Computer-assisted proofs: Introduction

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CNRS
équipe MC2, LIP, ENS Lyon

18/11/2025

1.	mar	18/11	15h45	Introduction
2.	jeu	20/11	10h15	Disc packings in containers
3.	mar	25/11	15h45	Interval arithmetic
4.	jeu	27/11	10h15	Interval arithmetic
5.	mar	2/12	15h45	Disc packings on the plane
6.	jeu	4/12	10h15	Triangulated packings and 3-disc packings
7.	mar	9/12	15h45	Kepler conjecture and dodecahedral conjecture
8.	jeu	11/12	10h15	2-sphere packings: rock salt cojecture and bounds
9.	mar	16/12	15h45	Tammes problem, kissing number
10.	jeu	18/12	10h15	Four-color theorem and tilings
11.				Four-color theorem and tilings
12.				Four-color theorem and tilings
13.				Proof verification
14.				Proof verification
15.				Proof verification

Lecturers

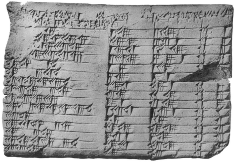
Daria Pchelina, Michael Rao, Damien Pous, Nathalie Revol

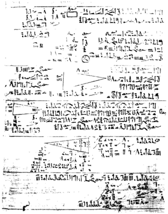
Grading: 100% homework

Small homework after each class, to be submitted by email before the next session.
At the start of each class, a random student will briefly present their solution.
Two catch-up sessions: opportunity to still get points for homework exercises with a penalty, dates to be confirmed.

Example: today's homework is due on 20/11/25 at 10h15.

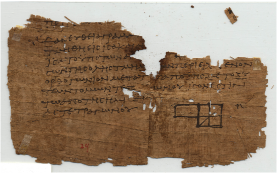
Before Euclid: practical life observation, e.x. land surveying → geometry.



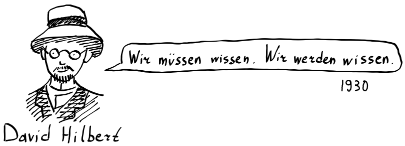


Plimpton 322 Babylonian clay tablet ≈ 1800 BC   Rhind Mathematical Papyrus, Egypt ≈ 1550 BC

Elements 300 BC Euclid: definitions, theorems, proofs, but gaps in reasoning.



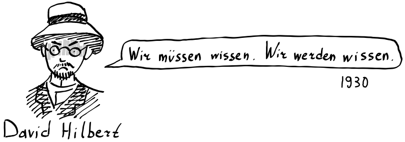
19th century: Cauchy, Riemann, Weierstrass (rigorous calculus); Abel, Galois (algebra); Cantor (foundations of set theory).



1900 Hilbert's problems
1st: Continuum Hypothesis
2nd: consistency of the arithmetic axioms

1910–1913 *Principia Mathematica*
Whitehead, Russell

1920s Hilbert's program



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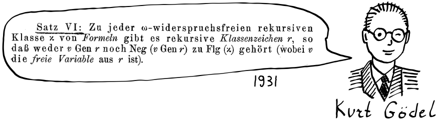
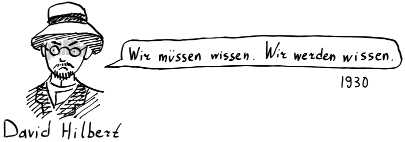
- 1st any consistent formal system containing basic arithmetic is incomplete
  (a statement that is not provable nor disprovable)
- 2nd consistency of a system can not be proved within the system





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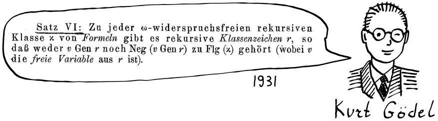
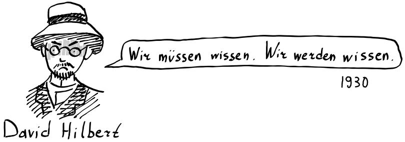


Examples:

Gödel 1941

Continuun Hypothesis is not disprovable in ZFC
Axiom of Choice is not disprovable in ZF



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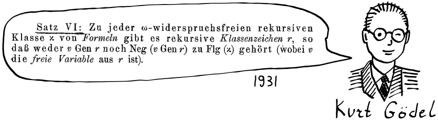


Examples:

Gödel 1941 Cohen 1963

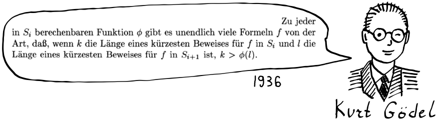
Continuun Hypothesis is not disprovable nor provable in ZFC

Axiom of Choice is not disprovable nor provable in ZF

We might never know...

And even if we do, the proof might be long.

1936 *On the length of proofs*
statements with arbitrary long shortest proofs



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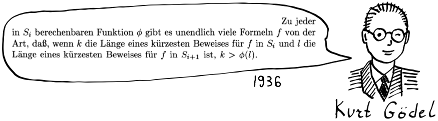

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Examples for Peano arithmetic:

"This statement has no proof in Peano arithmetic that contains fewer than N symbols."

1980 Friedman's Finite Form (FFF) of Kruskal's Theorem
$\forall k > 1$, $\exists n$ such that, if $T_1, \ldots T_n$ is a sequence of trees with the cardinality of T_i at most $k + i$, then there are $i < j \leq n$ such that T_i is homeomorphically embeddable in T_j.
FFF $\forall k \exists n\ A(k, n)$ is unprovable in PA. Any proof of $\exists n\ A(10, n)$ has length at least

$$\left.2^{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}\right\}1000$$

Smoryński 1982

History of proofs

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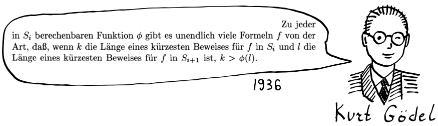

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Possible examples for mathematics:

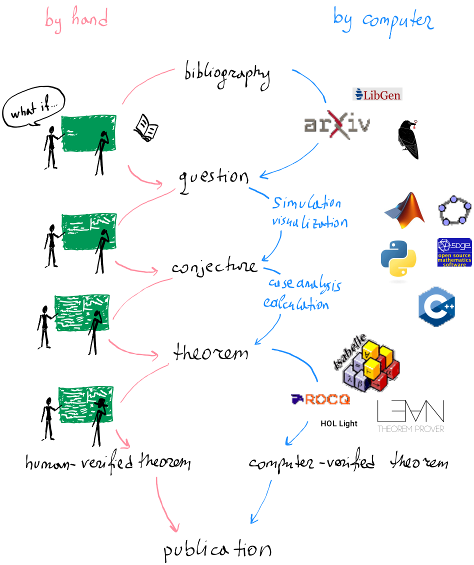
1974 4-color theorem: 139 pages and computer code (1997 Robertson et al: shorter)

1998–2000 Kepler conjecture: 250 pages and $>$ 180000 lines of code

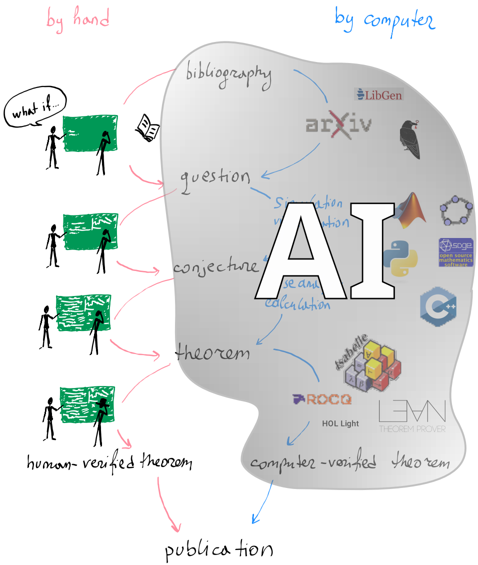
1983–2004 Robertson-Seymour theorem: 20 papers, 500 pages

1832–2008 Classification of finite simple groups: hundreds of papers, 20000 pages

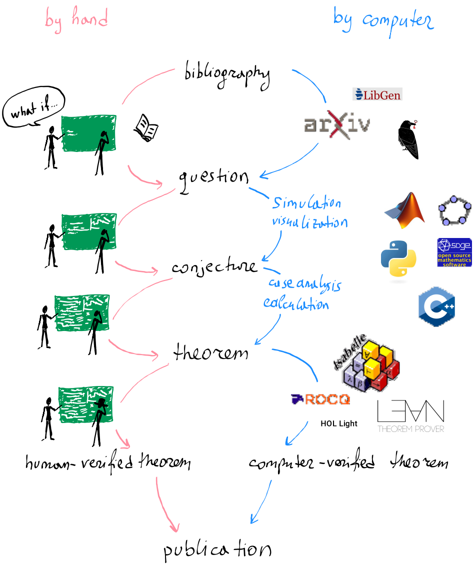
Life of a modern mathematician



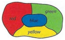
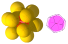
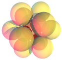
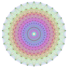
Life of a future mathematician?



Life of a modern mathematician



Conjectured without proved with

	4-color theorem	1850s	1976
	Kepler conjecture	1611	1998
	Dodecahedral conjecture	1943	1998
	Tammes problem for 13 and 14 spheres	1953	2012, 2015
	Optimal sphere packings (E_8 (1873) and Leech lattices (1967)) in $\mathbb{R}^8, \mathbb{R}^{24}$		2017

Conjectured with proved without

- 2D bootstrap percolation 1989 2003

  square $n \times n$, each cell is infected with initial probability p

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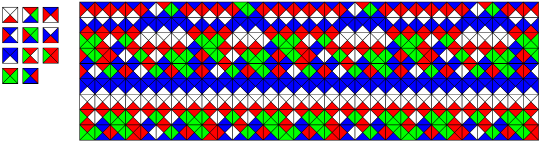
- Aperiodic set of 11 Wang tiles 2015



Jeandel, Rao

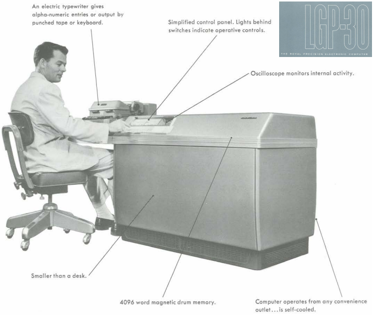
Conjectured with proved with

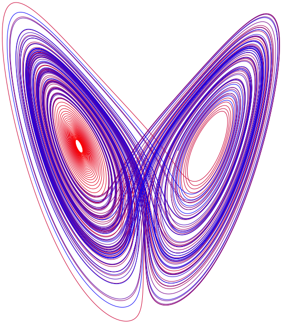
- No aperiodic tileset of size less than 11 Jeandel, Rao 2015

Conjectured with proved with

- No aperiodic tileset of size less than 11 — Jeandel, Rao 2015
- Lorenz attractor — 1963 — Tucker 2002
  Atmospheric convection model:

$$\dot{x} = \sigma(y - x), \quad \dot{y} = \rho x - y - xz, \quad \dot{z} = xy - \beta z$$





robust strange attractor for $\sigma = 10, \rho = 28, \beta = 8/3$

AI

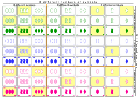
Conjecture generation (mostly in graph theory):

- HR, HR2
- TxGraffiti (former Graffiti, Graffity.pc)

If G is connected and regular, then $\texttt{matching\_number}\,(G) \geq \texttt{independence\_number}\,(G)$.
This bound is sharp on 3 graphs. Caro et al. 2020

New records (DeepMind) :

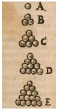
- 2022 AlphaTensor: algorithms for matrix multiplication
  e.x. 4×4-matrices, number of multiplications: $49 \to 47$
- 2024 FunSearch: cap set size record in dimension 8: $496 \to 512$

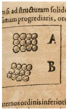


- 2025 AlphaEvolve: kissing number record in dimension 11: 592 (2022) $\to 593$

Sphere packings then

Kepler 1611: the best way to store cannonballs is this one

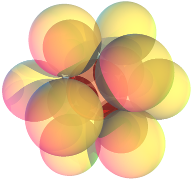




Lagrange 1773: equivalence classes of quadratic forms and reduced quadratic forms

Gauss 1831: introduces lattices

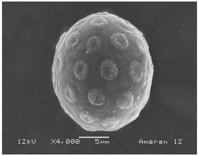
$\Rightarrow$ indirectly, Kepler conjecture for lattice packings

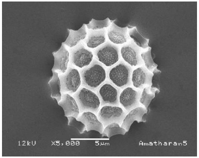


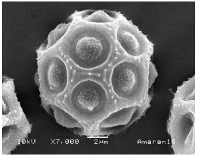
Newton 1690s: Only 12 unit spheres fit around a unit sphere.

Gregory: I can fit 13!

Tammes 1930: arrangements of "the places of exit" on the surface of pollen grains







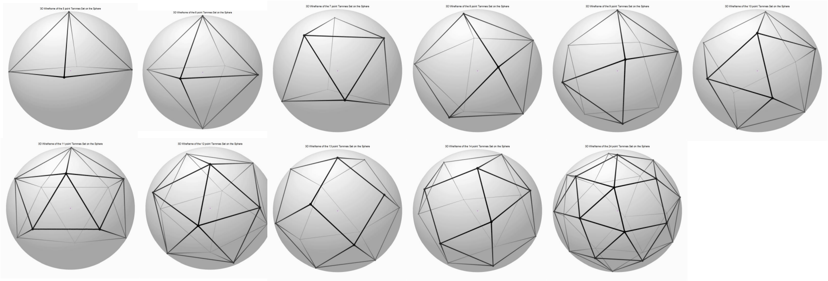
Sphere packings now

Hales and Ferguson 1998 Kepler conjecture + 2017 verified proof

Vyasovska et al. 2017 Fields Medal 2022: optimal sphere packing in dimension 8 and 24

1979–2003 Kissing number solved in dimensions 3, 4, 8, 24

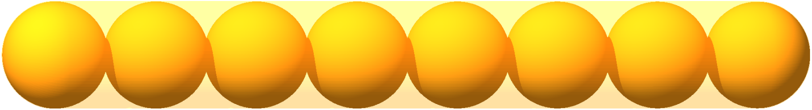
1943–1963 Tammes problem solved for 1-14, 24



Locally-optimal packings

Optimal **finite packing** of k unit spheres: minimize volume of convex hull — Thue 1992

sausage:



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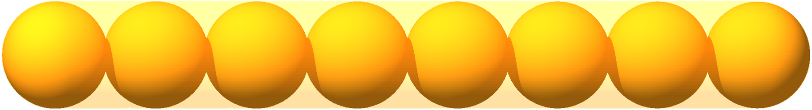
sausage: 

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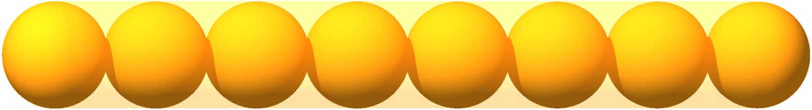
sausage: 

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$\mathbb{R}^2$: false for $k \geq 3$
$\mathbb{R}^3, \mathbb{R}^4$: true for $k \leq n_3^*, n_4^*$ false for $k \geq N_3^*, N_4^*$ **sausage catastrophe** (optimum is full-dimensional)

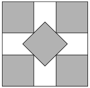
$5 \leq n_3^* \leq 56,\ 56 \leq N_3^* \leq 58$
$5 \leq n_4^* \leq 338196,\ 5 \leq N_4^* \leq 516964$ 1983–2023

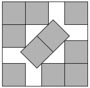
Des carrés dans des carrés Lastman s1 ep5

k^2, $k^2 - 1$ and $k^2 - 2$ unit squares: optimal square container is $k \times k$

also for some $k^2 - 3$: $6, 13, 22, 33, 46$. 1979–2018

Non-trivial solutions for $5, 10$ squares: 1979, 2003





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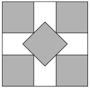
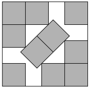
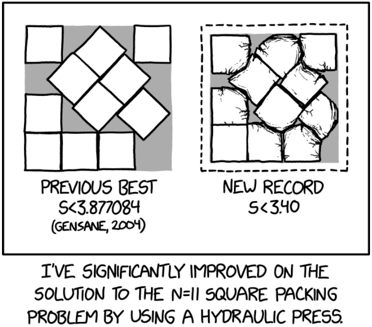




Lower & upper bounds for others

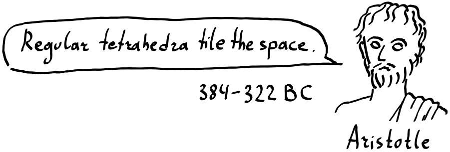
Example: best known packing of 11 squares: Trump 1979



xkcd

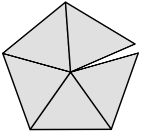
best lower bound on optimal square side: $2 + \frac{4}{\sqrt{5}} \approx 3.789$ Stromquist 2003

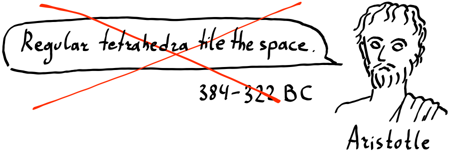
Ulam's conjecture



Ulam's conjecture

$\arccos(\frac{1}{3}) \approx 70.259^\circ$





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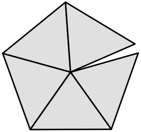

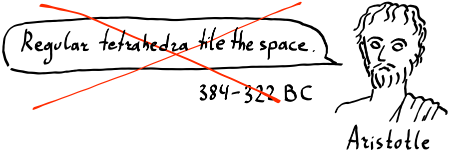

18. Building up of Space from Congruent Polyhedra.

. . . How can one arrange most densely in space an infinite number of equal solids of given form, *e. g.*, spheres with given radii . . .

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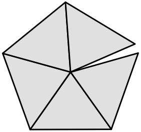

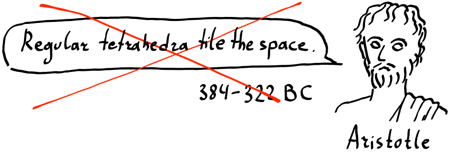

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DONE
Hales, Ferguson
1998

$\delta^* = \frac{\pi}{3\sqrt{2}} \approx 74\%$

Ulam suspected that **sphere is the worst case of dense packing of identical convex solids**, but that this would be difficult to prove. *1972*

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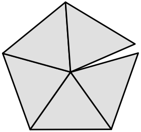

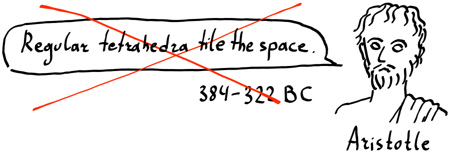

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Ulam's conjecture is proved for Johnson and Catalan solids *J. de Graaf et al 2011*
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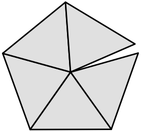

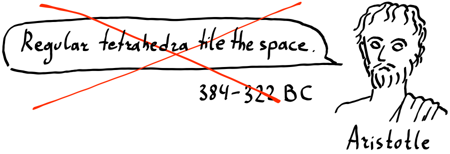

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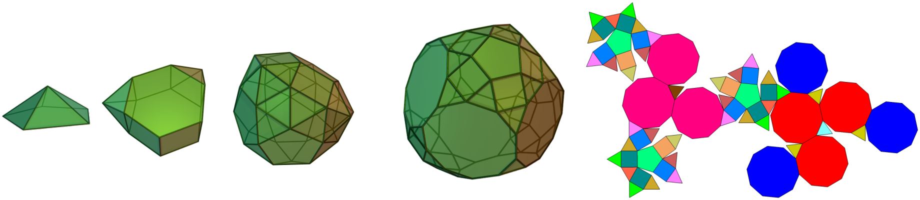
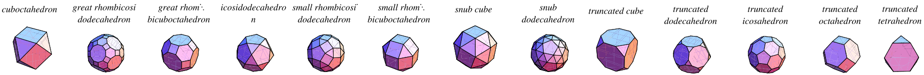
Johnson and Catalan solids

Johnson solid: convex polyhedron whose faces are regular polygones 92



Archimedean solid: vertex-transitive Johnson solid 13

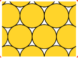
all vertices are surrounded by the same kinds of faces with the same angles

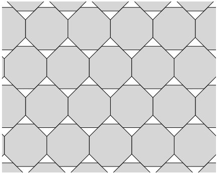


Catalan solid: dual of Archimedian solid (face-transitive) 13

Ulam's conjecture in other dimensions

$\mathbb{R}^2$: **false**

δ () $= \frac{\pi}{2\sqrt{3}} \approx 90.69\%$



Regular octagon
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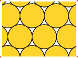
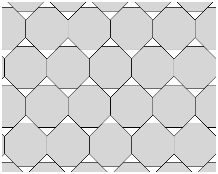
$\delta\Big($  $\Big) = \frac{\pi}{2\sqrt{3}} \approx 90.69\%$



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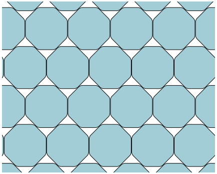

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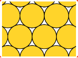
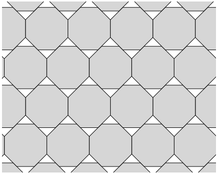
$\delta\Bigg($  $\Bigg) = \frac{\pi}{2\sqrt{3}} \approx 90.69\%$



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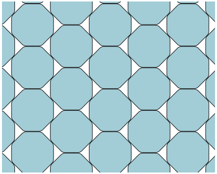

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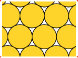
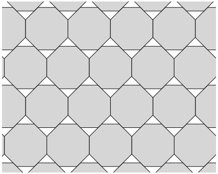
$\delta\Big($  $\Big) = \frac{\pi}{2\sqrt{3}} \approx 90.69\%$



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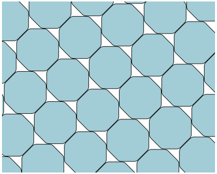

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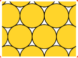
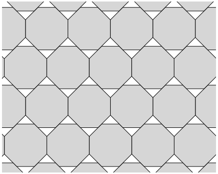
$\delta\Big($  $\Big) = \frac{\pi}{2\sqrt{3}} \approx 90.69\%$



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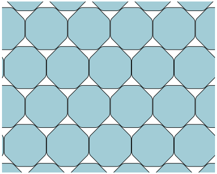

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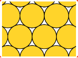
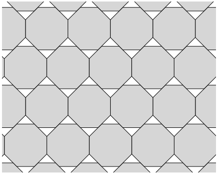
$\mathbb{R}^2$: **false** $\delta\Big($  $\Big) = \frac{\pi}{2\sqrt{3}} \approx 90.69\%$



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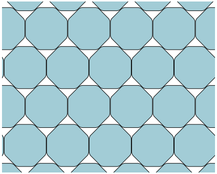

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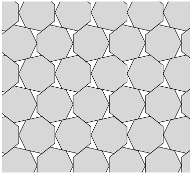

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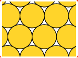
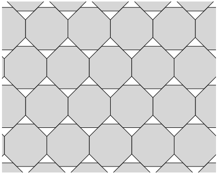
$\delta\Big($  $\Big) = \frac{\pi}{2\sqrt{3}} \approx 90.69\%$



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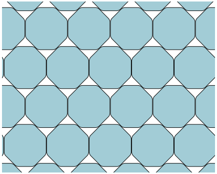

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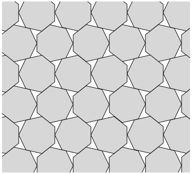

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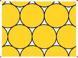
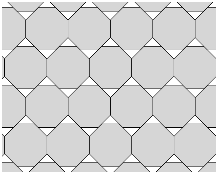
δ () $= \frac{\pi}{2\sqrt{3}} \approx 90.69\%$



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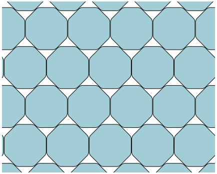

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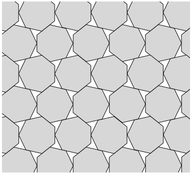

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Lattice density and regular octagons

For a convex body $B \subset \mathbb{R}^n$,

(congruent) packing: collection of congruent copies of B with disjoint interiors
translative packing: only translated copies of B (no rotations)
lattice packing: translative packing where the centers form a lattice in $\mathbb{R}^n$

density of a packing P is the proportion of space covered: $\delta(P)$.

Maximal densities:

$$\delta^l(B) \le \delta^t(B) \le \delta^c(B)$$

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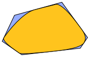

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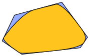

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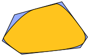

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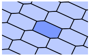
Any centrally symmetric hexagon tiles the plane as a lattice tiling:  $\Rightarrow$

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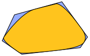

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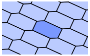
Any centrally symmetric hexagon tiles the plane as a lattice tiling:  $\Rightarrow$

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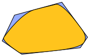

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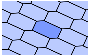
Any centrally symmetric hexagon tiles the plane as a lattice tiling:  $\Rightarrow$

Lemma

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$\Rightarrow$ Optimal packing of regular octagons is a lattice packing. Which one? (Homework)

1. Give an example of a body $B \subset \mathbb{R}^2$ for which $\delta^t(B) \neq \delta^c(B)$
2. * Give an example of a body $B \subset \mathbb{R}^2$ for which $\delta^l(B) \neq \delta^t(B)$
3. **lattice**: $L(u, v) := \{xu + yv | x, y \in \mathbb{Z}\}$ for non-collinear $u, v \in \mathbb{R}^2$
   (u, v) are **basis vectors** and form the fundamental domain $(0, 0), u, v, u + v$.
   1. Show that any lattice has an infinite number fundamental domains, describe them.
   2. Show that the areas of all fundamental domains are the same.
   3. Given a quadratic form $q(x, y) = ax^2 + bxy + cy^2$, consider a fundamental domain (u_q, v_q) such that for any point $xu_q + yv_q$ of its lattice, the square of distance to the origin equals $q(x, y)$. Describe (u_q, v_q).
   4. What is the area of the fundamental domain?
   5. What is the smallest area of a fundamental domain of a lattice whose closest points are separated by distance at least 1.
4. What is the maximal lattice density of unit disc?
5. What is the maximal congruent density of unit disc?

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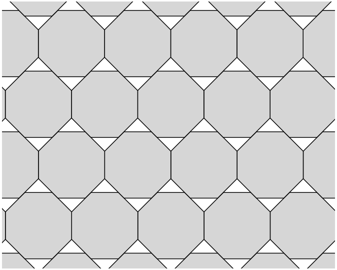

1. Prove that this packing of regular octagons is optimal. You can use the lemma on slide 17.

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Deadline: beginning of next lecture (20/11, 10h15)