Problem set 1 — Fundamental concepts in geophysical fluid dynamics

1 Balance relations

We recall that in a frame rotating with constant angular velocity Ω, a body of mass m experiences two pseudo-forces: the Coriolis force $-2m\mathbf{\Omega}\times\mathbf{v}$ where $\mathbf{v}$ is the velocity in the rotating frame, and the centrifugal force $-m\mathbf{\Omega}\times\mathbf{\Omega}\times\mathbf{r}$. In a gravity field $\mathbf{g}_{\text{grav}}$, the centrifugal acceleration, which is conservative, can be absorbed into a modified gravity term: $\mathbf{g}=\mathbf{g}_{\text{grav}}+\Omega^2\mathbf{r}_\perp$ where $\mathbf{r}_\perp$ is the projection orthogonal to the axis of rotation. We introduce the geopotential Φ defined by $\mathbf{g}=-\nabla\Phi$. The equations of motion for an ideal fluid in a rotating frame therefore become:

$$\frac{\partial \mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}+2\mathbf{\Omega}\times\mathbf{v}=-\frac{1}{\rho}\nabla p-\nabla\Phi, \tag{1}$$

with $\mathbf{v}=u\mathbf{e}_\lambda+v\mathbf{e}_\phi+w\mathbf{e}_r$ the 3D velocity field ($\mathbf{e}_\lambda$ and $\mathbf{e}_r$ are the usual azimuthal and radial vectors in spherical coordinates and $\mathbf{e}_\phi=-\mathbf{e}_\theta$ is the opposite of the polar vector, with $\phi=\frac{\pi}{2}-\theta$ the latitude and λ the longitude), ρ the density and p the pressure. We denote $\mathbf{u}=u\mathbf{e}_\lambda+v\mathbf{e}_\phi$ the horizontal velocity field. Although the concepts below apply both to the atmosphere and ocean, for simplicity we consider only the atmosphere here, and assume that it is a dry ideal gas.

1. **Hydrostatic balance.** We would like to show that to describe the large-scale motion of the atmosphere, we can simplify a little the above equations. To do so we will use observed estimates for the following quantities: horizontal velocity $U\sim 10$ m.s^{-1} with characteristic horizontal scale $L\sim 1000$ km, vertical velocity $W\sim 10^{-2}$ m.s^{-1} with characteristic scale $H\sim 1$ km.

   (a) We define the local vertical direction to be in the direction of the effective gravity, so that the geopotential reads $\Phi=gz$. Show that the equation of motion in the vertical direction reads

   $$\partial_t w+\mathbf{u}\cdot\nabla w+w\partial_z w-2\Omega\cos\phi u=-\frac{1}{\rho}\frac{\partial p}{\partial z}-g. \tag{2}$$

   (b) Estimate numerically the order of magnitude of each term on the left-hand side of Eq. (2). Conclude that, at leading order, gravity is balanced by the vertical pressure gradient:

   $$\frac{\partial p}{\partial z}=-\rho g. \tag{3}$$

   This relation is called *hydrostatic balance.*

   (c) Justify that, assuming hydrostatic balance holds, we can use pressure as a vertical coordinate. Show that the hydrostatic balance relation in pressure coordinates reads

   $$\frac{\partial \Phi}{\partial p}=-\frac{1}{\rho}. \tag{4}$$

   (d) Show that the component of the Coriolis force involving vertical velocity is negligible compared to the one involving horizontal velocity. Conclude that the horizontal velocity satisfies the equations:

   $$\partial_t\mathbf{u}+\mathbf{v}\cdot\nabla\mathbf{u}+2\Omega\sin\phi\mathbf{e}_r\times\mathbf{u}=-\frac{1}{\rho}\nabla p. \tag{5}$$

   We note $f=2\Omega\sin\phi$ the *Coriolis parameter.*

2. **Cartesian approximations.** It is often convenient to replace the spherical coordinates with Cartesian coordinates in the plane tangent to a fixed latitude: $x = a\lambda \cos\phi_0$ (*zonal* direction), $y = a(\phi - \phi_0)$ (*meridional* direction), with a the radius of the planet. This affects the ∇ operator (which we did not write in coordinates up to now) and the Coriolis parameter. Show that the Coriolis parameter can be expanded at order zero, $f = f_0$ (f-plane approximation), or at order one, $f = f_0 + \beta y$ (β-plane approximation) and give the expression of f_0 and β.

3. **Geostrophic balance.**

   (a) We define the *Rossby number* as $Ro = U/(fL)$. By considering (symbolically, not numerically) the magnitude of the different terms in the left-hand side of Eq. (5), give an interpretation of this nondimensional number. Estimate the Rossby number for the atmosphere.

   (b) When the Rossby number is small, we expect that the Coriolis force and the horizontal pressure gradient balance each other. This is called *geostrophic balance*. Show that the so-called *geostrophic wind* is given by:

$$fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \quad fv = \frac{1}{\rho}\frac{\partial p}{\partial x}. \tag{6}$$

   (c) Draw the streamlines of geostrophic wind close to a local pressure maximum (high) and close to a local minimum (low).

4. **Thermal wind.**

   (a) Show that, in pressure coordinates, geostrophic balance becomes

$$fu = -\frac{\partial \Phi}{\partial y}, \quad fv = \frac{\partial \Phi}{\partial x}. \tag{7}$$

   (b) Deduce from geostrophic and hydrostatic balance that the vertical shear of the horizontal wind is related to the horizontal temperature gradient by:

$$f\frac{\partial u}{\partial p} = \frac{R_d}{p}\frac{\partial T}{\partial y}, \quad -f\frac{\partial v}{\partial p} = \frac{R_d}{p}\frac{\partial T}{\partial x}. \tag{8}$$

   This relation is called *thermal wind*.

   (c) Based on Fig. 1, is the meridional structure of the atmosphere consistent with the thermal wind relation? Deduce the order of magnitude of the maximum zonal velocity at the tropopause from the order of magnitude of the horizontal temperature gradient (use $R_d = 287$ kg^{-1}.K^{-1}). Is this consistent with the observations?

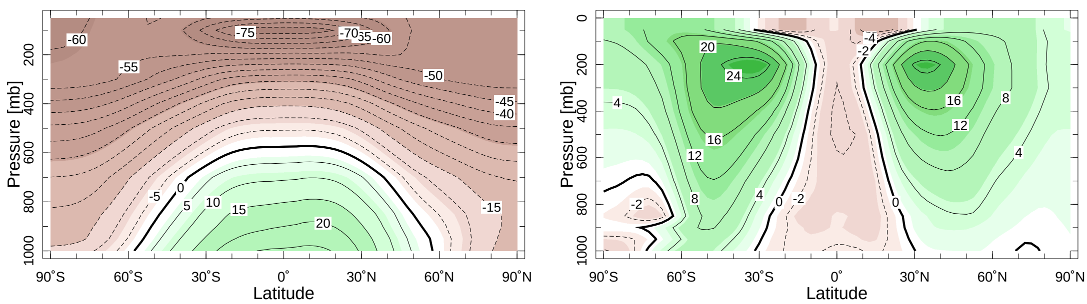

Figure 1: Observed zonally-averaged annual-mean temperature (left) and zonal wind (right). Figure from Marshall and Plumb (2008, Chap. 5).

2 Rossby Waves

The large-scale dynamics in the atmosphere and ocean is, to a good approximation, described by the *quasi-geostrophic equations*[1]. The simplest version of these equations is:

$$\partial_t q + \mathbf{u} \cdot \nabla q = 0, \tag{9}$$

with $q = -\Delta\psi + \beta y$ the *potential vorticity* and ψ the stream function for the horizontal velocity field: $\mathbf{u} = \nabla \times (\psi \mathbf{e}_z)$. We assume that all the fields are independent of the vertical coordinate.

1. Linearize Eq. (9) around a state of rest, and show that it admits plane-wave solutions $\psi(x, y, t) = \psi_0 e^{i(kx+ly-\omega t)}$ with dispersion relation $\omega = -\beta k/(k^2 + l^2)$. Such waves are called *Rossby waves.* In which direction do Rossby waves propagate?

2. The *relative vorticity* is defined by $\zeta = (\nabla \times \mathbf{u}) \cdot \mathbf{e}_z$. Show that $\zeta = -\Delta\psi$. We are going to give a physical interpretation of the mechanism for propagation of Rossby waves using potential vorticity conservation. Consider a horizontal material line with initially vanishing vorticity. Assume that we displace northward from this line a fluid parcel. By considering the vorticity induced by this displacement, explain that the initial displacement will propagate westward. Proceed similarly for southward displacements.

3. We can decompose any field A into its zonal mean $\bar{A} = \frac{1}{L}\int_0^L A dx$ and an *eddy* contribution A': $A = \bar{A} + A'$. Recalling that the zonal momentum equation reads $\partial_t u + \mathbf{u} \cdot \nabla u = -\partial_x \Phi + fv$, show that the evolution equation for the zonally averaged zonal wind is:

$$\partial_t \bar{u} = (f - \partial_y \bar{u})\bar{v} - \partial_y \overline{u'v'}. \tag{10}$$

   In question 1, we have assumed $\bar{u} = \bar{v} = 0$ and shown that Rossby waves are solutions of the linearized equations, corresponding to the *eddy* component here. Their properties would be different if we would linearize about an arbitrary mean flow. Conversely, Eq. (10) tells us that eddies, and in particular, Rossby waves, feed back on the mean flow through the $-\partial_y \overline{u'v'}$ term. Can you interpret this term in terms of transport of a physical quantity?

4. By computing explicitly the eddy momentum flux $\overline{u'v'}$ and the group velocity for a plane Rossby wave, show that such waves transport energy and momentum in opposite directions.

5. Explain that for plane Rossby waves, wave fronts are also streamlines. Show that if the wave front is oriented from north west to south east, the momentum flux is towards the south, and towards the north in the opposite case.

6. Fig. 2 shows the anomaly of geopotential height at a given level in the upper troposphere, on a specific day. Interpreting it as a stream function for the eddy part of the flow, and relying on question 5, suggest an interpretation of the role of Rossby waves in the maintenance of the Jet Stream[2].

References

Marshall, J. and R. A. Plumb (2008). *Atmosphere, Ocean, and Climate Dynamics: An Introductory Text.* Vol. 45. International Geophysics. Academic Press.

Singh, M. S. (2022). “The General Circulation of the Atmosphere”.

Vallis, G. K. (2017). *Atmospheric and Oceanic Fluid Dynamics.* Cambridge University Press.

[1] See Vallis (2017, Chap. 5), for a detailed derivation.

[2] The model considered here, called *barotropic*, does not consider the vertical structure of the flow. This is sufficient to capture Rossby wave dynamics, but it is not a particularly good model of the Jet Stream. See Vallis (2017) for more details.

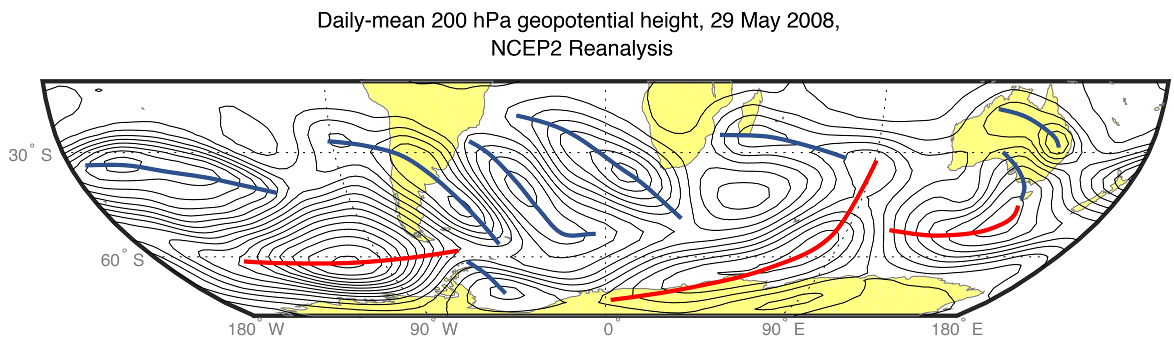

Figure 2: Daily-mean geopotential height anomaly for May 29, 2008 at 200 hPa (black lines). Figure from Singh (2022). Red and blue lines are added as an indication.