An aperiodic set of 11 Wang tiles

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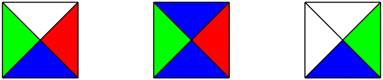
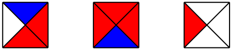
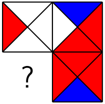
A Wang tile is a square tile with a color on each border

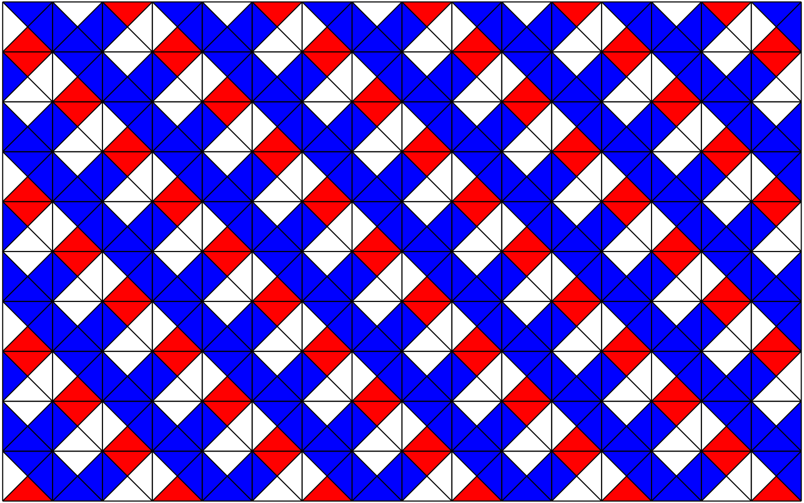


Given a set of Wang tiles, one try to tile the plane with copies of tiles in the set s.t. two adjacent sides have the same color (without rotations !)

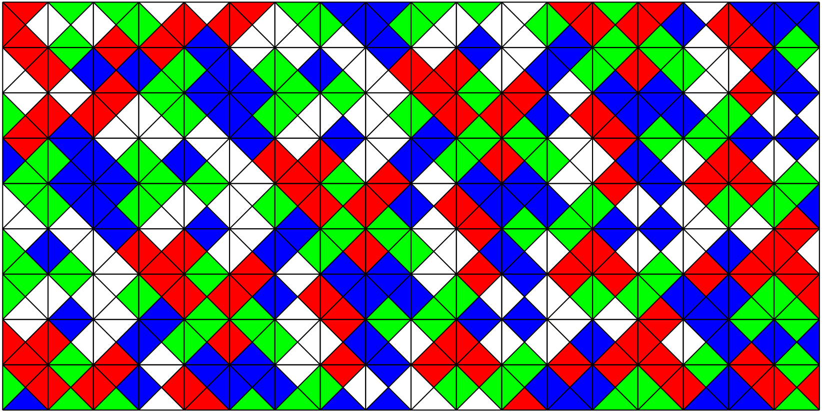
Tileset: 



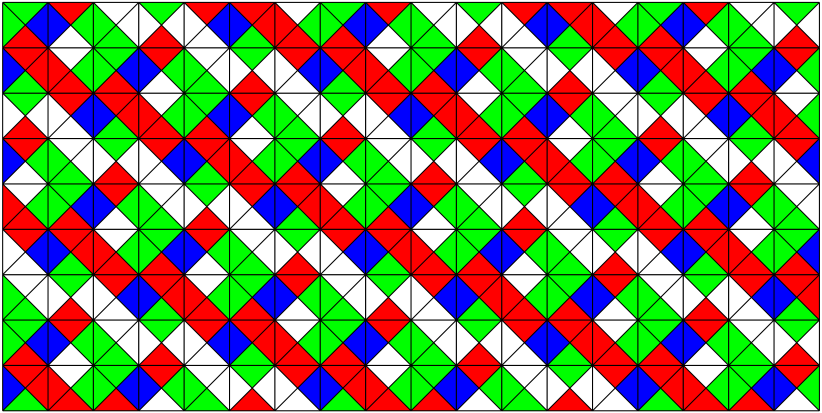
Tileset: 



A tiling of the plane is *periodic* if there is a translation vector which does not change the tiling



A tile set is *periodic* if there is a periodic tiling of the plane with this set



A set is periodic if and only if there is a tiling with 2 (not colinear) translation vectors

A set is *finite* if there is no tiling of the plane with this set

A set is *aperiodic* if it tiles the plane, but no tiling is periodic

Conjecture (Wang 1961)

Every set is either finite or periodic

False:

Theorem (Berger 1966)

There is an aperiodic set of Wang tiles

We can encode the halting problem of a Turing machine.

Eternity II

- $2 million for the first person to solve this puzzle...

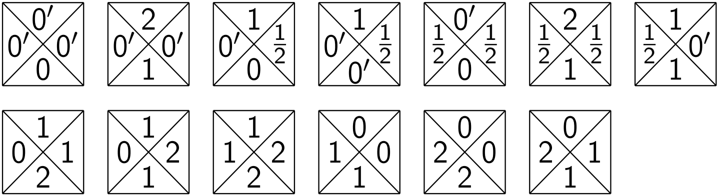
History

- Berger : 20426 tiles in 1966 (lowered down later to 104)
- Knuth : 92 tiles in 1968
- Robinson : 56 tiles in 1971
- Ammann : 16 tiles in 1971
- Grunbaum : 24 tiles in 1987
- Kari : 14 tiles in 1996 (with an interesting new method)
- Culik : (same method) 13 tiles in 1996
- Here : 11 tiles in 2015 and this is the fewest possible

"Kari & Culik" tile set

Theorem (Culik 1996)

The following set (13 tiles) is aperiodic



New results

Theorem

Every set with at most 10 Wang tiles is either finite or periodic

Theorem

There is a set with 11 Wang tiles which is aperiodic

Transducer

A set of Wang tiles can be seen as a transducer

A *transducer* is a finite automaton where each transition has an input letter and an output letter

$\mathcal{T} = (H, V, T)$ where $T \subseteq H^2 \times V^2$

We note $w \mathcal{T} w'$ if the transducer $\mathcal{T}$ writes w' when it reads w

(Note: Transducer on Σ = Automaton on Σ^2)

Note: we call it transducer, but it is more known as a "Mealy machine" in the literature...

"Kari & Culik" tile set

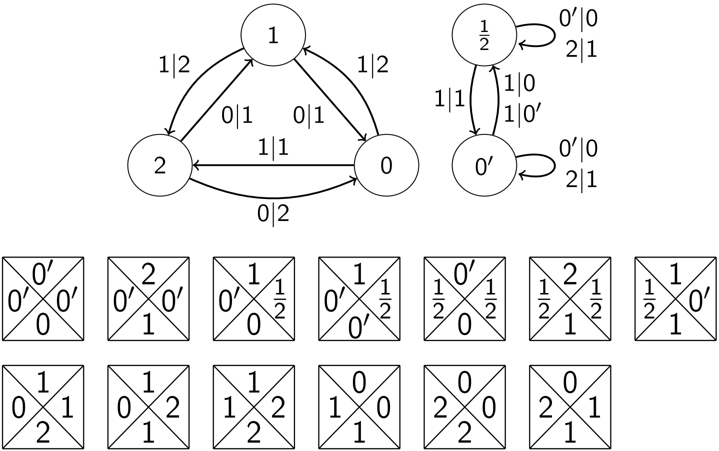


Figure: Kari & Culik tile set.

Simplification

If $\mathcal{T}$ has a transition a between two strongly connected components, then $\mathcal{T}$ is finite (resp. periodic) if and only if $\mathcal{T} \setminus \{a\}$ is finite (resp. periodic)

Let $s(\mathcal{T})$ be the union of strongly connected components of $\mathcal{T}$

$\mathcal{T}$ is finite (resp. periodic, aperiodic) if and only if $s(\mathcal{T})$ is finite (resp. periodic, aperiodic)

Composition and power

Let $\mathcal{T} = (H, V, T)$ and $\mathcal{T}' = (H', V, T)$ be two transducers

Then $\mathcal{T} \circ \mathcal{T}' = (H \times H', V, T'')$ where:

$$T'' = \{((w, w'), (e, e'), s, n') : (w, e, s, x) \in T, (w', e', x, n')\}$$

$\mathcal{T}^k = \mathcal{T}^{k-1} \circ \mathcal{T}$

Power

Proposition

There is $k \in \mathbb{N}$ *s.t.* $s(\mathcal{T}^k)$ *is empty iff* $\mathcal{T}$ *is finite*

Proposition

There is $k \in \mathbb{N}$ *s.t. there is a bi-infinite word* w *such that* $w\mathcal{T}^k w$ *iff* $\mathcal{T}$ *is periodic*

Enumeration (I)

To enumerate all sets with n tiles, we compute all oriented graphs with n arrows (with loops and multiple arrows)

For every pair of graphs G and G', we try every $n!$ bijections between the arrows of G and G'

We only consider graphs without arrows between two strongly connected components.

n	nb. graphs
8	2518
9	13277
10	77810
11	493787

Enumeration (II)

For every generated set $\mathcal{T}$, we compute $s(\mathcal{T}^k)$ until:

- $s(\mathcal{T}^k)$ is empty $\rightarrow$ the set is finite
- $\exists w$ s.t. $ws(\mathcal{T}^k)w$ $\rightarrow$ is periodic
- The computer run out of memory $\rightarrow$ the computer cannot conclude...

Optimizations :

- Cut branches in the exploration of $n!$ bijections
- Make tests on $\mathcal{T}$ and $\mathcal{T}^{\text{tr}}$ on the same time
- Use (sometimes) bi-simulation to simplify transducers

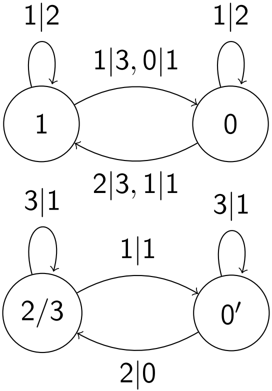
Result ($n \leq 10$)

Theorem

Every set of n Wang tile, $n \leq 10$, is finite or periodic

- 3 differents programs (2 public)
- $\sim$ 4 days on $\sim$ 100 cores for the first one
- About 23 cpu years for the last one ($\sim$ one week on $\sim$ 10 computing machines)
- Kind of certificate to "prove" that the computation has be done
- Only one problematic case

The only problematic case with 10 tiles



Kari type construction with $\times 2$, $\times \frac{1}{3}$

- If it tiles, then it is aperiodic
- It is a strict subset of the Kari's construction. Does it tiles ?
- We have to use compactness to show that : No

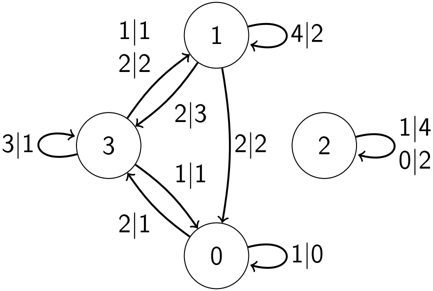
Exploration of the 11 tile sets

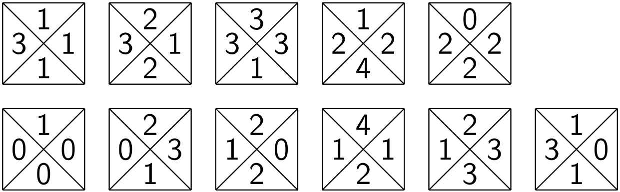
Do the same thing for 11 tiles, and look at the results...

Several complicated cases, but 4 "strange" tilesets...

One can show that some are aperiodic "by hand"

Aperiodic set of 11 tiles





Proof (squetch.)

Theorem

$\mathcal{T}$ is aperiodic

Ideas:

- It is the union if two transducers $\mathcal{T}_0$ and $\mathcal{T}_1$ (as for Kari & Culik)
- In a tiling by $\mathcal{T}$, we can merge layers into $\mathcal{T}_{10000}$ and $\mathcal{T}_{1000}$
- We get a new transducer $\mathcal{T}_D$ with 28 transitions, which is the union of $\mathcal{T}_a \simeq \mathcal{T}_{10000}$ and $\mathcal{T}_b \simeq \mathcal{T}_{1000}$
- We define the family T_n, with $T_{n+3} = T_{n+1} \circ T_n \circ T_{n+1}$
- We show $\mathcal{T}_b = T_0$, $\mathcal{T}_{aa} = T_1$, $\mathcal{T}_{bab} = T_2$
- The only admissible vertical word for $\mathcal{T}_D$ is the Fibonacci word

$\mathcal{T}$ is the union of $\mathcal{T}_0$ and $\mathcal{T}_1$, with (resp.) 9 and 2 tiles
For $w \in \{0,1\}^* \setminus \{\epsilon\}$, let $\mathcal{T}_w = \mathcal{T}_{w[1]} \circ \mathcal{T}_{w[2]} \circ \dots \mathcal{T}_{w[|w|]}$

Fact

$s(\mathcal{T}_{11})$, $s(\mathcal{T}_{101})$, $s(\mathcal{T}_{1001})$ and $s(\mathcal{T}_{00000})$ are empty

If t is a tiling by $\mathcal{T}$, then there exists a bi-infinite word $w \in \{1000, 10000\}^{\mathbb{Z}}$ s.t. $t(x,y) \in T(\mathcal{T}_{w[y]})$

Let $\mathcal{T}_A = s(\mathcal{T}_{1000} \cup \mathcal{T}_{10000})$

There is a bijection between tilings by $\mathcal{T}$ and tilings by $\mathcal{T}_A$

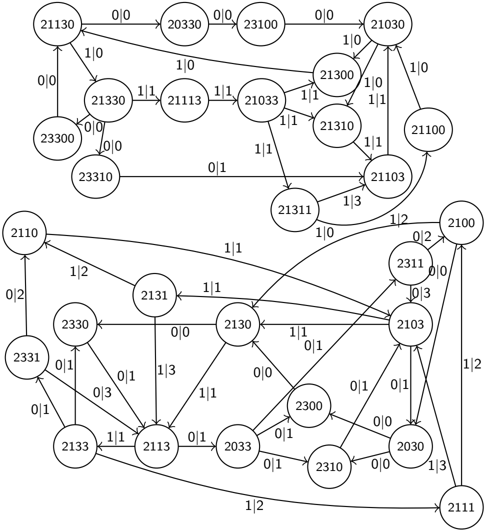


Figure: $\mathcal{T}_A$, the union of $s(\mathcal{T}_{10000})$ (top) and $s(\mathcal{T}_{1000})$ (bottom).

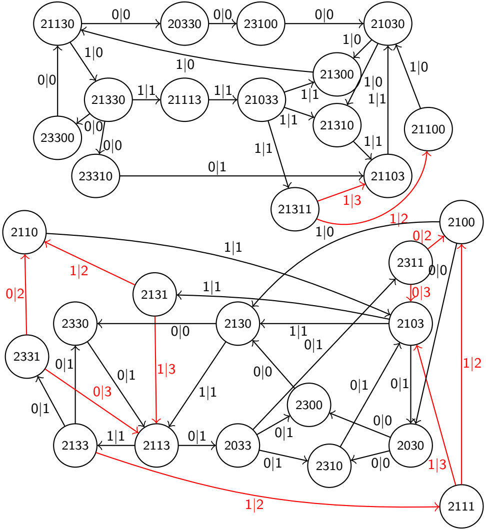


Figure: $\mathcal{T}_A$, the union of $s(\mathcal{T}_{10000})$ (top) and $s(\mathcal{T}_{1000})$ (bottom).

Elimination of transitions with 2, 3 or 4

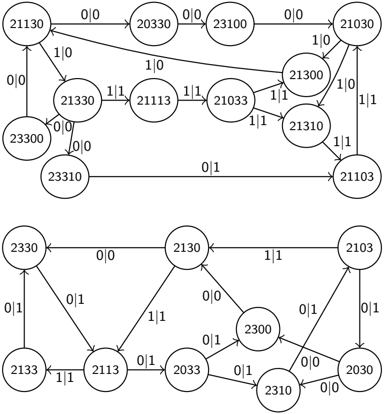


Figure: $\mathcal{T}_B = s(s(\mathcal{T}_A{}^{\mathrm{tr}})^{\mathrm{tr}})$.

Simplification by bi-simulation

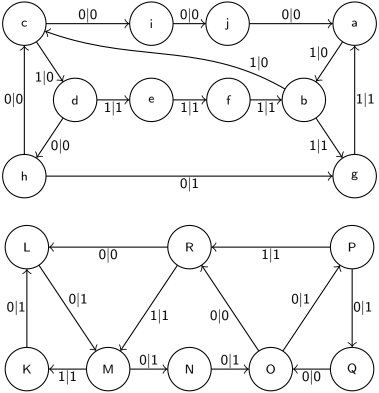


Figure: $\mathcal{T}_C$, “simplification” of $\mathcal{T}_B$.

Proposition

Let $(w_i)_{i\in\mathbb{Z}}$ be a bi-infinite sequence of bi-infinite words s.t. $w_i \mathcal{T}_C w_{i+1}$ for every $i \in \mathbb{Z}$.
Then for every $i \in \mathbb{Z}$, w_i is $(010, 101)$-free

This follow from the fact than $s((\mathcal{T}_C{}^{\mathrm{tr}})^3)$ does not contains the state 010, nor the state 101

Every tiling by $\mathcal{T}_C$ is in bijection with a tiling by $\mathcal{T}_D$

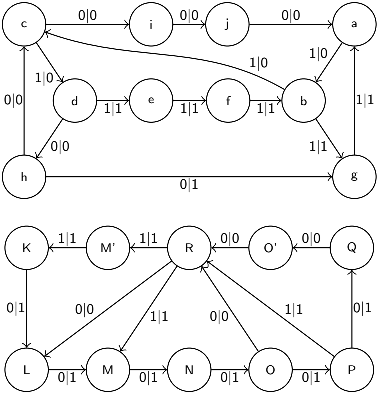
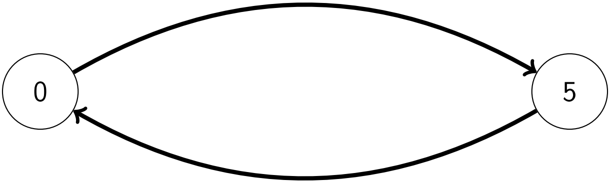


Figure: $\mathcal{T}_D$, the union of $\mathcal{T}_a$ (top) and $\mathcal{T}_b$ (bottom)

T_n for even n:

$\alpha : 0^{g(n+1)-3} | 1^{g(n+1)-3}$

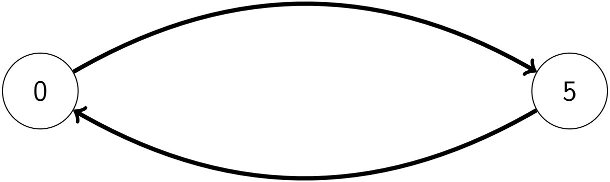


$$
\begin{array}{ll}
\beta : 0^{g(n)+3} & |(100)1^{g(n)} \\
\gamma : 0^{g(n+2)+3} & |1^{g(n+1)}(000)1^{g(n)} \\
\delta : 0^{g(n)}(111)0^{g(n+1)} & |1^{g(n+2)+3} \\
\epsilon : 0^{g(n)}(110) & |1^{g(n)+3} \\
\omega : 0^{g(n+2)}(110)0^{g(n)} & |1^{g(n)}(100)1^{g(n+2)}
\end{array}
$$

$g(n)$ is the $(n+2)$-th Fibonacci number: $g(0) = 2$, $g(1) = 3$, $g(n+2) = g(n+1) + g(n)$

T_n for odd n:

$$\mathbb{A} : 1^{g(n+1)-3} | 0^{g(n+1)-3}$$



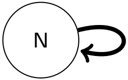
$$\mathbb{B} : 1^{g(n)+3} \quad | (110)0^{g(n)}$$
$$\mathbb{C} : 1^{g(n+2)+3} \quad | 0^{g(n+1)}(111)0^{g(n)}$$
$$\mathbb{D} : 1^{g(n)}(000)1^{g(n+1)} | 0^{g(n+2)+3}$$
$$\mathbb{E} : 1^{g(n)}(100) \quad | 0^{g(n)+3}$$
$$\mathbb{O} : 1^{g(n+2)}(100)1^{g(n)} | 0^{g(n)}(110)0^{g(n+2)}$$

$g(n)$ is the $(n+2)$-th Fibonacci number: $g(0) = 2$, $g(1) = 3$, $g(n+2) = g(n+1) + g(n)$

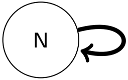
Case of $\mathcal{T}_b$

In $\mathcal{T}_b$, every long enough path passes through "N"

Thus $\mathcal{T}_b$ is equivalent to:



00000	10011
00000000	11100011
00111000	11111111
00110	11111
0000011000	1110011111
0010	1011
001000	111011
0000010	1110011
0011000	1011111



00000	10011
00000000	11100011
00111000	11111111
00110	11111
0000011000	1110011111

Case of $\mathcal{T}_{aa}$

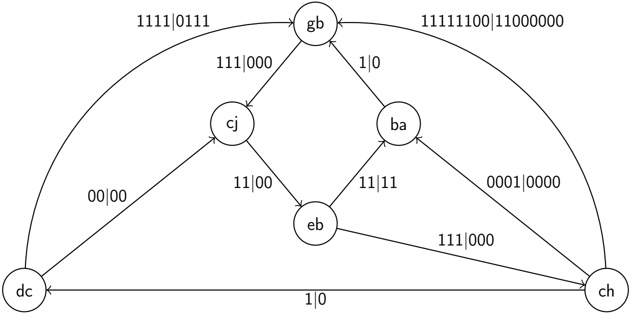
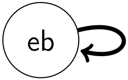


Figure: $s(\mathcal{T}_{aa})$

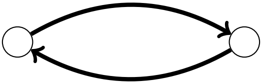
In $s(\mathcal{T}_{aa})$, every long path passes through "eb". It is equivalent to:



11111111	11000000
1111111111111	0000011100000
1110001111111	0000000000000
11110011	00000000
1111111110011111	0001100000000000

$\mathcal{T}_{aa}$ is equivalent to T_1

$1^{5-3}|0^{5-3}$



1^{3+3}	$(110)0^3$
1^{8+3}	$0^5(111)0^3$
$1^3(000)1^5$	$0^{8+3)}$
$1^3(100)$	0^{3+3}
$1^8(100)1^3$	$0^3(110)0^8$

Case of $\mathcal{T}_{bab}$

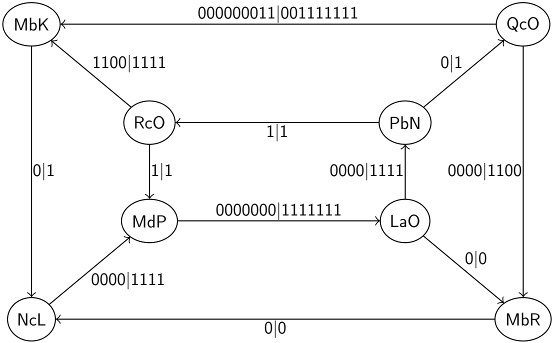
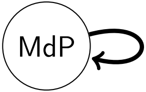


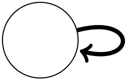
Figure: $s(\mathcal{T}_{bab})$

Every long path passes through "MdP". Thus it is equivalent to :




```
0000000000000             |1111111001111
000000000000000000000     |111111111111110001111
000000000001110000000     |111111111111111111111
0000000000011             |1111111111111
00000000000000000001100000|11111111111100111111111111
                          |
```

If we shift the input (3 times) and the output (6 times), we get:



```
0000000000000             |1001111111111
000000000000000000000     |111111110001111111111
000000000111000000000     |111111111111111111111
0000000011000             |1111111111111
00000000000000001100000000|11111100111111111111111111
                          |
```

$\mathcal{T}_{bab}$ is equivalent to T_2

Fact

$s(\mathcal{T}_{bb})$, $s(\mathcal{T}_{aaa})$ and $s(\mathcal{T}_{babab})$ are empty

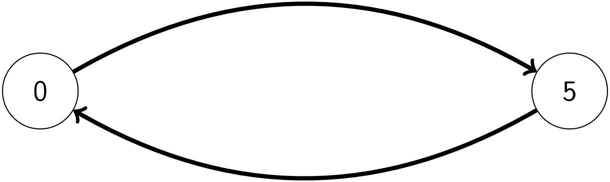
If t is a tiling by $\mathcal{T}_D$, then there is a bi-infinite word $w \in \{b, aa, bab\}^{\mathbb{Z}}$ s.t. $t(x,y) \in T(\mathcal{T}_{w[y]})$

That is, the tilings with $\mathcal{T}_D$ are images of the tilings by

$$\mathcal{T}_b \cup \mathcal{T}_{aa} \cup \mathcal{T}_{bab} \simeq T_0 \cup T_1 \cup T_2.$$

T_n for even n:

$\alpha : 0^{g(n+1)-3} | 1^{g(n+1)-3}$



$\beta : 0^{g(n)+3} | (100)1^{g(n)}$

$\gamma : 0^{g(n+2)+3} | 1^{g(n+1)}(000)1^{g(n)}$

$\delta : 0^{g(n)}(111)0^{g(n+1)} | 1^{g(n+2)+3}$

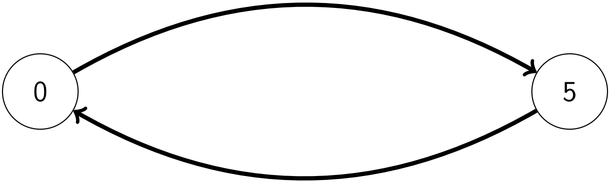
$\epsilon : 0^{g(n)}(110) | 1^{g(n)+3}$

$\omega : 0^{g(n+2)}(110)0^{g(n)} | 1^{g(n)}(100)1^{g(n+2)}$

$g(n)$ is the $(n+2)$-th Fibonacci number: $g(0) = 2$, $g(1) = 3$, $g(n+2) = g(n+1) + g(n)$

T_n for odd n:

$\mathbb{A} : 1^{g(n+1)-3} | 0^{g(n+1)-3}$



$\mathbb{B} : 1^{g(n)+3} \quad | (110)0^{g(n)}$

$\mathbb{C} : 1^{g(n+2)+3} \quad | 0^{g(n+1)}(111)0^{g(n)}$

$\mathbb{D} : 1^{g(n)}(000)1^{g(n+1)} | 0^{g(n+2)+3}$

$\mathbb{E} : 1^{g(n)}(100) \quad | 0^{g(n)+3}$

$\mathbb{O} : 1^{g(n+2)}(100)1^{g(n)} | 0^{g(n)}(110)0^{g(n+2)}$

$g(n)$ is the $(n+2)$-th Fibonacci number: $g(0) = 2$, $g(1) = 3$, $g(n+2) = g(n+1) + g(n)$

One supposes than n is even. (The odd case is similar.)

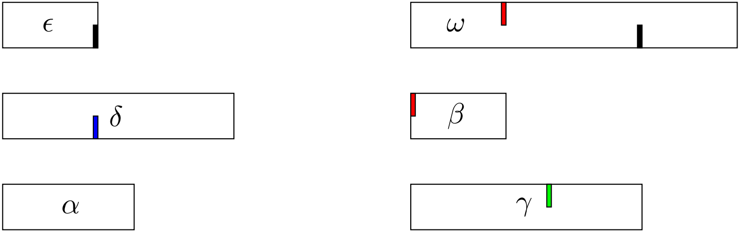
One supposes that $T_n \cup T_{n+1} \cup T_{n+2}$ tiles the plane

We show that $T_{n+1} \cup T_{n+2} \cup T_{n+3}$ tiles, and that:

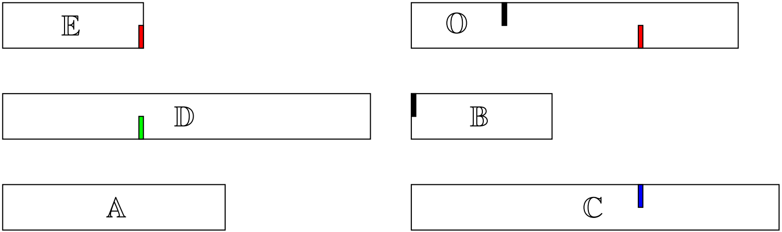
$$T_{n+3} \simeq T_{n+1} \circ T_n \circ T_{n+1}.$$

T_n is surrounded by T_{n+1}
(output of T_{n+1} has more 0's than 1's, T_n and output T_{n+2} have more 1's than 0's)

Transitions of T_n:



Transitions of T_{n+1}:



Let's take T_n. (We forget α.)

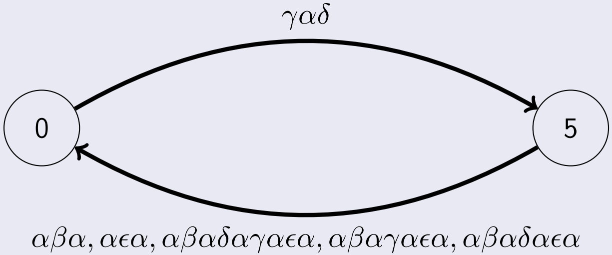
Lemma

The following words cannot appear:

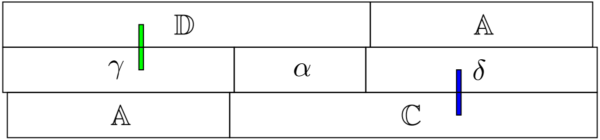
- $\gamma\omega$,$\gamma\gamma$,$\gamma\beta$, $\beta\omega$, $\beta\beta$, $\beta\epsilon\beta$, $\gamma\epsilon\beta$, $\beta\delta\epsilon\beta$, $\gamma\delta\epsilon\beta$
- $\omega\delta$, $\delta\delta$, $\epsilon\delta$, $\omega\epsilon$, $\epsilon\epsilon$, $\epsilon\beta\epsilon$, $\epsilon\beta\delta$, $\epsilon\beta\gamma\epsilon$, $\epsilon\beta\gamma\delta$
- ω

Lemma

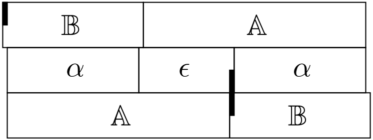
Every infinite path in the transducer T_n can be seen as an infinite path in the following transducer:



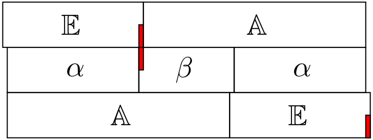
$\gamma\delta \to \mathbb{A}'$



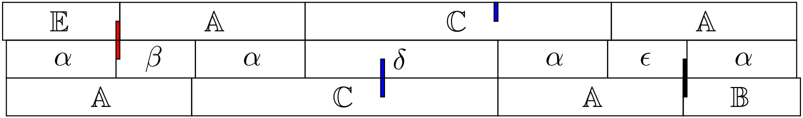
$\epsilon \to \mathbb{B}'$



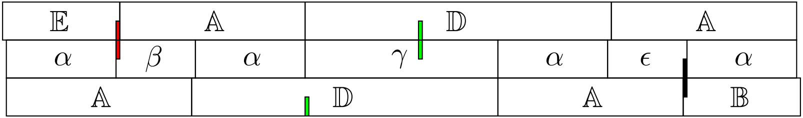
$\beta \to \mathbb{E}'$:



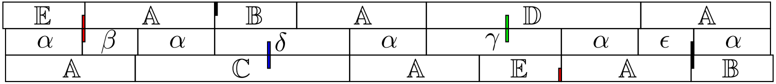
$\beta\gamma\epsilon \to \mathbb{C}'$:



$\beta\delta\epsilon \to \mathbb{D}'$:



$\beta\delta\gamma\epsilon \to \mathbb{O}'$:



It remains to show that one cannot have a stack of layers $T_{n+1}, T_n, T_{n+1}, T_n, T_{n+1}$

We can merge layers of a tiling with $T_n \cup T_{n+1} \cup T_{n+2}$ by:

1. T_{n+1}
2. T_{n+2}
3. $T_{n+1} T_n T_{n+1} \quad \simeq \quad T_{n+3}$

$T_n \simeq \mathcal{T}_{u_n}$ where:

$$u_0 = b, u_1 = aa, u_2 = bab$$

and

$$u_{n+3} = u_{n+1} u_n u_{n+1}$$

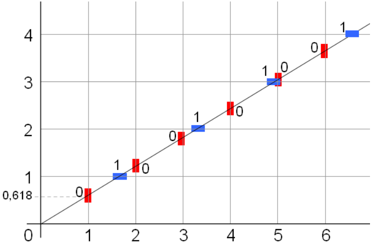
$$(u_n)_{n \geq 0} = (b, aa, bab, aabaa, babaabab, \ldots)$$

is a sequence of factors of the Fibonacci word
(the "singular factors")

Fibonacci Word

Aperiodic word

$$w_f = abaababaabaababaababaabaababaabaababaababaa...$$



Fixed point of the morphism $a \to ab, b \to a$

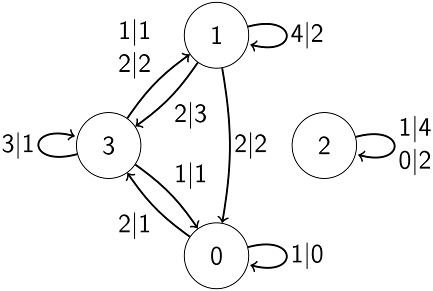
$v_0 = a, \;\; v_1 = ab$ and $v_{n+2} = v_{n+1}v_n$

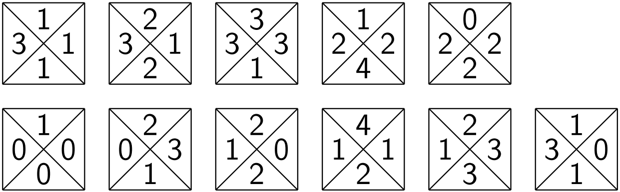
v_i converges to w_f

$(v_n)_{n\geq 0} = (a, \underline{a}b, \underline{ab}a, \underline{abaa}b, \underline{abaabab}a, \underline{abaababaabaa}b, \ldots)$

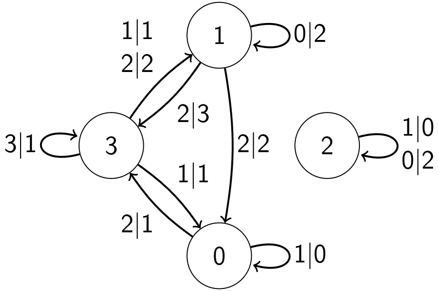
$(u_n)_{n\geq 0} = (b, a\underline{a}, b\underline{ab}, a\underline{abaa}, b\underline{abaabab}, a\underline{abaababaabaa}, \ldots)$

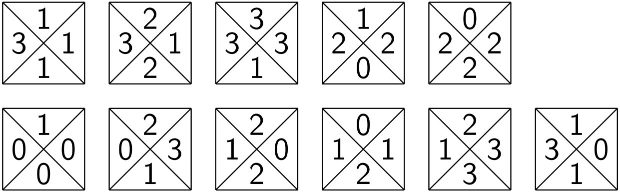
Aperiodic set with 11 tiles

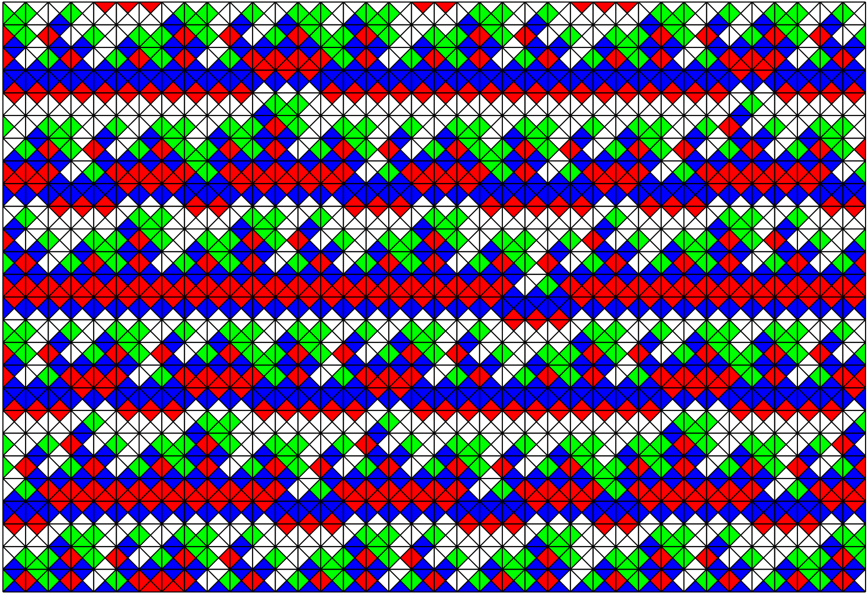




Aperiodic set with 11 tiles and 4 colors



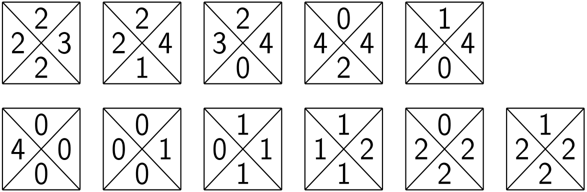


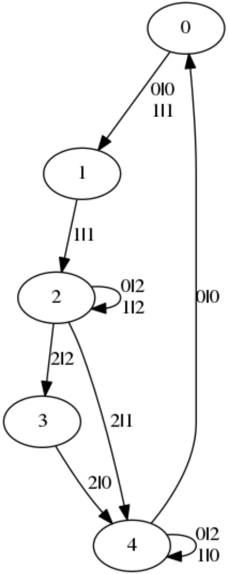


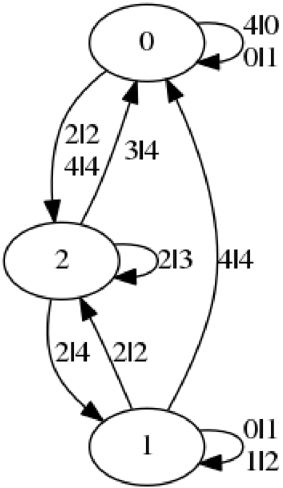
Open question 1 : Another aperiodic set ?

Tiles sets with 11 tiles:

- 3 aperiodic
- 23 others "candidates"
- 9 of "Kari" type, and probably finite
- 14 not "Kari" type
- 1 strange (interesting) candidate:




0
1
2
3
4
0|0
1|1
1|1
0|2
1|2
2|2
2|1
2|0
0|0
0|2
1|0


0
4|0
0|1
2|2
4|4
3|4
2
2|3
4|4
2|4
2|2
1
0|1
1|2

Open question 2 : “proof from the book” ?

If we look at densities of 1 on each line on an infinite tiling, one transducer add $\varphi - 1$ and the other add $\varphi - 2$.

$\rightarrow$ “additive-Kari-type” ?