Computer-assisted proofs: Disk packings on the plane

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4/12/2025

1. Introduction
2. Definitions
3. Hexagonal packing is optimal
4. Multi-disk packings
5. Homework

1 Introduction

2 Definitions

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Optimal coin packing

Given infinite number of identical coins (),

how to place them on an infinite plane without overlap to maximize the covered surface?

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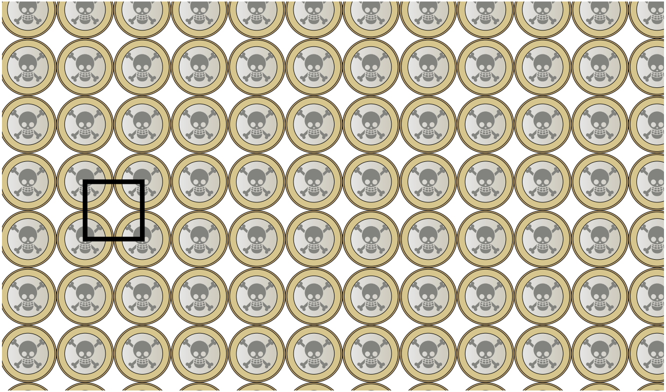
square coin packing:
covers 78% of the surface



Optimal coin packing

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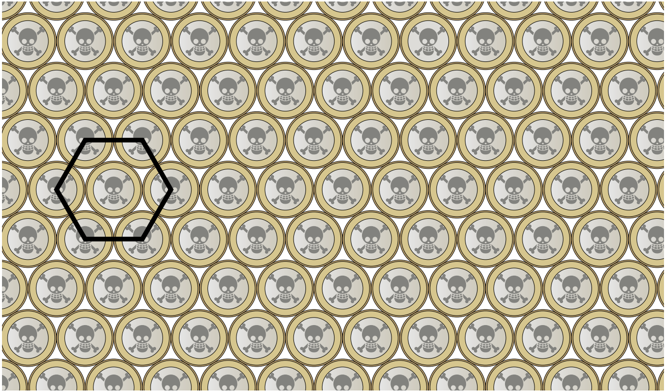
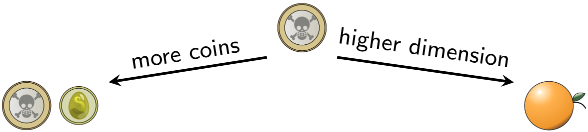
hexagonal coin packing:
covers 90% of the surface

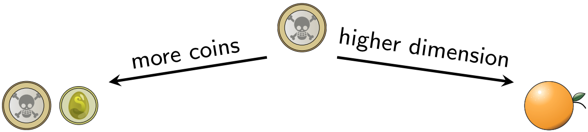


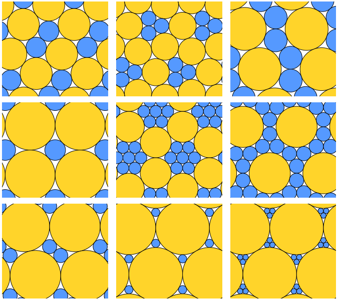
1910–1940

The hexagonal coin packing is optimal.

Introduction







(proved optimal in 2000-2022)

Kepler conjecture, 1611

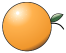
The "cannonball" packing is optimal:

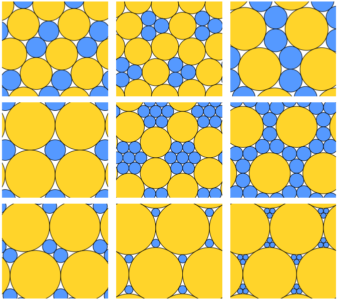
(proved in 1998–2014)

Introduction

more coins

higher dimension





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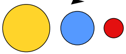

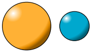

$\mathbb{R}^8, \mathbb{R}^{24}$

(Viazovska, Fields Medal 2022)

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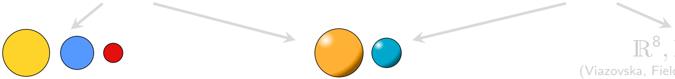

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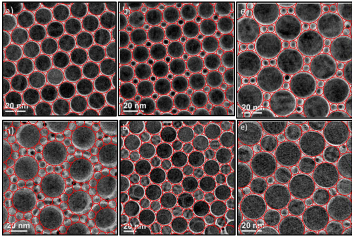
Nanomaterials and packings

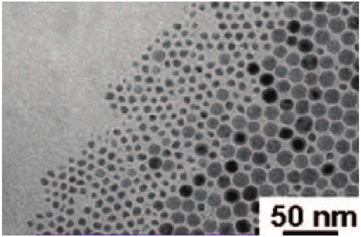
combine different types of nanoparticles
self-assembly

new material



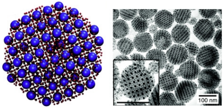
Paik et al 2015

phase separation



Cheon et al 2006

Also in 3D:



Wu, Fan, Yin 2022

Chemists' question : **which sizes and concentrations allow for new materials?**

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Definitions

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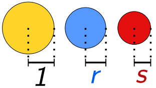
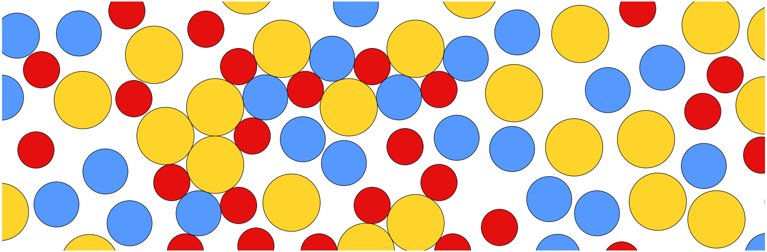


Packing P:
(in $\mathbb{R}^2$)



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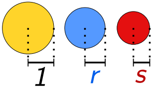

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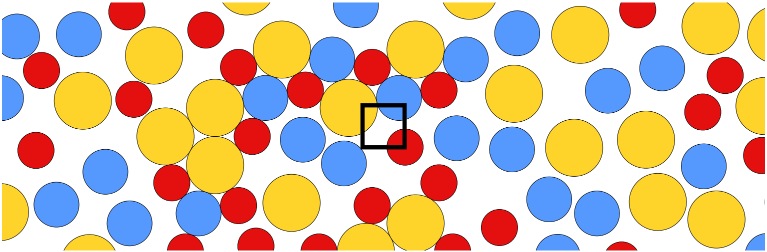

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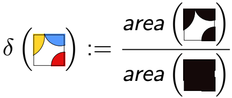

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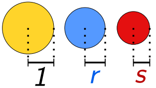

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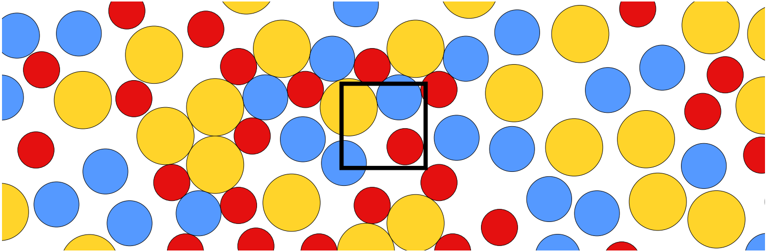

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$$\delta\left(n\blacksquare \cap P\right) := \frac{area\left(n\blacksquare \cap P\right)}{area\left(n\blacksquare\right)}$$

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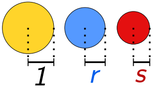

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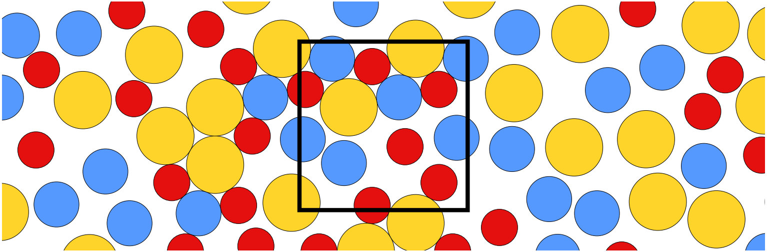

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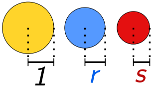

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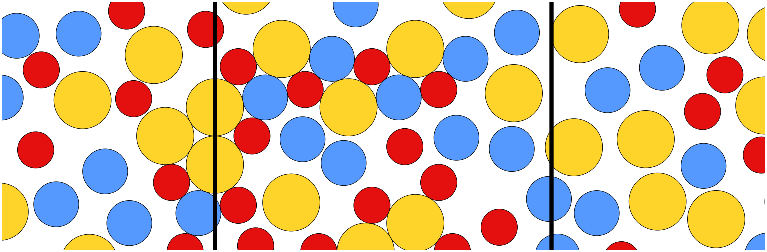

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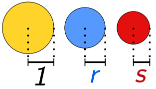

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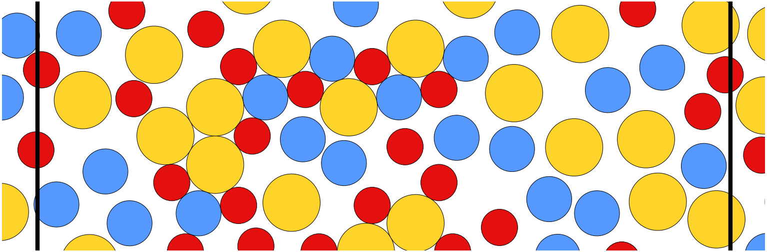

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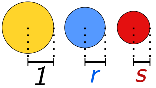

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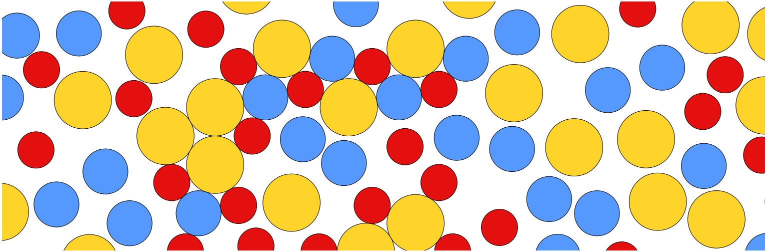

Density:

$$\delta(P) := \limsup_{n\to\infty} \frac{area\left(n\updownarrow \overset{\overset{n}{\leftrightarrow}}{\blacksquare} \cap P\right)}{area\left(n\updownarrow \overset{\overset{n}{\leftrightarrow}}{\blacksquare}\right)}$$

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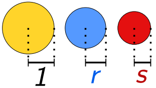

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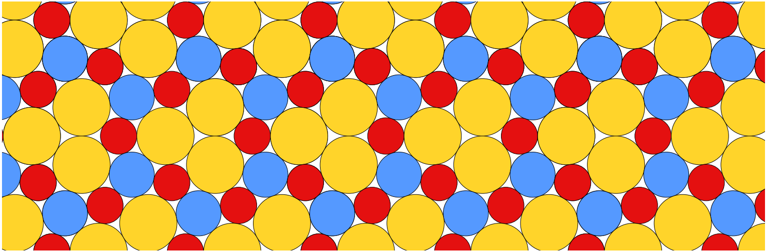

Density:

$$\delta^* \approx 90.9\%$$

Main Question

Given a finite set of disks (e.g., ),
what is the maximal density δ^* of a packing?

$$\delta^* := \sup_P \delta(P)$$

Existence of optimal packing

$\delta^* =$ sup: does a packing of density δ^* always exist?

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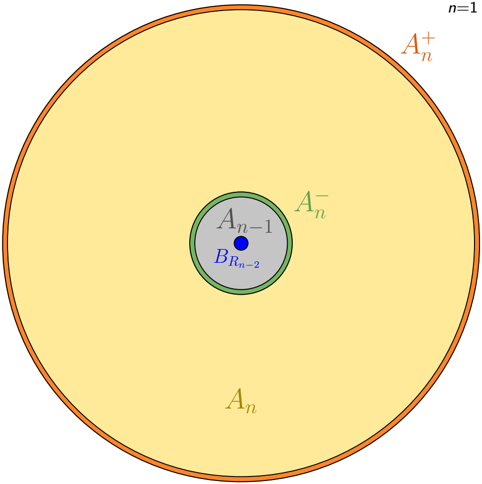

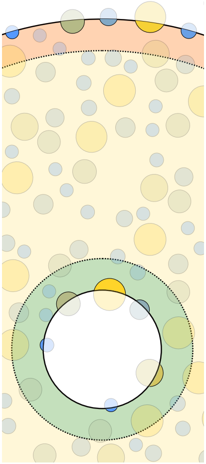

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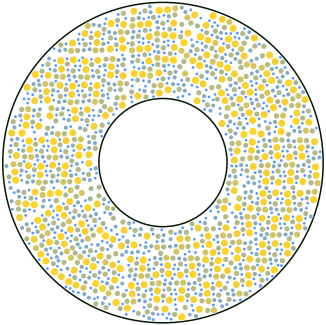
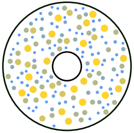
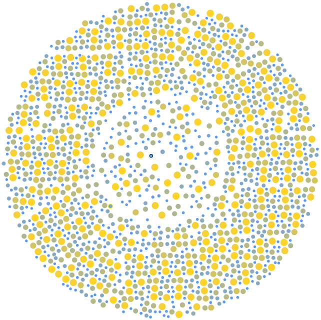


$P_n|_{A_2}$

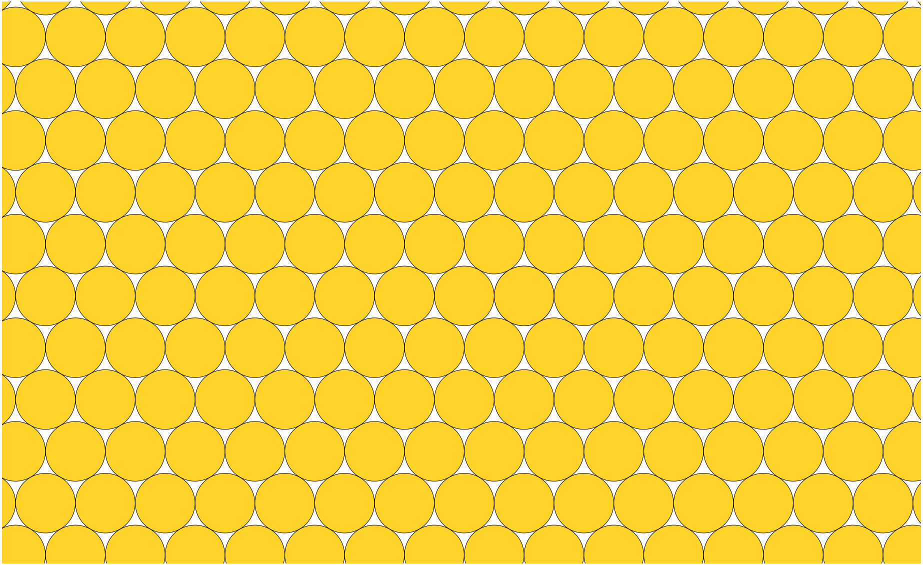




$P_n|_{A_1} \cup P_n|_{A_2} \cup P_n|_{A_3}$

Number of optimal packings

How many packings of density δ^*?



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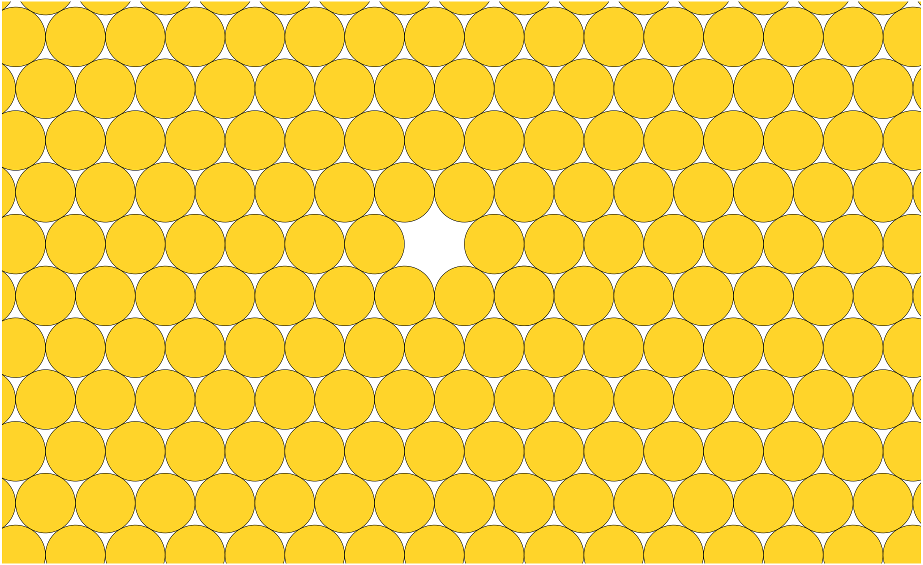

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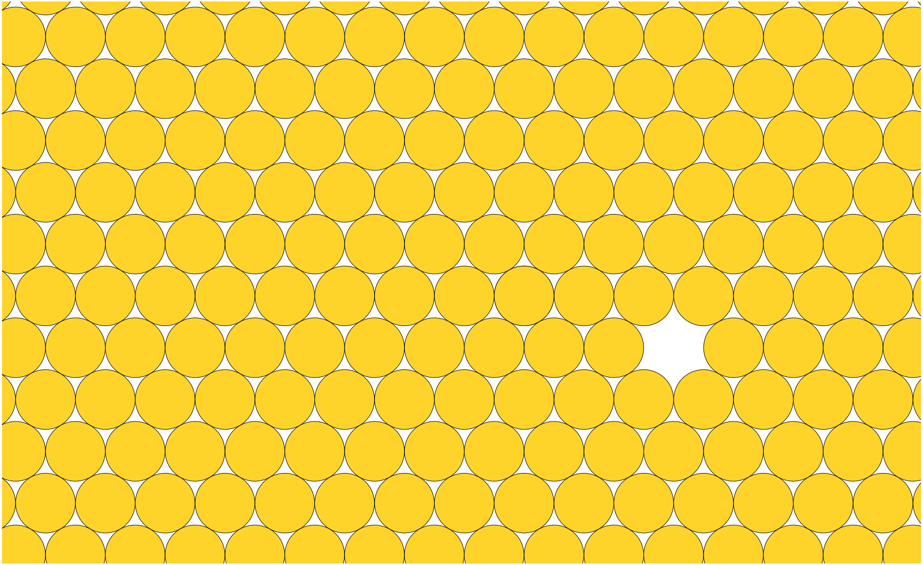

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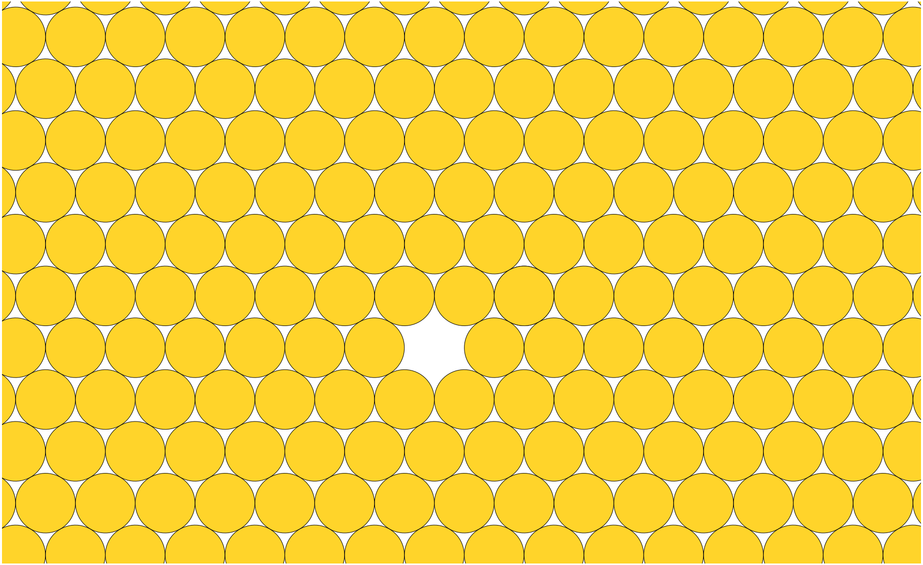

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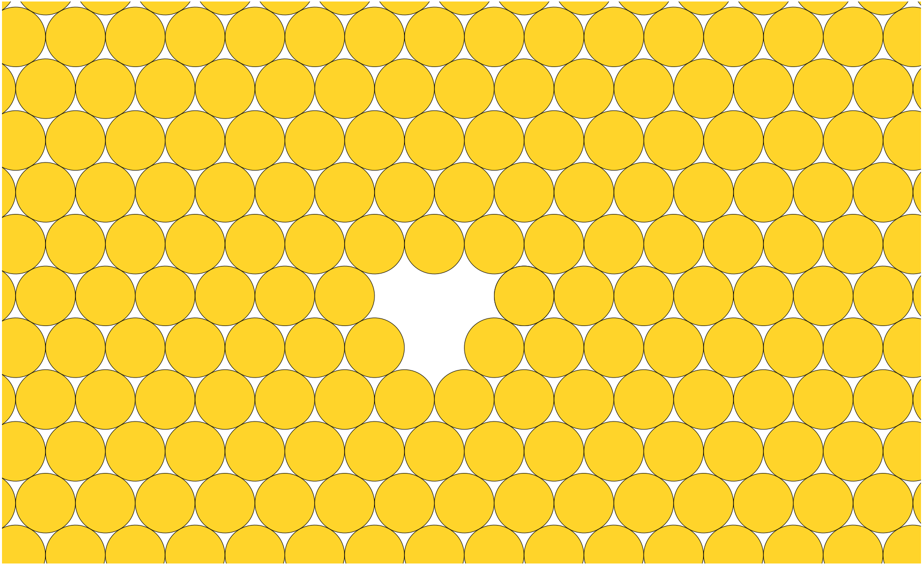

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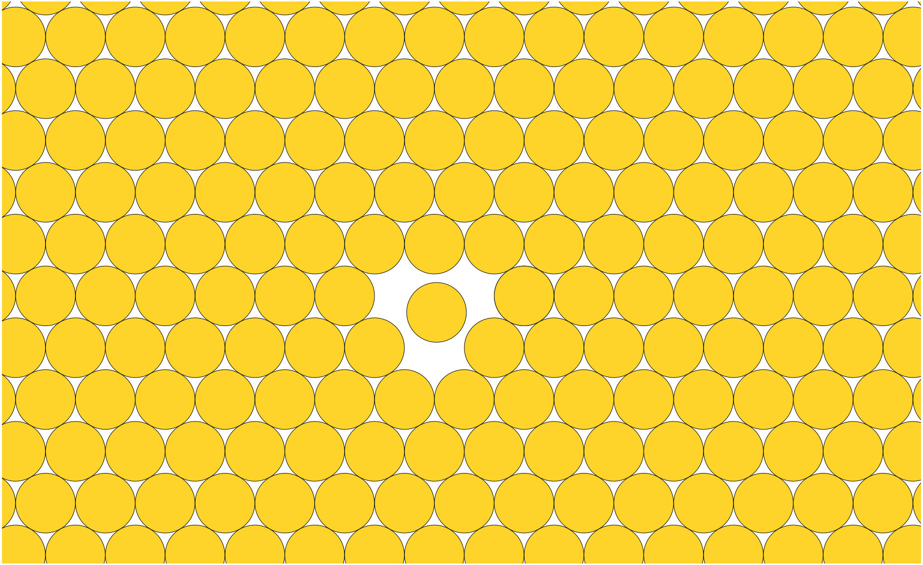

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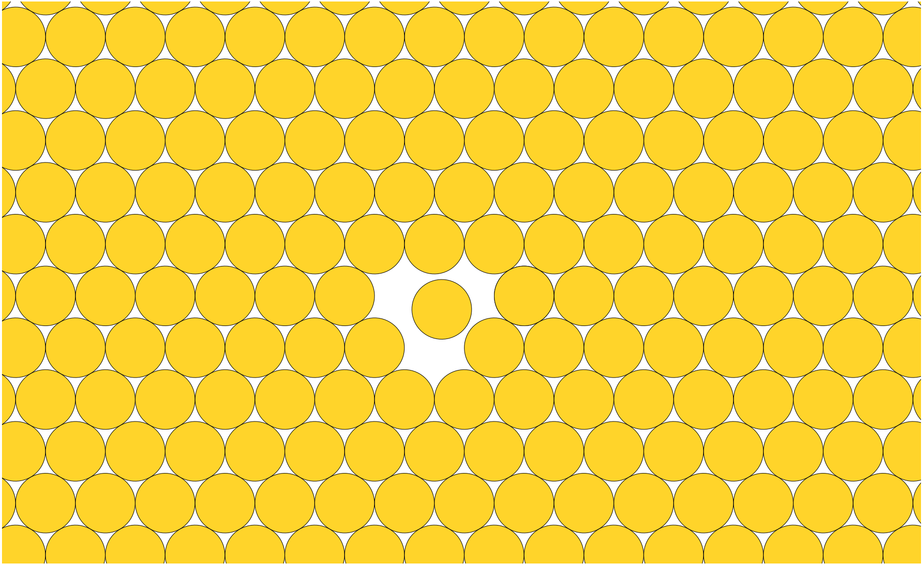

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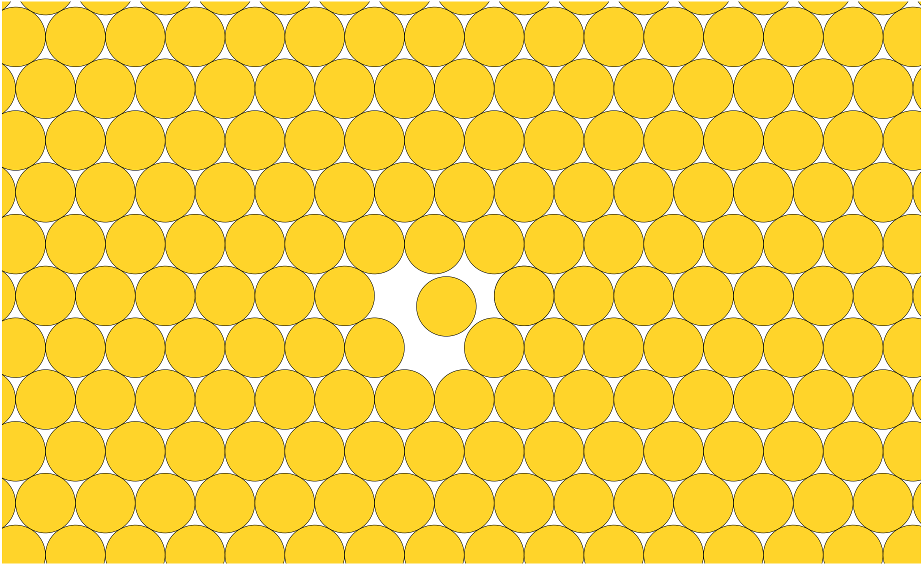
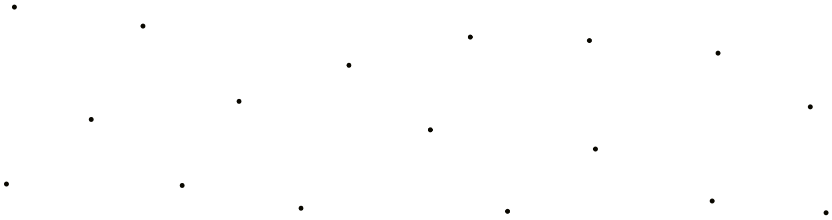


Delaunay triangulation

Given $S \subset \mathbb{R}^2$ set of points,

three points form a **Delaunay triangle** if no other point from S in circumcircle



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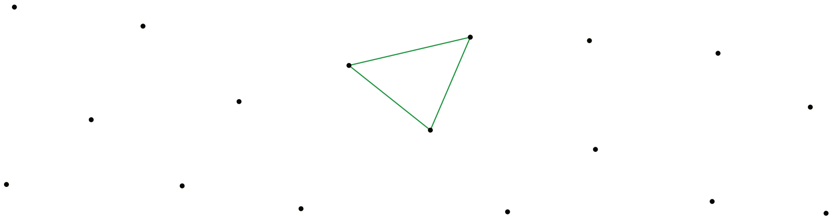

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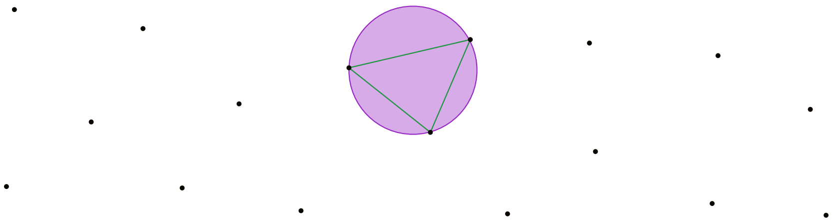

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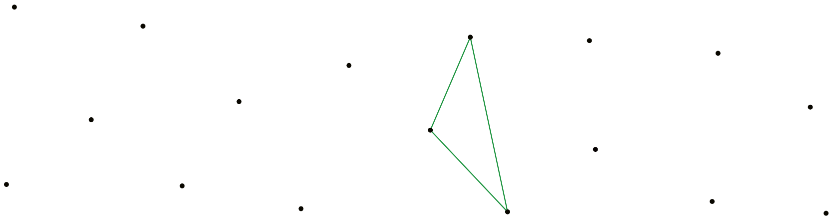

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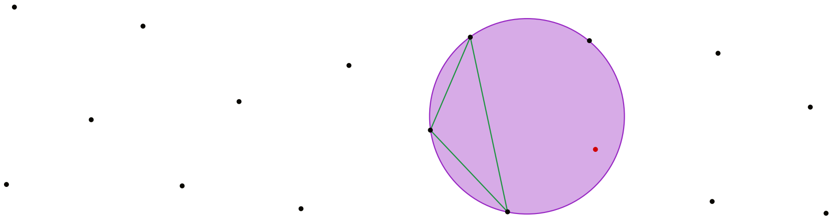

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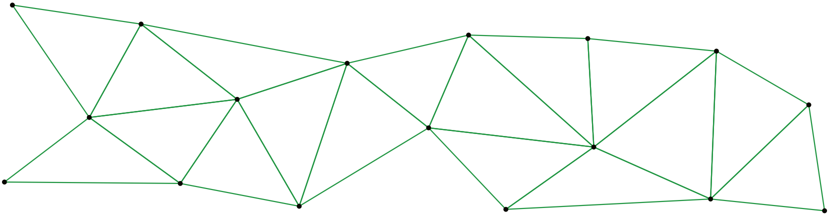

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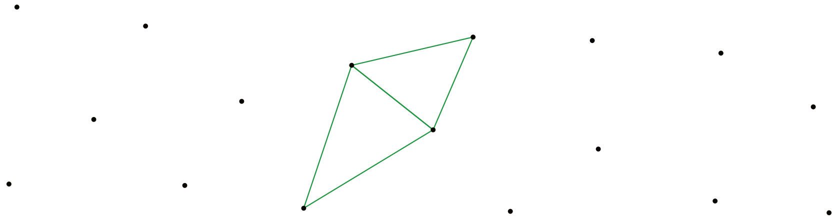

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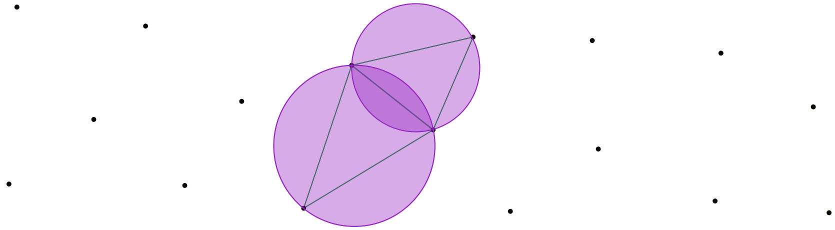

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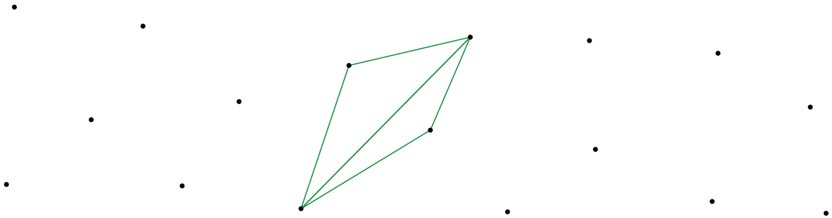

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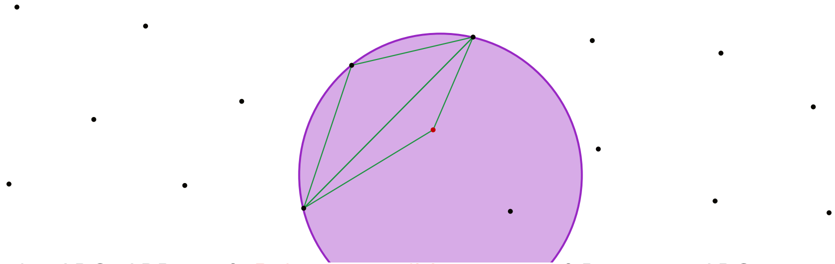

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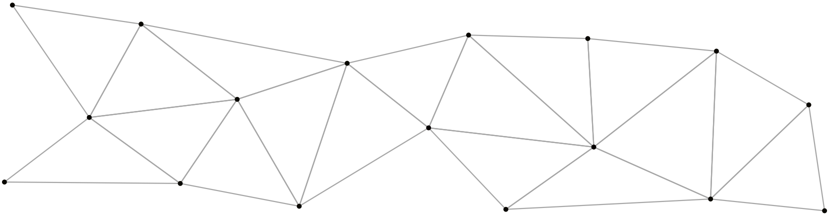

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If each pair of adjacent triangles satisfies then the whole triangulation does. Delaunay, 1934

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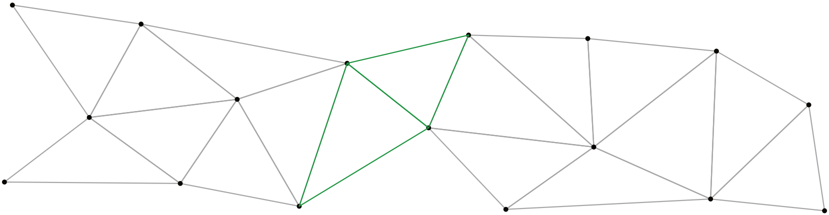

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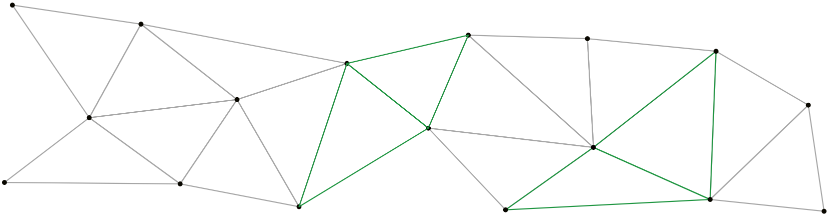

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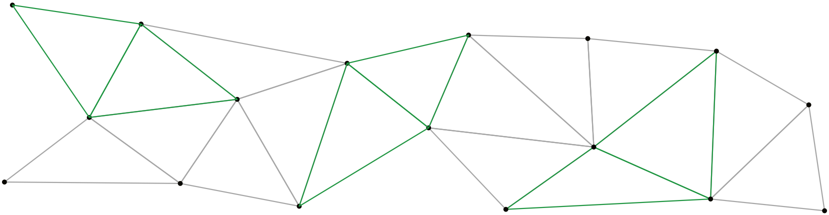

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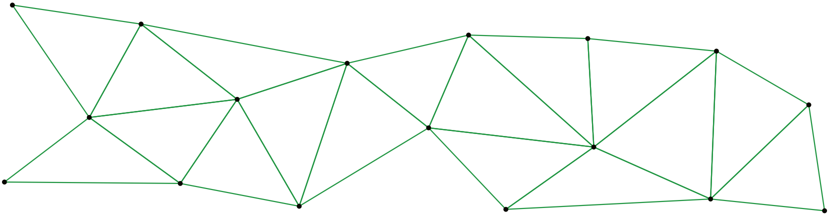

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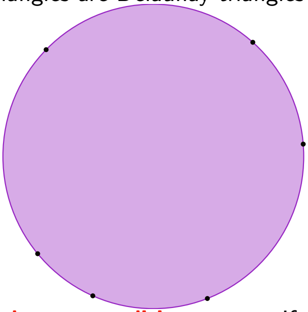

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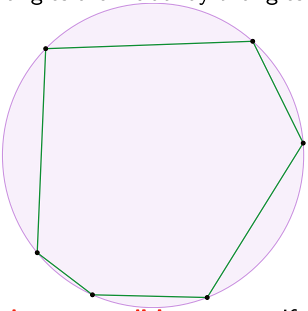

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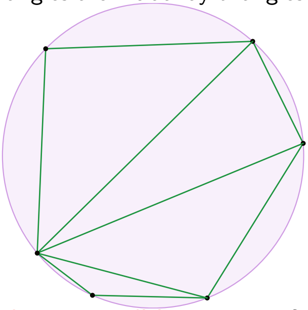

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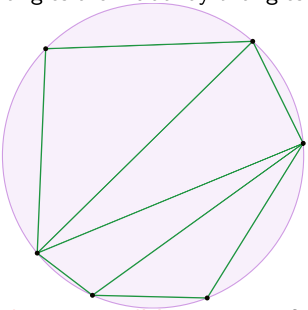

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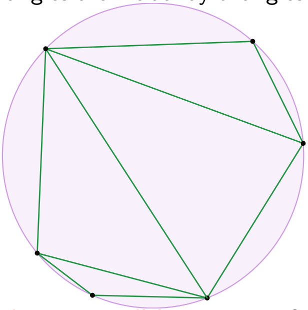

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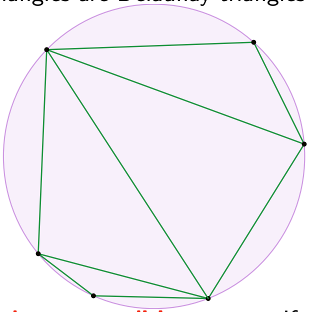

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all triangulations are connected by flips Lawson 1972

flipping non-Delaunay edges, we obtain a Delaunay triangulation in at most $\binom{n}{2}$ flips

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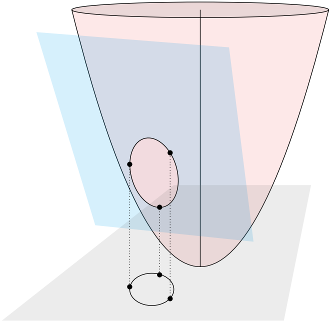

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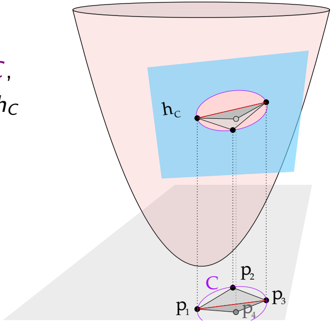

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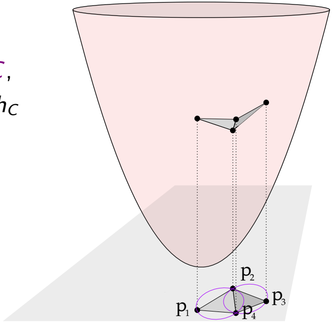

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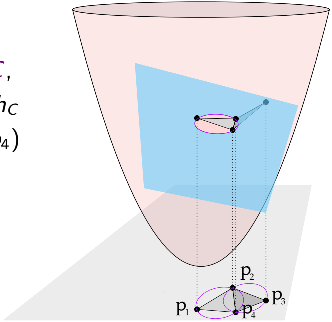

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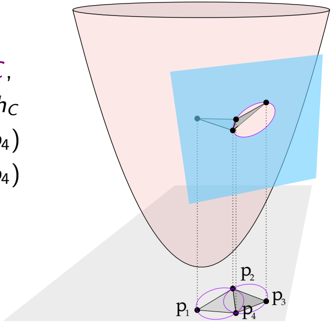

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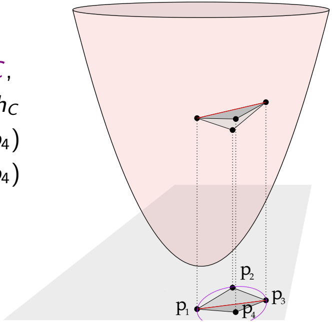
Delaunay flip: two top faces $\to$ two bottom faces
$\Rightarrow$ the surface becomes lower (point-wise)



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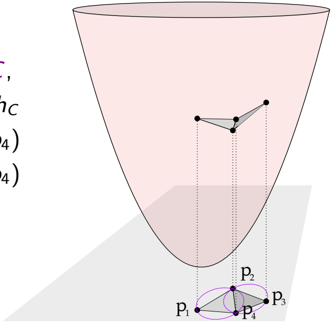

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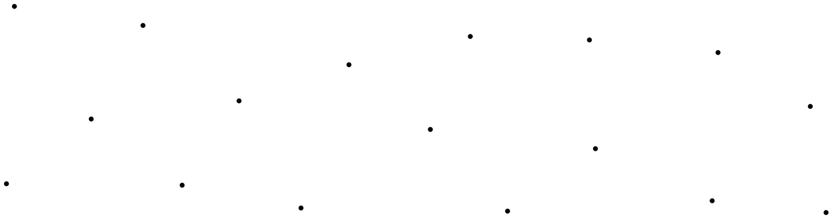
Voronoi cell is **convex** polygonal domain (possibly unbounded)



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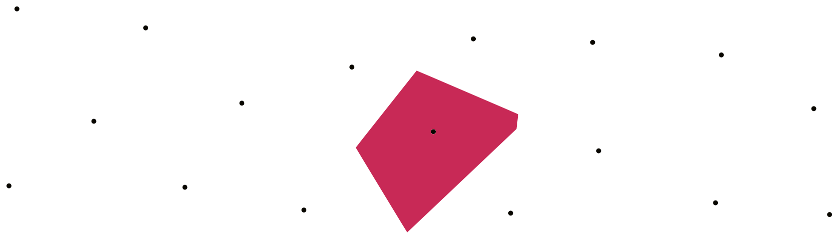

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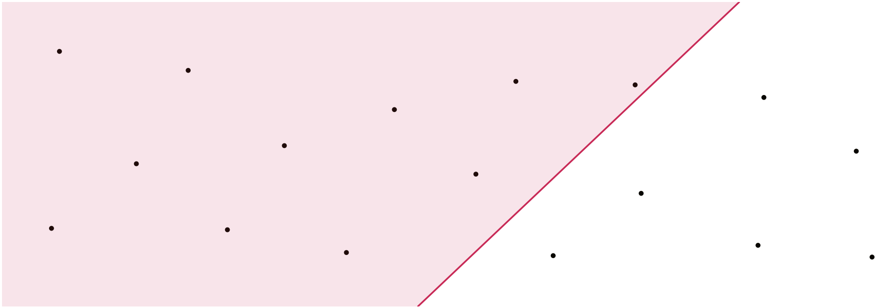

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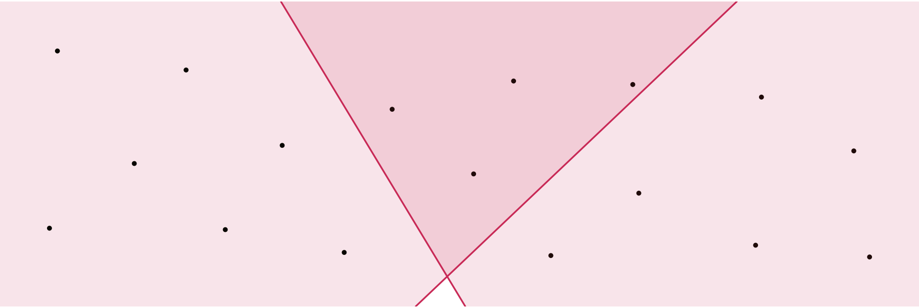

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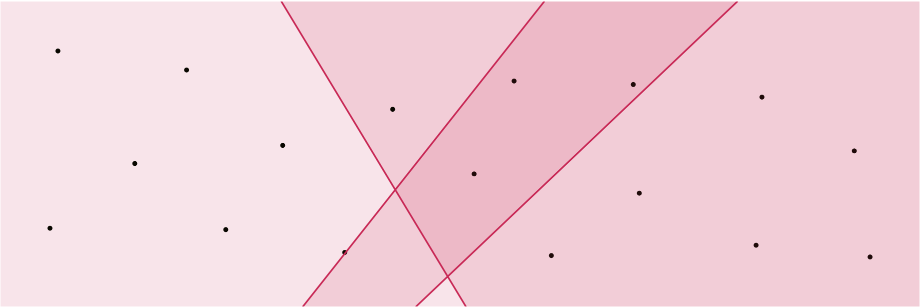

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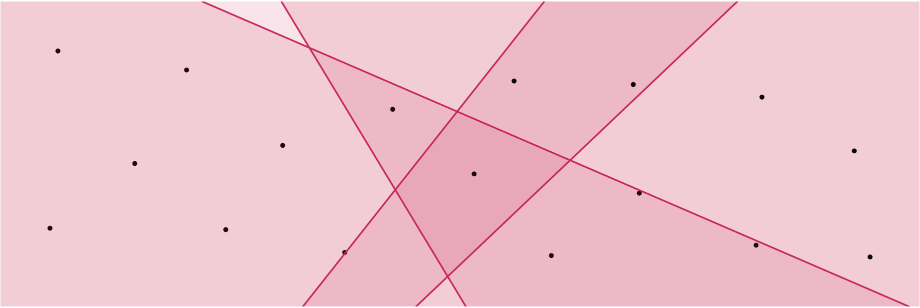

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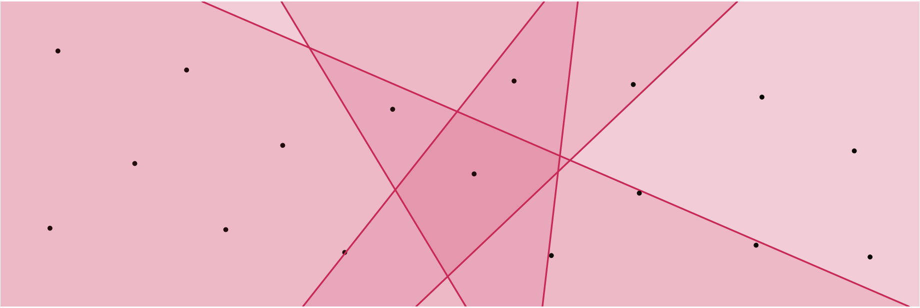

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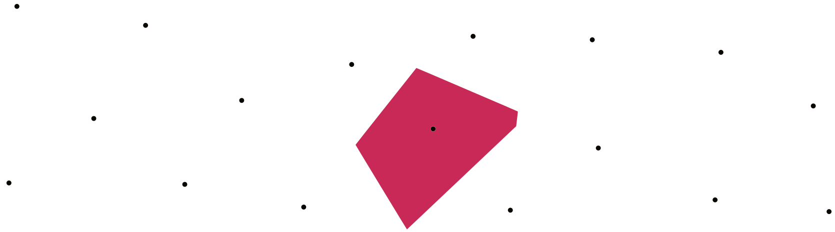

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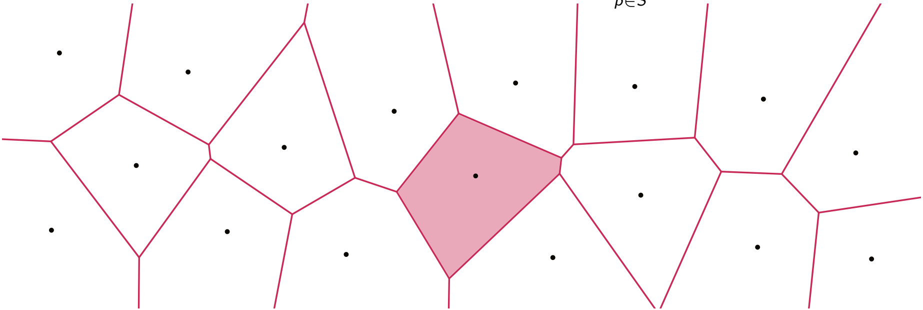

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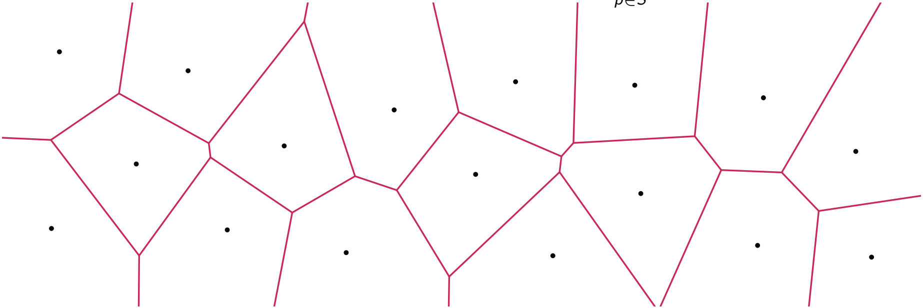

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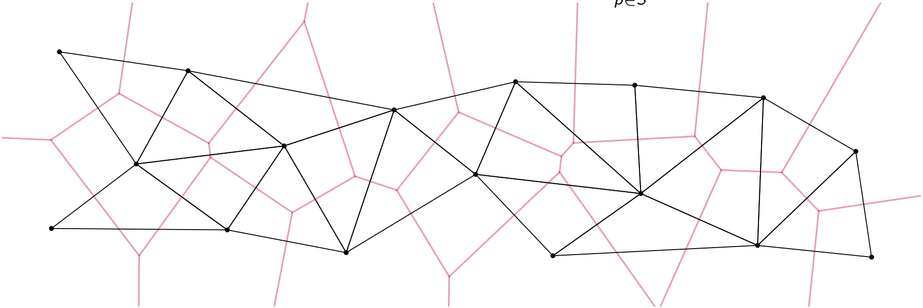

all vertices of degree $\leq 3 \Rightarrow$ dual graph is a triangulation

Delaunay triangulation and Voronoi diagram

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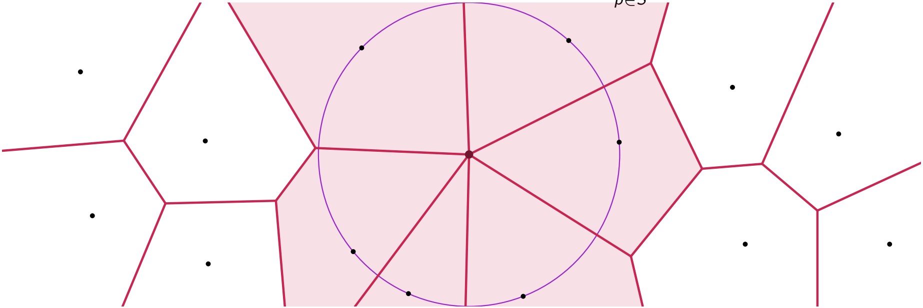

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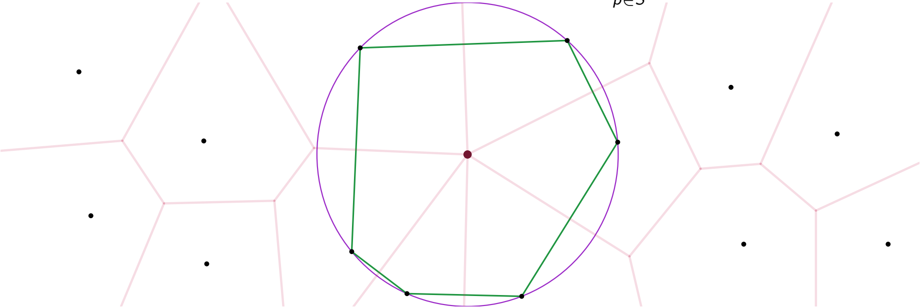

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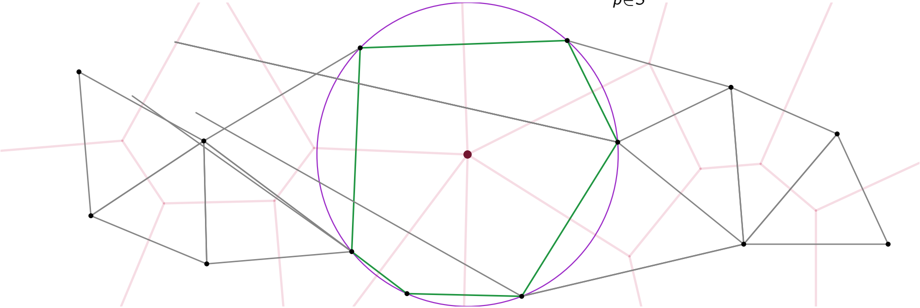

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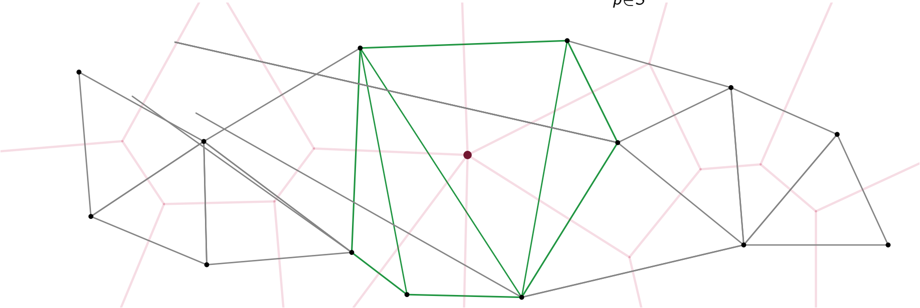

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dual graph of a Voronoi diagram can be completed into a Delaunay triangulation

Properties of Delaunay triangulations

- does not exist if three collinear points
- circumscribed circle of each triangle contains no other points
- edges do not intersect: **exercise**
- maximizes minimal angle in the triangulation **exercise**

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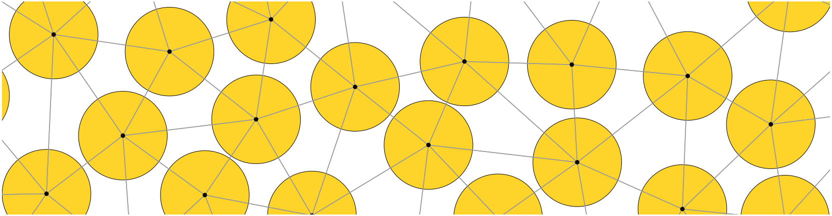

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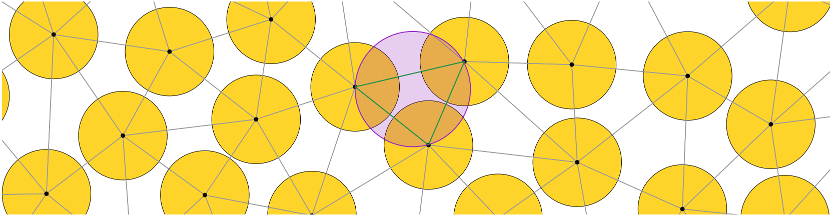

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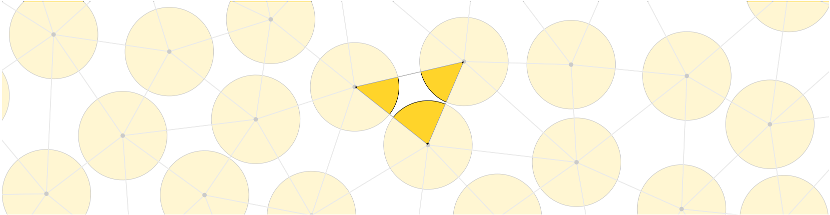

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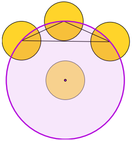

Density of a triangle Δ in a packing = its proportion covered by disks

$$\delta_\Delta = \frac{area(\Delta \cap P)}{area(\Delta)}$$

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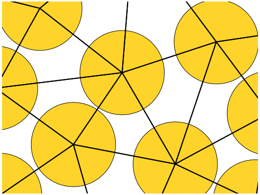
Proof with triangles



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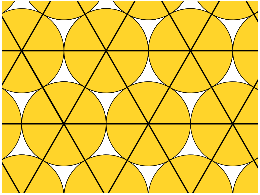

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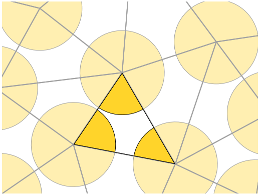

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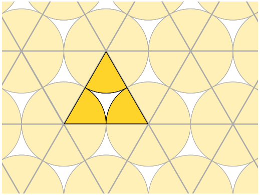

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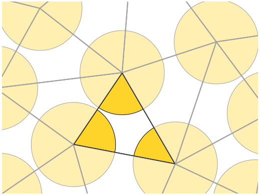

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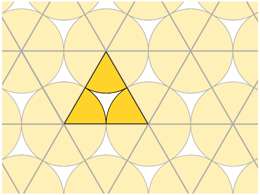

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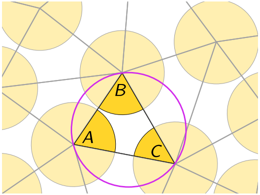

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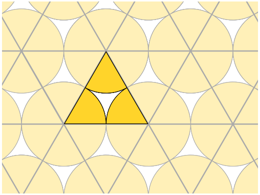

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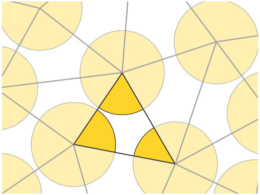

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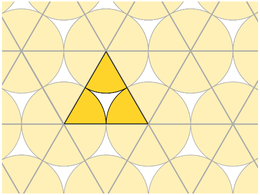

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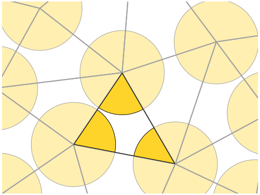

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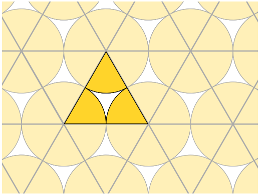

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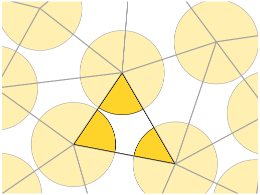

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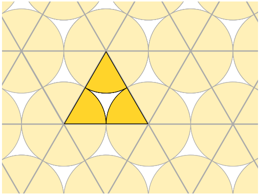

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Hexagonal packing is optimal

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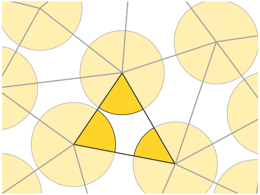

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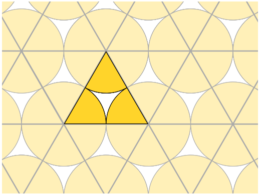

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additively weighted Voronoi diagram of S: Voronoi diagram with modified distance

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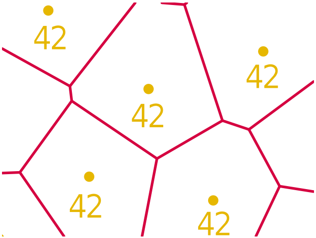
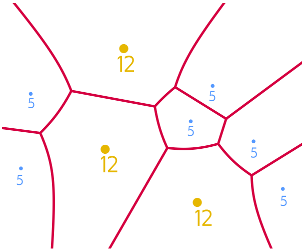
the heavier the point the larger its cell





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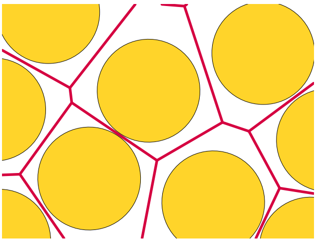

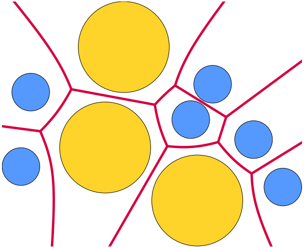

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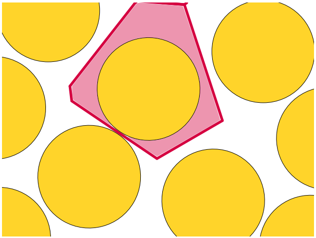

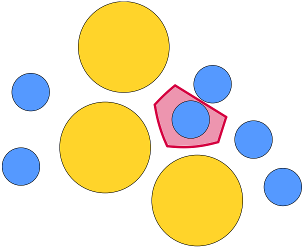

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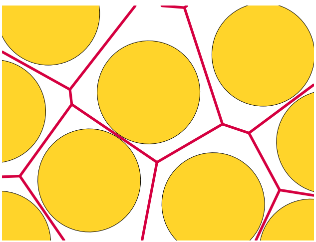

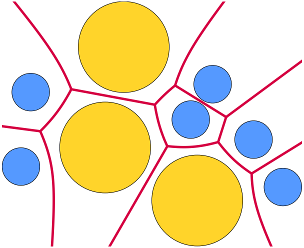

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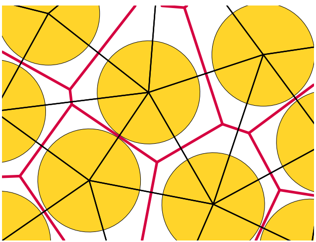

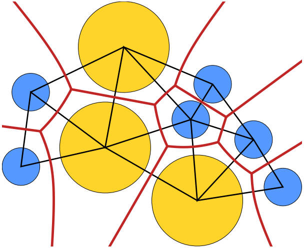

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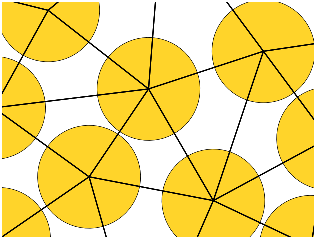

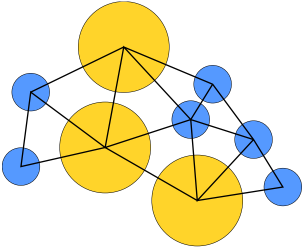

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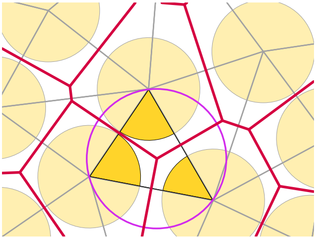

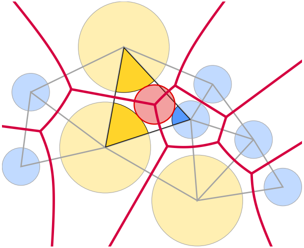

support circle: centered in the Voronoi vertex, tangent to the three disks of FM-Δ

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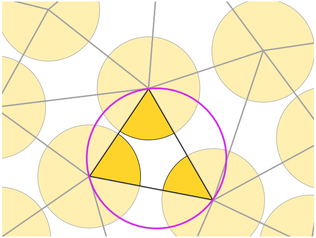
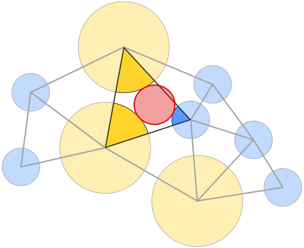
FM-triangulation of a packing: dual graph of the Voronoi diagram — Fejes Tóth, Mólnar 1958



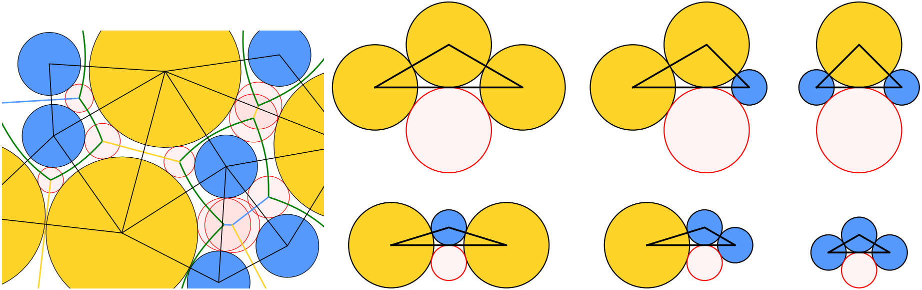


support circle: centered in the Voronoi vertex, tangent to the three disks of FM-Δ

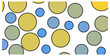
if packing is saturated, support circle is of radius at most min radius of disk ∈ packing

Properties of FM-triangulations

in FM-Δ of a saturated packing, none of its discs can intersect the opposite edge



Uniformity and Florian bound



Packing of **uniformity** q: packing of the plane by disks of radii $\in [q, 1]$.

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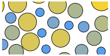

Florian, 1960

The density of a packing of uniformity q never exceeds $\bar{\delta}_F(q) := \delta$ ():

$$\bar{\delta}_F(q) := \frac{\pi q^2 + 2(1-q^2)\arcsin\left(\frac{q}{1+q}\right)}{2q\sqrt{2q+1}}.$$

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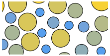

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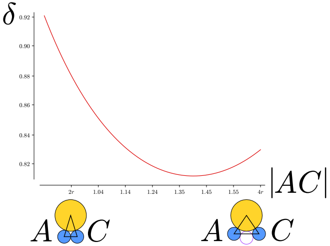
Proof:

- Among all triangles with 2 contacts

between disks, 

is the densest.

$$\delta(r,x) = \frac{2\left(2\,r^2 \arccos\left(\frac{x}{2\,(r+1)}\right) + \arccos\left(\frac{2\,r^2 - x^2 + 4\,r + 2}{2\,(r^2 + 2\,r + 1)}\right)\right)}{\sqrt{-x^4 + 4\,(r^2 + 2\,r + 1)x^2}}$$



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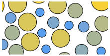

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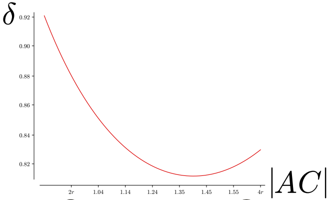


- For any triangle, there is a denser triangle with at least two contacts.

Reduce the dimension of the set of triangles $(3 \to 1)$

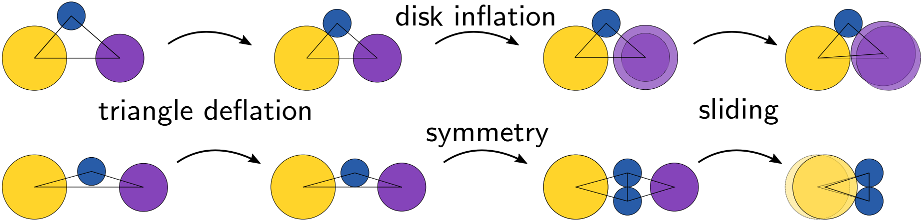
Fejes Tóth, Mólnar, 1958

Dimension reduction

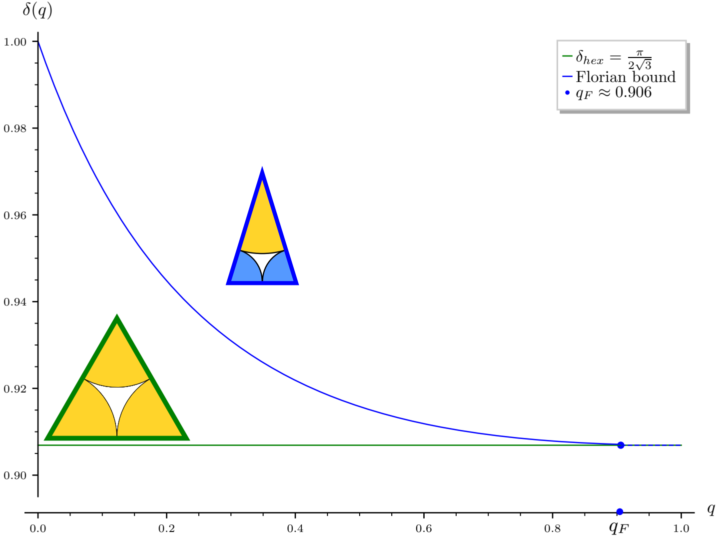
Given uniformity q,

for any FM-triangle, there is a denser FM-triangle with at least two contacts

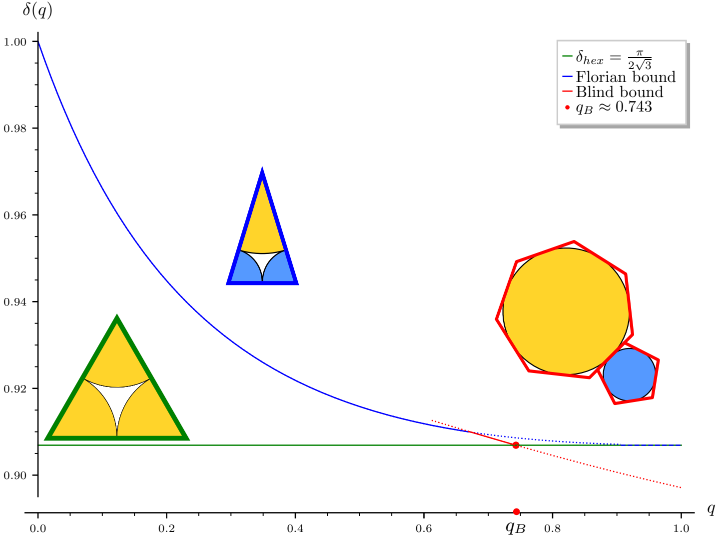
Proof: each transformation does not diminish the density of the triangle



Florian bound 1960



Blind bound 1969



Power (Laguerre) diagram

Given a disk D of radius r centered in O in packing P,

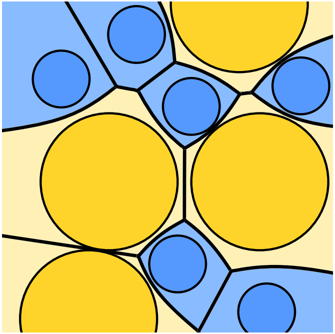
Voronoi cell(D): points closer to D than to any other disk in P in Euclidean distance:

$$dist_D(X) = d(O, X) - r$$

Power cell(D): points closer to D than to any other disk in P in power distance

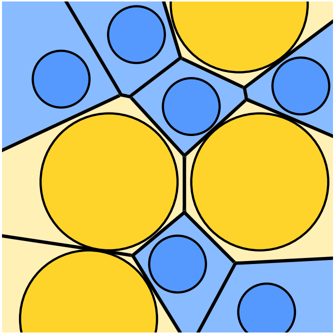
$$\Pi_D(X) = d(O, X)^2 - r^2$$

Voronoi diagram



$\longrightarrow$

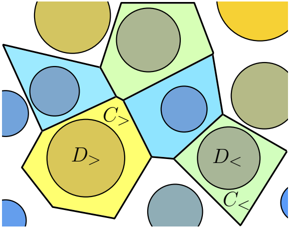
Power diagram



power cells are convex and polygonal (**exercise**)

Blind bound

1. the mean number of edges of power cells in a packing is ≤ 6
2. D disk, C its power cell with k edges



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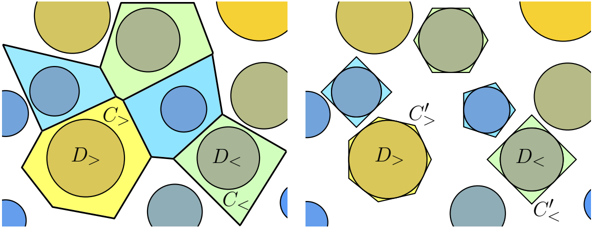
3. C' regular circumscribed version of C area$(C') <$ area(C)



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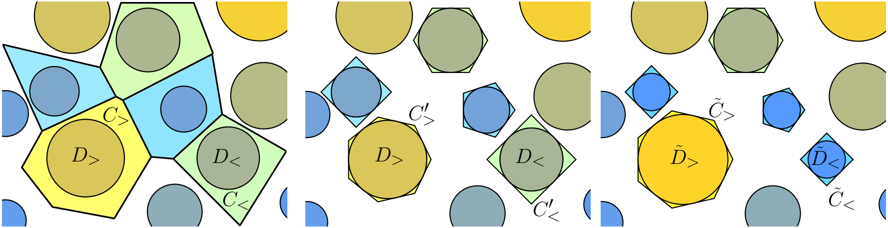

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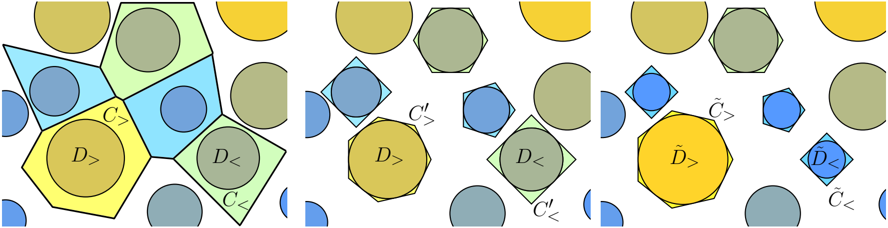
$$q \in [0.612, 0.74] \qquad \delta(P) \leq \delta_B(q) := \frac{\pi(q^2+1)}{q^2 a(5) + a(7)}$$

$$q > 0.74 \qquad \delta(P) \leq \frac{\pi}{a(6)} = \delta_{\text{hex}}.$$



1 Introduction

2 Definitions

3 Hexagonal packing is optimal

4 Multi-disk packings

5 Homework

Homework

4/12–11/12

1. Given an FM-triangle with disks of radii r_A, r_B, r_C of a saturated packing of uniformity q, find lower and upper bounds on its edge lengths.

2. **Use interval arithmetic in any programming language supporting it**. You know the value of the optimal density of a triangle of uniformity r (Florian bound, slide 14). Find an enclosure e of the length of the rr-edge of a triangle formed by one unit disk and two r-disks, where the unit disk is tangent to both small disks for $r = \frac{2}{\sqrt{3}} - 1$, such that $\delta(e)$ contains the optimum (and it is certified). You can use the formula from slide 14.

3. * Prove that the edges in the Delaunay triangulation do not intersect.

4. ** Write an algorithm, that takes $\{(c_i, r_i)\}_{i=1}^{n}$ where $c_i = (x_i, y_i)$ are disk center coordinates and r_i are disk radii as input and constructs an FM-triangulation of the packing. The output is $\{(i, j)\}_{\text{there is an edge between } c_i \text{ and } c_j}$.

LaTeX-generated pdfs, txt, anything except handwriting to be submitted by email to: daria.pchelina@ens-lyon.fr

Deadline: beginning of the lecture in one week (11/12, 10h15)