Comet, exercises 6

Bring your answers to next course (Oct 26)

1 Universal Coalgebra

Let $F = 2 \times Id^A$ (i.e., $FX = 2 \times X^A$) be the functor for deterministic automata.

Question 1.1. *Give its action on morphisms (i.e., what is Ff for some $f : X \to Y$?). Prove that it is a functor.*

Answer. Ff is the following function from $FX = 2 \times X^A$ to $FY = 2 \times Y^A$:

$$\begin{aligned} Ff :\ & 2 \times X^A \to 2 \times Y^A \\ & \langle o, t \rangle \mapsto \langle o, f \circ t \rangle \end{aligned}$$

Given this definition, that $Fid = id$ and $F(f \circ g) = Ff \circ Fg$ is obvious. □

Recall that the final coalgebra for this functor is the coalgebra of formal languages on the alphabet A, with derivatives describing the dynamics: $\langle \mathcal{P}(A^*), \langle \epsilon, \delta \rangle \rangle$ with $\epsilon(L) =$ "$\epsilon \in L$" and $\delta_a(L) = a^{-1}L = \{w \mid aw \in L\}$.

Question 1.2. *Describe the final coalgebra for the functors $B \times Id^A$ and $B \times Id$, where B is an arbitrary set (justify your answers).*

Answer. For $B \times Id^A$, the final algebra consists of the set $B^{A^\star}$ of functions from finite words on A, to B. (The case $B = 2$ gives back formal languages, represented by their characteristic function.)

The coalgebra structure is given as follows:

$$\begin{aligned} z :\ & B^{A^\star} \to B \times (B^{A^\star})^A \\ & f \mapsto \langle f(\epsilon), (a \mapsto w \mapsto f(aw)) \rangle \end{aligned}$$

Given a coalgebra $f : X \to B \times X^A$, one defines the following function $[\cdot] : X \to B^{A^\star}$ by induction on words:

$$\begin{aligned} [x](\epsilon) &= \pi_1(f(x)) \\ [x](aw) &= [(\pi_2(f(x)))(a)](w) \end{aligned}$$

(Writing $f = \langle o, t \rangle$ with $o = \pi_1 \circ f$ and $t = \pi_2 \circ f$, we get the more friendly notations $[x](\epsilon) = o(x)$ and $[x](aw) = [t(x)(a)](w)$.)

I let you check that this is the unique function such that $z \circ [\cdot] = F[\cdot] \circ f$ (i.e., the unique coalgebra homomorphism from $\langle X, f \rangle$ to $\langle B^{A^\star}, z \rangle$.)

For the functor $B \times Id$, just apply the previous answer to $A = 1$ (any singleton set). The final coalgebra is thus $B^{1^\star}$, but $1^\star \simeq \mathbb{N}$ so that we get functions from natural numbers to B, i.e., infinite streams of elements of B. $\square$