Problem set 3 — Ekman layer

We start with the equations of motion for a fluid with constant density ρ_0 in a rotating frame of reference:

$$\frac{D\mathbf{v}}{Dt} + f\mathbf{k}\times\mathbf{v} = -\nabla\phi + \mathbf{g} + \mathbf{F},$$
$$\nabla\cdot\mathbf{v} = 0,$$

with $\mathbf{v} = (u, v, w)$ the 3D velocity field, f the Coriolis parameter, $\mathbf{k}$ the unit vector in the local vertical direction, $\mathbf{g} = -g\mathbf{k}$ the acceleration of gravity, ϕ the geopotential, and $\mathbf{F}$ the divergence of the stress tensor: $F_i = \partial^j \tau_{ij}/\rho_0$.

For simplicity, we will work in Cartesian coordinates, on an f-plane. The domain is a half-space bounded by a flat horizontal surface, which we will always denote as $z = 0$. In atmospheric and oceanic applications this surface can either be the continental surface or the sea floor (in both cases the half-space of interest is then $z > 0$) or the sea surface (then the half-space of interest is $z > 0$ for the marine atmospheric boundary layer, or $z < 0$ for the surface layer of the ocean).

This problem is inspired after McWilliams (2011, Chap. 6).

1 General properties

We first establish the main balance equations for Ekman layers and study their generic properties.

1. Show that if we neglect the advective terms in the horizontal momentum equations, and assume that the divergence of the stress tensor is dominated by its vertical component in each direction, we obtain the frictional-geostrophic balance:

$$-fv = -\frac{\partial\phi}{\partial x} + \frac{1}{\rho_0}\frac{\partial\tau_{xz}}{\partial z}, \tag{1}$$
$$fu = -\frac{\partial\phi}{\partial y} + \frac{1}{\rho_0}\frac{\partial\tau_{yz}}{\partial z}. \tag{2}$$

2. Justify that the stress can either be interpreted as a viscous stress or as Reynolds stresses. Which one is the most relevant here?

3. We note $\boldsymbol{\tau}^s = (\tau^s_{xz}, \tau^s_{yz})$ the vertical component of the stress tensor at the surface ($z = 0$). We further assume that the stress tensor decays away from the boundary: $\tau_{xz}, \tau_{yz} \to 0$ as $z \to \pm\infty$, and we decompose the velocity field into a boundary-layer and interior components: $u = u^i + u^b$, with $u^b \to 0$ as $z \to \pm\infty$ (and similarly for v, w and ϕ). Justify that the interior solution satisfies geostrophic balance: $fu^i = -\partial_y\phi^i$, $fv^i = \partial_x\phi^i$, $w^i = 0$, and that it is depth-independent.

4. We define the horizontal mass transport in the Ekman layer as the vector field $\mathbf{T} = (T_x, T_y)$ with $T_x = \rho_0 \int u^b dz$, $T_y = \rho_0 \int v^b dz$, where the integral extends over the half-space of interest (either $z < 0$ or $z > 0$). Show that

$$f\mathbf{T} = \mathbf{k}\times(\boldsymbol{\tau}^B - \boldsymbol{\tau}^T), \tag{3}$$

where $\boldsymbol{\tau}^B$ is the bottom surface stress (0 for a top Ekman layer) and $\boldsymbol{\tau}^T$ the top surface stress (0 for a bottom Ekman layer). What is the direction of the vertically integrated mass transport relative to the surface stress, for a bottom Ekman layer and for a top Ekman layer, in the Northern Hemisphere? How is this modified in the Southern Hemisphere?

5. Show that the vertical velocity at the edge of the Ekman layer ($z = +\infty$ for a bottom layer and $z = -\infty$ for a top layer) is given by:

$$w^b(\pm\infty) = \mathbf{k}\cdot\nabla\times\frac{\boldsymbol{\tau}^s}{\rho_0 f}. \tag{4}$$

This vertical motion is referred to as *Ekman pumping*.

2 The laminar Ekman layer

Now we will search an explicit solution of the Ekman layer equations in the case where the stress is a viscous stress: $\tau_{ij} = \mu \partial_j u_i$ with μ the dynamic viscosity.

1. We first consider the case of a semi-infinite fluid (in the $z < 0$ part of the domain) with imposed stresses at the top boundary ($z = 0$), i.e. a top boundary layer, in the Northern Hemisphere ($f > 0$).

   In the laminar case, the equations are:

$$-fv^b = \nu \frac{\partial^2 u^b}{\partial z^2}, \tag{5}$$

$$fu^b = \nu \frac{\partial^2 v^b}{\partial z^2}, \tag{6}$$

   with $\nu = \mu/\rho_0$ the kinematic viscosity, and with boundary conditions

$$\mu \frac{\partial u^b}{\partial z} = \tau^s_{xz} \text{ at } z = 0, \qquad \text{and } u^b(z) \to 0 \text{ at } z \to -\infty, \tag{7}$$

$$\mu \frac{\partial v^b}{\partial z} = \tau^s_{yz} \text{ at } z = 0, \qquad \text{and } v^b(z) \to 0 \text{ at } z \to -\infty. \tag{8}$$

   (a) We introduce the complex variable $U = u^b + iv^b$: show that it satisfies the equation $\nu U'' = ifU$ with boundary conditions $\nu U'(0) = \frac{\tau^s_{xz} + i\tau^s_{yz}}{\rho_0}$ and $U(-\infty) = 0$. Deduce that the general solution is the linear combination $U(z) = Ae^{\alpha z} + Be^{-\alpha z}$ where $\alpha = \sqrt{f/\nu}(1+i)/\sqrt{2}$ and A and B are constants.

   (b) Using the boundary conditions, compute A and B and conclude that

$$u^b = \frac{e^{z/\delta}}{\sqrt{\nu f}} \left[\frac{\tau^s_{xz}}{\rho_0} \cos\left(\frac{z}{\delta} - \frac{\pi}{4}\right) + \frac{\tau^s_{yz}}{\rho_0} \cos\left(\frac{z}{\delta} + \frac{\pi}{4}\right) \right], \tag{9}$$

$$v^b = \frac{e^{z/\delta}}{\sqrt{\nu f}} \left[\frac{\tau^s_{yz}}{\rho_0} \cos\left(\frac{z}{\delta} - \frac{\pi}{4}\right) - \frac{\tau^s_{xz}}{\rho_0} \cos\left(\frac{z}{\delta} + \frac{\pi}{4}\right) \right], \tag{10}$$

   with $\delta = \sqrt{2\nu/f}$. What is the shape of the hodograph? What is the angle between the velocity in the Ekman layer at the surface and the surface stress?

   (c) Estimate the size of the Ekman layer using molecular viscosity $\nu \sim 10^{-6}$ m^2.s^{-1} and using an eddy viscosity $\nu \sim 0.1$ m^2.s^{-1}. Which one is more realistic for the ocean surface and why?

2. In the case of a bottom Ekman layer, the problem is a bit different: we consider that it is not a given surface stress which controls the Ekman layer, but rather the given interior geostrophic flow, and the surface stress adjusts accordingly. The equation to solve is $\nu U'' = ifU$ with boundary conditions $U(0) = -U^i$, so that the total velocity vanishes at the surface (no-slip boundary condition) and $U(+\infty) = 0$.

   (a) Show that the solution in the bottom Ekman layer is:

$$u^b = -e^{-z/\delta} \left[u^i \cos\left(-\frac{z}{\delta}\right) - v^i \sin\left(-\frac{z}{\delta}\right) \right], \tag{11}$$

$$v^b = -e^{-z/\delta} \left[u^i \sin\left(-\frac{z}{\delta}\right) + v^i \cos\left(-\frac{z}{\delta}\right) \right]. \tag{12}$$

   (b) Show that the surface stress can be expressed in terms of the interior flow as:

$$\frac{\tau^s_{xz}}{\rho_0} = \nu \frac{u^i - v^i}{\delta}, \tag{13}$$

$$\frac{\tau^s_{yz}}{\rho_0} = \nu \frac{u^i + v^i}{\delta}. \tag{14}$$

   (c) Similarly, compute the mass transport and Ekman pumping:

$$T_x = -\frac{\nu \rho_0}{f\delta}(u^i + v^i), \tag{15}$$

$$T_y = \frac{\nu \rho_0}{f\delta}(u^i - v^i), \tag{16}$$

$$w^b = \frac{\nu}{f\delta}\zeta^i, \tag{17}$$

under the approximations that $\varepsilon = \nu/\delta$ does not vary spatially, and that the divergence of the interior flow is small due to its small Rossby number.

(d) Assuming that the fluid is a single homogeneous layer of thickness H on top of the Ekman layer, with vanishing vertical velocity at the top and with constant f, show that the Ekman layer acts as a linear drag in the quasi-geostrophic vorticity equation:

$$\frac{D\zeta}{Dt} \approx -\frac{f\delta}{2H}\zeta. \tag{18}$$

(e) Draw a schematic of the hodograph for the velocity in the bottom Ekman layer.

References

McWilliams, J. C. (2011). *Fundamentals of Geophysical Fluid Dynamics.*