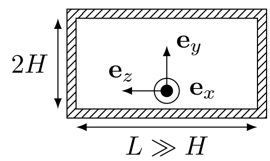
Problem: Channel Flow

1 Velocity Profile in a Turbulent Channel Flow



We consider a flow in a closed channel with rectangular section. We assume that the density of the fluid is constant. The governing equations are

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \nu \Delta \mathbf{u},$$
$$\nabla \cdot \mathbf{u} = 0.$$

We impose a no-slip boundary condition $\mathbf{u} = 0$ on the walls.

1. Let us write the velocity field $\mathbf{u} = \bar{\mathbf{u}} + \mathbf{u}'$, where $\bar{\cdot}$ denotes ensemble averaging, so that $\overline{\mathbf{u}'} = 0$. Show that the mean flow obeys the Reynolds equations:
$$\partial_t \bar{u}_i + \bar{u}_j \partial^j \bar{u}_i = -\partial_i \bar{p} + \nu \partial_j \partial^j \bar{u}_i - \partial^j \tau_{ij},$$
$$\partial^i \bar{u}_i = 0,$$
with $\tau_{ij} = \overline{u_i u_j} - \bar{u}_i \bar{u}_j = \overline{u'_i u'_j}$ the Reynolds stress tensor.

2. We assume that the flow statistics are stationary and homogeneous in x and z, and we denote $U = \bar{\mathbf{u}} \cdot \mathbf{e}_x$ and $V = \bar{\mathbf{u}} \cdot \mathbf{e}_y$. Show that $V = 0$.

3. Using the Reynolds equations in the direction y normal to the wall, show that the mean axial pressure gradient is uniform across the flow:
$$\frac{\partial \bar{p}}{\partial x} = \frac{dp_w}{dx},$$
with $p_w = \overline{p(x,0,0)}$ the mean pressure on the bottom wall.

4. Show that the Reynolds equations in the axial direction x becomes:
$$\frac{d\tau}{dy} = \frac{dp_w}{dx},$$
with $\tau = \nu \frac{dU}{dy} - \overline{u'v'}$ the total shear stress. Deduce that $\frac{d\tau}{dy}$ is a constant.

5. Using symmetry arguments, show that $\tau(y) = \tau_w(1 - y/H)$, with $\tau_w = \tau(0)$ the wall shear stress. Show that $\tau_w = \nu \left(\frac{dU}{dy}\right)_{y=0}$.

6. Using τ_w and ν, build from dimensional analysis a velocity u_τ and a length scale δ characterizing the flow close to the wall. We define the y coordinate in *wall units* $y^+ = y/\delta$. Show that y^+ can be interpreted as a local Reynolds number, which is of order one at scale δ, and which we denote Re_τ at scale H. Justify that we expect two different regimes, as a function of the distance to the wall, corresponding to the different terms of τ.

7. List the dimensional parameters on which the flow depends. Deduce that
$$\frac{dU}{dy} = \frac{u_\tau}{y} \Phi\left(\frac{y}{\delta}, \frac{y}{H}\right),$$
where Φ is a universal non-dimensional function. Does Φ depend upon Re_τ? We assume that, close to the wall, the flow does not depend explicitly on H: $\Phi(y/\delta, y/H) \approx \Phi_I(y/\delta)$ for $y \ll H$. Check that, with the notation $u^+ = U/u_\tau$, we have $\frac{du^+}{dy^+} = \frac{1}{y^+}\Phi_I(y^+)$.

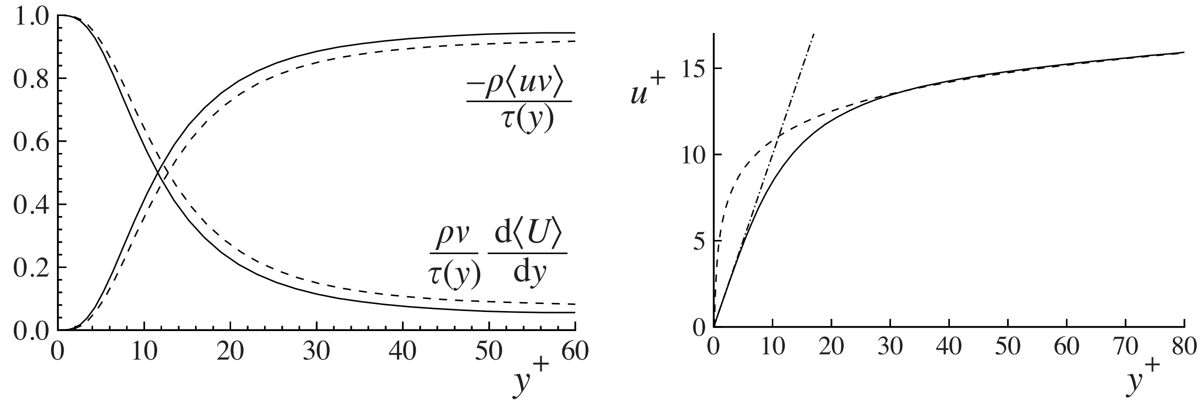


Figure 1: Relative contributions of the viscous and Reynolds stresses (left) and mean axial velocity (right) as a function of the distance to the wall, obtained in Direct Numerical Simulations of a channel flow at Re= 13750.

8. We denote $f_W(y^+) = \int_0^{y^+} \Phi_I(Y)/Y dY$. Using boundary conditions, show that $u^+ = f_W(y^+)$, then that $f'_W(0) = 1$. Deduce from this that there exists a *viscous sublayer* where the velocity profile is linear: $u^+ = y^+$.

9. While $y \ll H$ still holds, we now assume $y \gg \delta$, so that $\Phi_I(y^+) = 1/\kappa$ is a constant, called the *von Kármán constant*. Show that the velocity has a logarithmic profile: $u^+ = \frac{1}{\kappa} \ln y^+ + B$.

10. Interpret figure 1 based on the above questions.