Comet, exercises 11

Bring your answers to next course, in two weeks (Dec 7)

1 Streams

Let us denote by $\mathbb{R}^\omega$ the set of *streams*, i.e., infinite sequences $\sigma, \tau \dots$ of real numbers.

Together with the following function associating to each stream its first element and its tail, $\mathbb{R}^\omega$ is a final coalgebra for the functor $FX = \mathbb{R} \times X$

$$\begin{aligned} \mathbb{R}^\omega &\to \mathbb{R} \times \mathbb{R}^\omega \\ \sigma &\mapsto \langle \sigma_0, \sigma' \rangle \end{aligned}$$

One can thus define streams by *behavioural differential equations* (i.e., F-coalgebras). For instance, the everywhere-0 stream $\hat{0}$ can be defined by the following equations:

$$\hat{0}_0 = 0 \qquad\qquad \hat{0}' = \hat{0}$$

Similarly, pointwise addition of streams can be defined by

$$(\sigma + \tau)_0 = \sigma_0 + \tau_0 \qquad\qquad (\sigma + \tau)' = \sigma' + \tau'$$

Shuffle product. Things become more interesting for more complex operations. Take for instance the *shuffle product* of streams, usually defined by the following formula:

$$(\sigma \otimes \tau)_n = \sum_{k=0}^{n} \binom{n}{k} \times \sigma_k \times \tau_{n-k}$$

This operation can alternatively be defined using the following differential equations, which no longer involve binomial coefficients:

$$(\sigma \otimes \tau)_0 = \sigma_0 \times \tau_0 \qquad\qquad (\sigma \otimes \tau)' = \sigma' \otimes \tau \ + \ \sigma \otimes \tau'$$

Proving a simple property like associativity can be difficult with the former definition, as it would involve double summations of terms with several binomial coefficients. In contrast, one can give a straightforward coinductive proof.

Let $\mathbf{b}$ be the following (monotone) function on binary relations on streams:

$$\mathbf{b} : \ \mathcal{R} \mapsto \left\{ \langle \sigma, \ \tau \rangle \mid \sigma_0 = \tau_0 \text{ and } \sigma' \ \mathcal{R} \ \tau' \right\}$$

One easily checks that its greatest fixpoint is just the identity relation: $\langle \sigma, \tau \rangle \in \nu\mathbf{b}$ iff $\sigma = \tau$. One can thus prove stream equalities by coinduction.

Question 1.1. *Give (again?) a simple bisimulation that proves commutativity of stream addition.*

Question 1.2. *Prove that the following function is compatible*

$$\mathbf{c}^+ : \ \mathcal{R} \mapsto \{\langle \sigma + \rho,\ \tau + \omega \rangle \mid \sigma \,\mathcal{R}\, \tau,\ \rho \,\mathcal{R}\, \omega\}$$

Question 1.3. *Assuming distributivity of shuffle product over addition, prove associativity of shuffle product using a simple bisimulation up-to.*

Question 1.4. *Is the following function compatible? Prove that it is below the companion.*

$$\mathbf{c}^\otimes : \ \mathcal{R} \mapsto \{\langle \sigma \otimes \rho,\ \tau \otimes \omega \rangle \mid \sigma \,\mathcal{R}\, \tau,\ \rho \,\mathcal{R}\, \omega\}$$

Exponentiation. Let us finally consider a third natural operation on streams: *exponentiation*, defined by the following differential equation:

$$e^\sigma_0 = e^{\sigma_0} \qquad\qquad {e^\sigma}' = \sigma' \otimes e^\sigma$$

Question 1.5. *Prove the following law, using a bisimulation up to the companion.*

$$e^{\sigma+\tau} = e^\sigma \otimes e^\tau$$