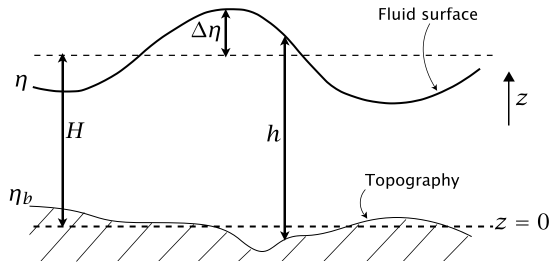
Problem set 2 — Equatorial waves

1 Shallow-water equations

In this exercise, we derive the shallow-water equations, which is a very useful model in geophysical fluid dynamics. See Vallis (2017, Chap. 3) for more information.



We consider a layer of fluid with constant density ρ. The fluid is bounded below by a rigid boundary with equation $z = \eta_b(x, y)$. The surface of the fluid layer has equation $z = \eta(x, y)$. Hence the thickness of the fluid layer is given by $h = \eta - \eta_b$. We denote H the average thickness and $\Delta\eta$ the position with respect to this average thickness: $\eta = H + \Delta\eta$. We denote $\mathbf{v} = u\mathbf{e}_x + v\mathbf{e}_y + w\mathbf{e}_z$ the three-dimensional velocity field and $\mathbf{u} = u\mathbf{e}_x + v\mathbf{e}_y$ the horizontal velocity.

We first assume that the fluid layer has a free surface. Above it lies a fluid with negligible inertia ($\rho \approx 0$). In this exercise the fluid is non-rotating.

1. We assume that hydrostatic balance holds. Show that the pressure at any point is given by $p = \rho g(\eta(x, y) - z)$.
2. Deduce that the equations for the horizontal velocity field become
$$\frac{D\mathbf{u}}{Dt} = -g\boldsymbol{\nabla}_z\eta, \tag{1}$$
where $\boldsymbol{\nabla}_z$ denotes the gradient operator on a horizontal surface. Deduce that if $\mathbf{u}$ is initially independent of z, it remains so.
3. Show that the mass continuity equation simplifies to $\boldsymbol{\nabla} \cdot \mathbf{v} = 0$. Deduce the evolution equation for the layer thickness:
$$\frac{Dh}{Dt} + h\boldsymbol{\nabla} \cdot \mathbf{u} = 0. \tag{2}$$

2 Waves on the equatorial beta plane

In this exercise, we study the properties of waves propagating in a simple model for the atmosphere or the upper ocean close to the equator. These waves play an important role in tropical meteorology and in phenomena such as El Niño. This problem is based on Vallis (2017, Chap. 8); refer to the book for more information.

Here we assume that the fluid has a flat bottom ($\eta_b = 0$).

1. Show that the shallow-water equations on a beta plane centered at the equator (i.e. expanding the Coriolis parameter as $f = \beta y$) can be written as:
$$\frac{Du}{Dt} - \beta y v = -g\frac{\partial h}{\partial x}, \tag{3}$$
$$\frac{Dv}{Dt} + \beta y u = -g\frac{\partial h}{\partial y}, \tag{4}$$
$$\frac{Dh}{Dt} + h\boldsymbol{\nabla} \cdot \mathbf{u} = 0, \tag{5}$$

where x and y represent the zonal and meridional directions, respectively.

2. We decompose the velocity and fluid thickness into a mean value and perturbations: $u = U + u', v = V + v', h = H + h'$. Show that small perturbations about a state of rest ($U = V = 0$) evolve according to the following equations:

$$\partial_t u' - \beta y v' = -g \partial_x h', \tag{6}$$
$$\partial_t v' + \beta y u' = -g \partial_y h', \tag{7}$$
$$\partial_t h' + H \partial_x u' + H \partial_y v' = 0, \tag{8}$$

We will drop the primes in the following.

3. Introducing the time and length units $T = 1/\sqrt{\beta c_g}$ and $L = \sqrt{c_g/\beta}$, with $c_g = \sqrt{gH}$ the velocity of pure gravity waves, show that the equations become:

$$\partial_t u - yv = -\partial_x \phi, \tag{9}$$
$$\partial_t v + yu = -\partial_y \phi, \tag{10}$$
$$\partial_t \phi + \partial_x u + \partial_y v = 0, \tag{11}$$

with $\phi = gh$ the geopotential.

4. In the atmosphere, $c_g \approx 25$ m.s^{-1}, and in the ocean, $c_g \approx 2$ m.s^{-1}. Estimate the equatorial radius of deformation L and the associated timescale T in both cases.

5. Show that

$$\partial_t(\zeta - y\phi) + v = 0, \tag{12}$$

with $\zeta = \partial_x v - \partial_y u$ the vorticity.

6. Acting on (9) with ∂_t, on (10) with ∂_{tt}, on (11) with ∂_{ty} and on (12) with ∂_x, show that the meridional velocity satisfies the equation:

$$\frac{\partial^3 v}{\partial t^3} + y^2 \frac{\partial v}{\partial t} - \frac{\partial}{\partial t}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - \frac{\partial v}{\partial x} = 0. \tag{13}$$

7. We seek wave solutions of the form $v = \tilde{v}(y)e^{i(kx-\omega t)}$. Show that the wave amplitude satisfies the differential equation:

$$\tilde{v}''(y) + \left[\omega^2 - k^2 - y^2 - \frac{k}{\omega}\right]\tilde{v}(y) = 0. \tag{14}$$

8. To put this equation in a standard form, we make the change of variable $\tilde{v}(y) = \Psi(y)e^{-y^2/2}$. Show that Ψ satisfies the *Hermite equation*:

$$\Psi''(y) - 2y\Psi'(y) + \lambda\Psi(y) = 0, \tag{15}$$

with $\lambda = \omega^2 - k^2 - \frac{k}{\omega} - 1$. It is known that this equation admits solution if and only if $\lambda = 2m$ is an even integer. The solutions are the Hermite polynomials $\Psi = H_m$; they are a family of orthogonal polynomials with respect to the scalar product $\langle f, g\rangle = \int_{-\infty}^{+\infty} f(y)g(y)e^{-y^2}dy$.

9. Show that for high-frequency waves the dispersion relation becomes $\omega^2 = k^2 + 2m + 1$, and for low-frequency waves, $\omega = -k/(k^2 + 2m + 1)$. Which types of waves are those dispersion relation reminiscent of?

10. We consider the special case $m = 0$.
    (a) Show that there are two solutions to the dispersion relation, $\omega = -k$ and $\omega = k/2 \pm \sqrt{k^2/4 + 1}$.
    (b) Show that necessarily, $\tilde{v}(y) = \tilde{v}(0)e^{-y^2/2}$.
    (c) In the case $\omega = -k$, show that u and ϕ can be written as $u(x, y, t) = G(x + t)ye^{-y^2/2} + H(x + t, y)$, $\phi(x, y, t) = G(x + t)ye^{-y^2/2} - H(x + t, y)$ where H satisfies an equation of the form $\partial_y H - yH = K(x + t)e^{-y^2/2}$. Show that H diverges as $|y| \to +\infty$, and conclude that such solutions are unphysical.

(d) In the case $\omega = k/2 + \sqrt{k^2/4+1}$, show that the wave behaves like an inertia-gravity wave for $k \to +\infty$ and like a Rossby wave for $k \to -\infty$ (the converse is true for $\omega = k/2 - \sqrt{k^2/4+1}$). For this reason these waves are sometimes called *mixed Rossby-gravity waves*, but they are also known as *Yanaï* waves.

11. We note that $v = 0$ was a trivial solution of (13). Hence we search for solutions of the linear shallow-water equations for which the meridional velocity identically vanishes. Such waves are called *Kelvin waves.*

(a) Show that the equations become:

$$\partial_t u = -\partial_x \phi, \tag{16}$$

$$yu = -\partial_y \phi, \tag{17}$$

$$\partial_t \phi + \partial_x u = 0. \tag{18}$$

(b) Show that the general solution is the sum of two traveling waves: $u = F_1(x-t, y) + F_2(x+t, y)$ and $\phi = F_1(x-t, y) - F_2(x+t, y)$.

(c) Using geostrophic balance, show that $F_1(x-t, y) = F(x-t)e^{-y^2}$ and $F_2(x+t, y) = G(x+t)e^{y^2}$. Conclude that Kelvin waves can only propagate eastward, and that the geopotential is proportional to the zonal velocity.

(d) Are Kelvin waves dispersive? Is the dispersion relation a solution of the general dispersion relation for some value of m?

12. Make a sketch of the dispersion relation $\omega(k)$ for all the waves we have studied here, using the asymptotic formulas obtained above when necessary.

References

Vallis, G. K. (2017). *Atmospheric and Oceanic Fluid Dynamics.* Cambridge University Press.