* Quest. 22

(i) $\mathcal{F}$ is an ultrafilter,

(ii) $\mathcal{F}$ is prime,

(iii) for each $a \in B$, we have ($\neg a \in \mathcal{F}$ iff $a \notin \mathcal{F}$).

We conclude this §3.1 with an easy but important characterization of filters. Write $\mathbf{2}$ for the Boolean algebra $\{0,1\}$ with $0 \leq 1$. Given a set A, the **characteristic function** of a subset $S \subseteq A$ is the function

$$\chi_S \quad : \quad A \quad \longrightarrow \quad \mathbf{2}$$

such that $\chi_S(a) = 1$ if and only if $a \in S$.

Question 23. *Let $(A, \leq)$ be a partial order and consider some $\mathcal{F} \subseteq A$.* Quest. 23

(1) Show that $\mathcal{F}$ is upward-closed if and only if $\chi_{\mathcal{F}}$ is monotone.

(2) Assume that A is a meet semilattice. Show that $\mathcal{F}$ is a filter if and only if $\chi_{\mathcal{F}} \colon A \to \mathbf{2}$ is a morphism of meet semilattices.

(3) Assume that A is a lattice. Show that $\mathcal{F}$ is a prime filter if and only if $\chi_{\mathcal{F}} \colon A \to \mathbf{2}$ is a morphism of lattices.

3.2 The Spectrum of a Boolean Algebra

Definition 3.5 (Spectrum of a Boolean Algebra (1/2))**.** *Given a Boolean algebra B, we let $\mathbf{Sp}(B)$ be the set of prime filters on B.*

Definition 3.6. *Given a Boolean algebra B and $a \in B$ we let*

$$\mathsf{ext}(a) \quad := \quad \{\mathcal{F} \in \mathbf{Sp}(B) \mid a \in \mathcal{F}\}$$

Question 24. *Let $(B, \leq)$ be a Boolean algebra. Show that we have* Quest. 24

$$\begin{array}{rcl}
\mathsf{ext}(a \wedge b) & = & \mathsf{ext}(a) \cap \mathsf{ext}(b) \\
\mathsf{ext}(a \vee b) & = & \mathsf{ext}(a) \cup \mathsf{ext}(b) \\
\mathsf{ext}(\neg a) & = & \mathbf{Sp}(B) \setminus \mathsf{ext}(a) \\
\mathsf{ext}(\top) & = & \mathbf{Sp}(B) \\
\mathsf{ext}(\bot) & = & \emptyset
\end{array}$$

Consider a set X together with a family of sets $\mathcal{B} \subseteq \mathcal{P}(X)$ which is closed under finite intersections. Let ΩX consist of all the $\bigcup_{i \in I} U_i$ for $(U_i)_{i \in I}$ a (possibly infinite or empty) family of elements of $\mathcal{B}$. We have seen in the course that $(X, \Omega X)$ is a topological space. The family $\mathcal{B}$ is a **base** of the topology ΩX.

Definition 3.7 (Spectrum of a Boolean Algebra (2/2))**.** *Given a Boolean algebra B, we equip $\mathbf{Sp}(B)$ with the topology $\Omega(\mathbf{Sp}(B))$ induced by the base $\mathcal{B} := \{\mathsf{ext}(a) \mid a \in B\}$. The space $(\mathbf{Sp}(B), \Omega(\mathbf{Sp}(B)))$ is the **spectrum** of B.*

Our next task is to prove that $\mathbf{Sp}(B)$ is always a Stone space.

Question 25. *Show that the spectrum $(\mathbf{Sp}(B), \Omega(\mathbf{Sp}(B)))$ of a Boolean algebra B is T_0 and zero-dimensional.* Quest. 25

It remains to show that $\mathbf{Sp}(B)$ is compact. For this, we rely on the following.

Lemma 3.8 (Ultrafilter Lemma). *Let $(L, \leq)$ be a lattice. If $F \subseteq L$ has the finite intersection property, then $F \subseteq \mathcal{F}$ for some ultrafilter $\mathcal{F}$ on L.*

The Ultrafilter Lemma 3.8 is discussed in §3.3 as a consequence of Zorn's Lemma (a formulation of the Axiom of Choice).

Question* 26. *Show that the spectrum $(\mathbf{Sp}(B), \Omega(\mathbf{Sp}(B)))$ of a Boolean algebra B is compact.* *Quest. 26

We now arrive at the simplified version of Stone's Representation Theorem alluded to in §1. First, it is easy to see that for a Boolean algebra B, the clopens $U \in \mathbf{K}\Omega(\mathbf{Sp}(B))$ are exactly the sets of the form $\bigcup_{a \in S} \mathsf{ext}(a)$ for some finite $S \subseteq B$. The argument is essentially the same as for the similar characterization of observable properties over $(\mathbf{2}^{\mathrm{AP}})^\omega$ seen in the course. It then follows from Question 24 that $\mathbf{K}\Omega(\mathbf{Sp}(B))$ is isomorphic to B, in the sense that we have morphisms of Boolean algebras

$$
\begin{array}{lrcrcl}
 & \mathsf{ext} & : & B & \longrightarrow & \mathbf{K}\Omega(\mathbf{Sp}(B)) \\
\text{and} & h & : & \mathbf{K}\Omega(\mathbf{Sp}(B)) & \longrightarrow & B
\end{array}
$$

such that

$$
h(\mathsf{ext}(a)) = a \qquad \text{and} \qquad \mathsf{ext}(h(U)) = U \qquad (\text{for all } a \in B \text{ and all } U \in \mathbf{K}\Omega(\mathbf{Sp}(B)))
$$

We focus on the representation of a Stone space (X, Ω) as the space of prime filters over $\mathbf{K}\Omega$.

Question 27. *Let (X, Ω) be a topological space. Show that (X, Ω) is compact if and only if we have $\bigcap \mathcal{F} \neq \emptyset$ for every family of closed sets $\mathcal{F}$ which has the finite intersection property (w.r.t. the inclusion (partial) order on closed sets).* Quest. 27

Question* 28. *Given a Stone space (X, Ω), consider the function* *Quest. 28

$$
\begin{array}{rcrcl}
\eta & : & X & \longrightarrow & \mathcal{P}(\mathbf{K}\Omega) \\
 & & x & \longmapsto & \{U \in \mathbf{K}\Omega \mid x \in U\}
\end{array}
$$

Show that η is a continuous bijection from X to $\mathbf{Sp}(\mathbf{K}\Omega)$.

Recall that a continuous function $f\colon (X, \Omega X) \to (Y, \Omega Y)$ is an **homeomorphism** if there is a continuous function $g\colon (Y, \Omega Y) \to (X, \Omega Y)$ such that

$$
g(f(x)) = x \qquad \text{and} \qquad f(g(y)) = y \qquad\qquad (\text{for all } x \in X \text{ and all } y \in Y)
$$

Question* 29. *Assume that $(X, \Omega X)$ and $(Y, \Omega Y)$ are compact Hausdorff spaces. Show that if $f\colon (X, \Omega X) \to (Y, \Omega Y)$ is a continuous bijection, then f is an homeomorphism.* *Quest. 29

Hence, a Stone space (X, Ω) is always homeomorphic to the spectrum of the Boolean algebra $(\mathbf{K}\Omega, \subseteq)$.

3.3 On the Ultrafilter Lemma 3.8

The Ultrafilter Lemma 3.8 follows from a formulation of the Axiom of Choice known as **Zorn's Lemma**. A **chain** in a partial order $(P, \leq)$ is a set $C \subseteq P$ such that for all $a, b \in C$, we have either $a \leq b$ or $b \leq a$.

Lemma 3.9 (Zorn's Lemma)**.** *Let $(P, \leq)$ be a partial order. If every chain in P has an upper bound in P, then P has a maximal element (i.e. some $a \in P$ such that $b \leq a$ whenever $a \leq b$).*

Note that a chain can be empty, so that P must be non-empty if every chain in P has an upper bound in P. Zorn's Lemma is equivalent to the Axiom of Choice.

Question* 30. *Prove the Ultrafilter Lemma 3.8 (assuming Zorn's Lemma 3.9).* *Quest. 30

* *
*