Computer-assisted proofs: Triangulated disk packings

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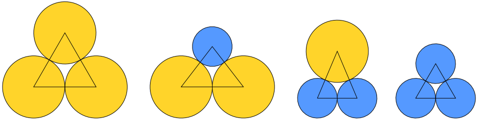
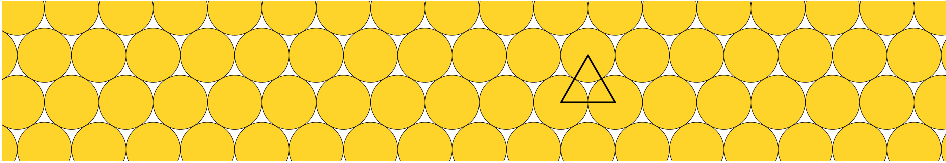
They are everywhere

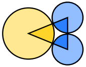
Looking for optimal packings...

tight triangle: a triangle formed by three pairwise tangent disks:



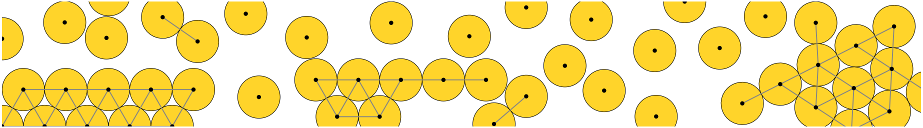
hexagonal packing: optimal, consists only of tight triangles



Florian bound: densest triangle is tight 

Definition

contact graph of a packing: **vertices**=centers, **edges** between centers of tangent disks



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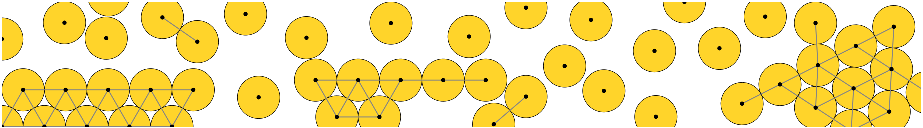
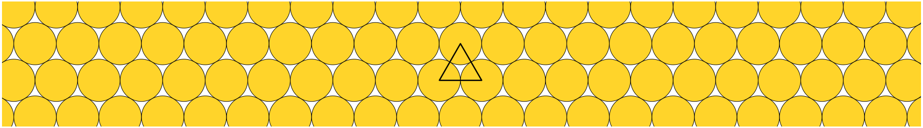


a packing is **triangulated** if its contact graph is a triangulation:



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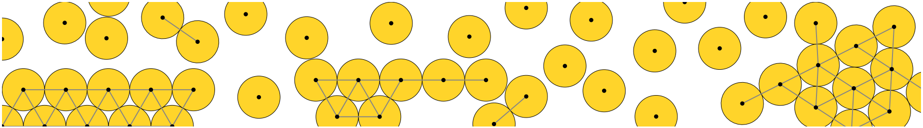

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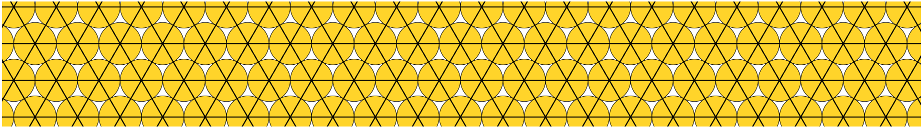

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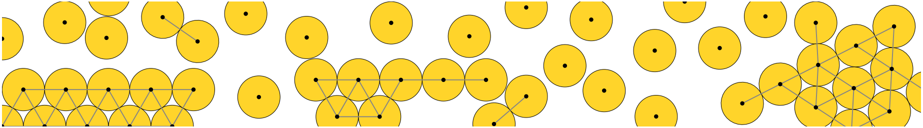

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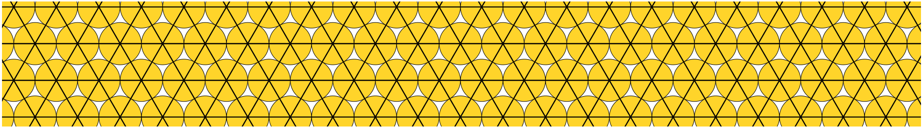
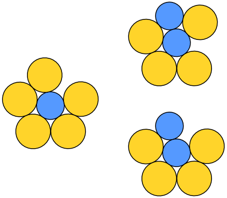


find a triangulated packing of disks of radii 1 and r: $r = \frac{4}{5} = 0.8$



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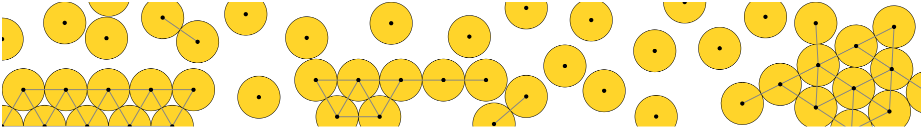

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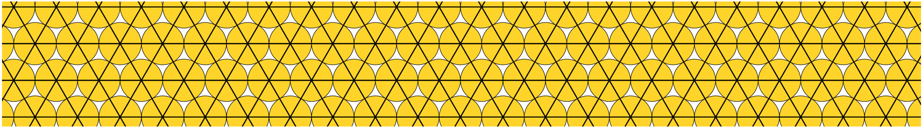

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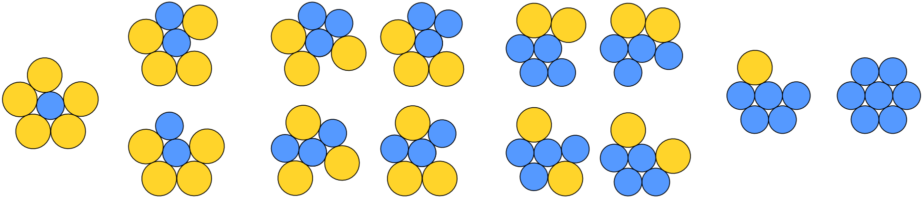
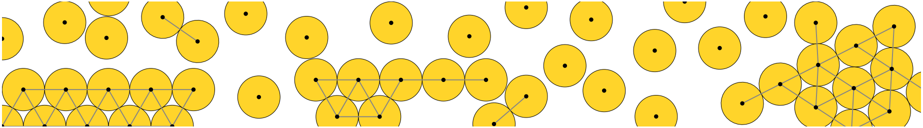


Definition

contact graph of a packing: **vertices**=centers, **edges** between centers of tangent disks



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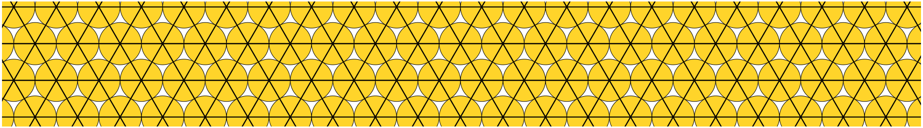
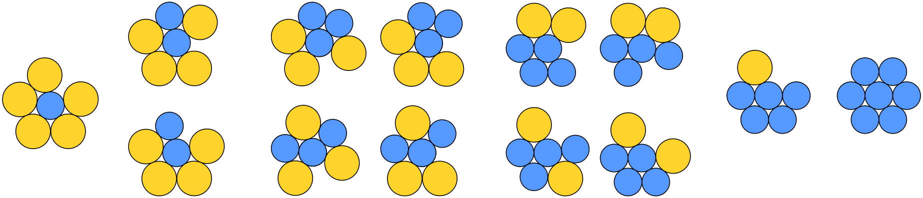


there exists no triangulated packing of disks of radii 1 and r: $r = \frac{4}{5} = 0.8$ 



2-disk triangulated packings

Which values of r allow triangulated packings?

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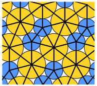
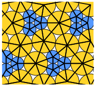
Kennedy, 2006

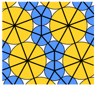
(Packings by disks of radii 1 and r) There are 9 values of r allowing triangulated packings:



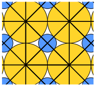
$r\approx0.63$ $\delta^*\approx91.1\%$



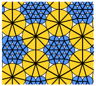
$r\approx0.54$ $\delta^*\approx91.1\%$



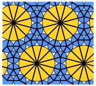
$r\approx0.53$ $\delta^*\approx91.4\%$



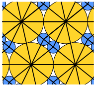
$r\approx0.41$ $\delta^*\approx92\%$



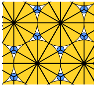
$r\approx0.38$ $\delta^*\approx92\%$



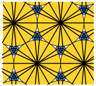
$r\approx0.34$ $\delta^*\approx92.5\%$



$r\approx0.28$ $\delta^*\approx93.2\%$



$r\approx0.15$ $\delta^*\approx95\%$



$r\approx0.1$ $\delta^*\approx96\%$

2-disk triangulated packings

Which values of r allow triangulated packings?

Kennedy, 2006

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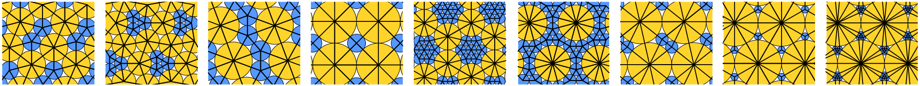
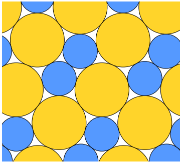
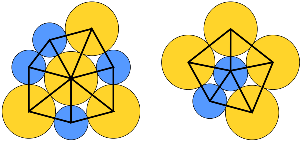


$r \approx 0.63$ $\delta^* \approx 91.1\%$ | $r \approx 0.54$ $\delta^* \approx 91.1\%$ | $r \approx 0.53$ $\delta^* \approx 91.4\%$ | $r \approx 0.41$ $\delta^* \approx 92\%$ | $r \approx 0.38$ $\delta^* \approx 92\%$ | $r \approx 0.34$ $\delta^* \approx 92.5\%$ | $r \approx 0.28$ $\delta^* \approx 93.2\%$ | $r \approx 0.15$ $\delta^* \approx 95\%$ | $r \approx 0.1$ $\delta^* \approx 96\%$

packing is triangulated $\Longrightarrow$ each disk has a **corona**





2-disk triangulated packings

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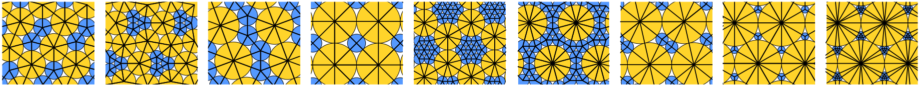

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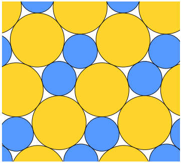
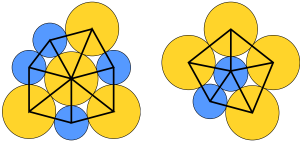


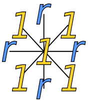


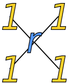
strategy: find r allowing a pair of coronas, then check if there is a packing of the plane

Find values of r allowing a pair of coronas: idea

symbolic corona: finite necklace of 1 and r — all rotations of a sequence are identical

1-corona: around 1-disk, **r-corona**: around r-disk

examples: 1-corona $1r1r1r1r$: 

r-corona 1111: 

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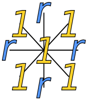
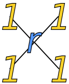
examples: 1-corona $1r1r1r1r$: 

r-corona 1111: 

1. if r allows a triangulated packing using both disks, there is $\epsilon > 0$ such that $r \geq \epsilon$ **exercise**

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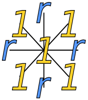
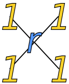
examples: 1-corona $1r1r1r1r$: 

r-corona 1111: 

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   exercise
2. for any $r \geq \epsilon$, there is at most N disks in an r-corona and M disks in a 1-corona
   exercise

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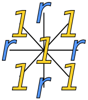
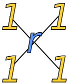
examples: 1-corona $1r1r1r1r$: 

r-corona 1111: 

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2. for any $r \geq \epsilon$, there is at most N disks in an r-corona and M disks in a 1-corona
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3. there is at most $2^N \cdot 2^M$ different pairs of coronas which can both be present in a triangulated packing — finitely many

Find values of r allowing a pair of coronas: idea

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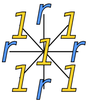
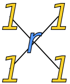
examples: 1-corona $1r1r1r1r$: 

r-corona 1111: 

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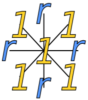
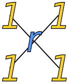
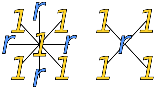
examples: 1-corona $1r1r1r1r$: 

r-corona 1111: 

1. if r allows a triangulated packing using both disks, there is $\epsilon > 0$ such that $r \geq \epsilon$ **exercise**
2. for any $r \geq \epsilon$, there is at most N disks in an r-corona and M disks in a 1-corona **exercise**
3. there is at most $2^N \cdot 2^M$ different pairs of coronas which can both be present in a triangulated packing — finitely many

for each pair of symbolic coronas: 

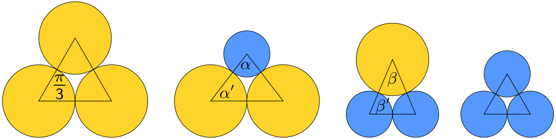
, find the value of r if it exists

in the end, we obtain a finite number of $r_1, \ldots, r_k$ with associated pairs of coronas

Corona → value of r

tight triangles: T_{111} T_{11r} T_{1rr} T_{rrr}

$\widehat{111} = \frac{\pi}{3}$ $\widehat{11r} = \alpha',\ \widehat{1r1} = \alpha$ $\widehat{1rr} = \beta',\ \widehat{r1r} = \beta$ $\widehat{rrr} = \frac{\pi}{3}$



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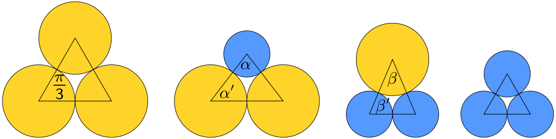

$\cos(\alpha') = \frac{1}{1+r},\ \alpha = \pi - 2\alpha'$ $\cos(\beta') = \frac{r}{1+r},\ \beta = \pi - 2\beta'$

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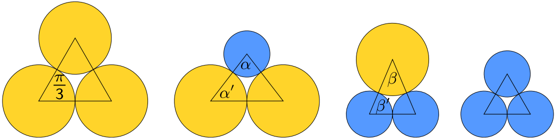

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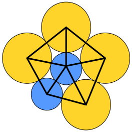
r-corona 

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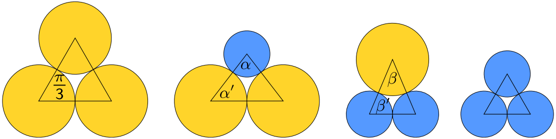

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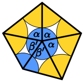
equation on α, β': $3\alpha + 2\beta' = 2\pi$

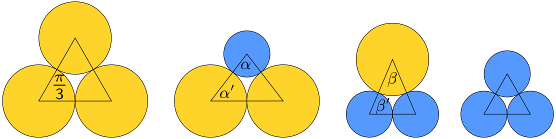
r-corona  $\iff$

Find triangulated binary packings

Corona → value of r

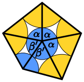
tight triangles: T_{111} T_{11r} T_{1rr} T_{rrr}

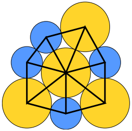
$\widehat{111} = \frac{\pi}{3}$ $\widehat{11r} = \alpha',\ \widehat{1r1} = \alpha$ $\widehat{1rr} = \beta',\ \widehat{r1r} = \beta$ $\widehat{rrr} = \frac{\pi}{3}$



$\cos(\alpha') = \frac{1}{1+r},\ \alpha = \pi - 2\alpha'$ $\cos(\beta') = \frac{r}{1+r},\ \beta = \pi - 2\beta'$

equation on α, β': $3\alpha + 2\beta' = 2\pi$

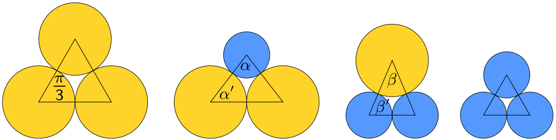
r-corona  $\iff$

1-corona 

Corona $\to$ value of r

tight triangles: T_{111} T_{11r} T_{1rr} T_{rrr}

$\widehat{111} = \frac{\pi}{3}$ $\quad \widehat{11r} = \alpha',\ \widehat{1r1} = \alpha$ $\quad \widehat{1rr} = \beta',\ \widehat{r1r} = \beta$ $\quad \widehat{rrr} = \frac{\pi}{3}$



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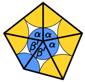
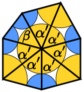
r-corona  $\iff$ equation on α, β': $3\alpha + 2\beta' = 2\pi$

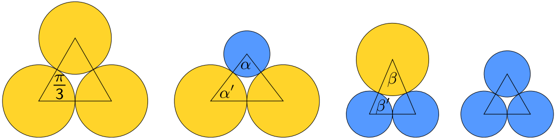
1-corona  $\iff$ equation on α', β: $\beta + 6\alpha' = 2\pi$

Find triangulated binary packings

Corona $\to$ value of r

tight triangles: T_{111} T_{11r} T_{1rr} T_{rrr}

$\widehat{111} = \frac{\pi}{3}$ $\widehat{11r} = \alpha'$, $\widehat{1r1} = \alpha$ $\widehat{1rr} = \beta'$, $\widehat{r1r} = \beta$ $\widehat{rrr} = \frac{\pi}{3}$



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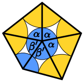
r-corona  $\iff$ equation on α, β':

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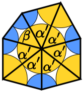
1-corona  $\iff$ equation on α', β:

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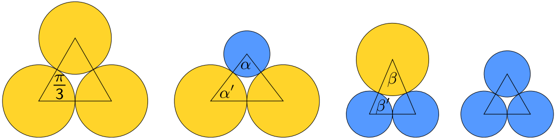

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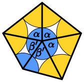
r-corona  $\Longleftrightarrow$ equation on α, β':

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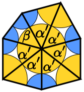
1-corona  $\Longleftrightarrow$ equation on α', β:

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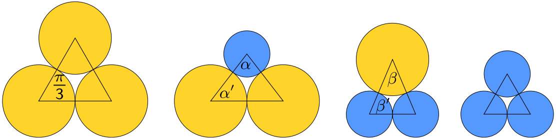

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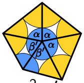
r-corona  $\iff$ equation on α, β':

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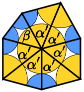
$$\cos^3\alpha\cos^2\beta' - 3\cos\alpha\cos^2\beta'\sin^2\alpha - 6\cos^2\alpha\cos\beta'\sin\alpha\sin\beta' + 2\cos\beta'\sin^3\alpha\sin\beta' - \cos^3\alpha\sin^2\beta' + 3\cos\alpha\sin^2\alpha\sin^2\beta' = 1$$

1-corona  $\iff$ equation on α', β:

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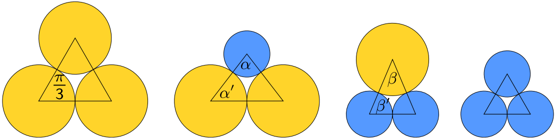

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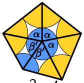
r-corona  $\iff$ equation on α, β':

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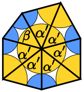
1-corona  $\iff$ equation on α', β:

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$$\cos(\beta + 6\alpha') = 1$$

$$\cos^6\alpha'\cos\beta - 15\cos^4\alpha'\cos\beta\sin^2\alpha' + 15\cos^2\alpha'\cos\beta\sin^4\alpha' - \cos\beta\sin^6\alpha' - 6\cos^5\alpha'\sin\alpha'\sin\beta + 20\cos^3\alpha'\sin^3\alpha'\sin\beta - 6\cos\alpha'\sin^5\alpha'\sin\beta = 1$$

Find triangulated binary packings

Corona → value of r

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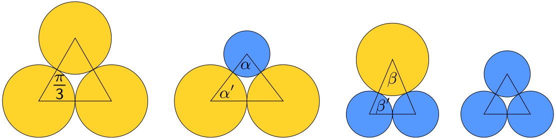

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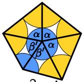
r-corona  $\Longleftrightarrow$ equation on α, β':

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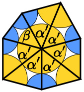
1-corona  $\Longleftrightarrow$ equation on α', β:

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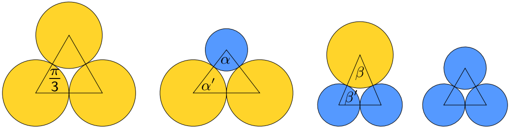
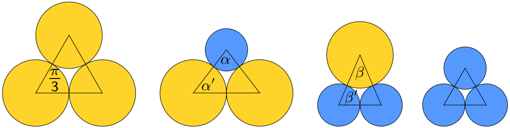
$(7+4\sqrt{3})\,r^4 + (20+12\sqrt{3})\,r^3 + (6+4\sqrt{3})\,r^2 + (-20-4\sqrt{3})\,r + 3 = 0$ $\qquad r \approx 0.5451510421$

Corona $\rightarrow$ value of r: properties



$\cos(\alpha') = \frac{1}{1+r},\ \alpha = \pi - 2\alpha'$ $\qquad$ $\cos(\beta') = \frac{r}{1+r},\ \beta = \pi - 2\beta'$

Corona $\to$ value of r: properties



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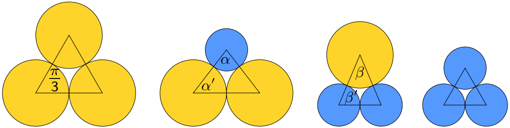
r-corona	$\Longrightarrow$	equation on α, β':	$i\,\alpha + j\,\beta' + k\,\frac{\pi}{3} = 2\pi$
1-corona	$\Longrightarrow$	equation on α', β:	$l\,\alpha' + m\,\beta + n\,\frac{\pi}{3} = 2\pi$

Corona → value of r: properties



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r-corona $\implies$ equation on α, β': $i\,\alpha + j\,\beta' + k\,\frac{\pi}{3} = 2\pi$

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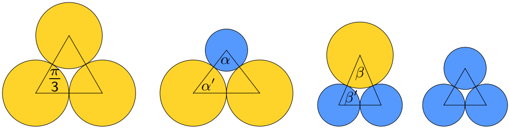
trivial solution: $i{=}j{=}l{=}m{=}0$, $k{=}n{=}6$ — phase separation no $1r$ contact, cannot use both disks

$\Rightarrow$ at least one of equations should have a non-trivial solution

Corona → value of r: properties



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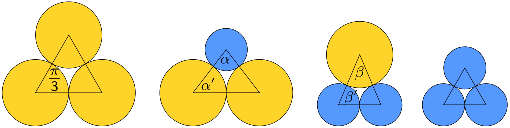
how many solutions (another method)

Corona $\to$ value of r: properties



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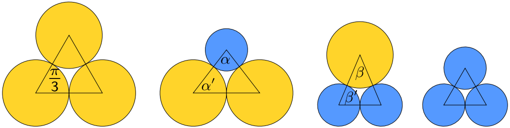
how many solutions (another method)

$F_{ijk} = i\,\alpha + j\,\beta' + k\,\frac{\pi}{3}$ is decreasing on $r \Rightarrow \forall(i,j,k)$, at most one value of r such that $F_{ijk}(r) = 2\pi$

Corona $\rightarrow$ value of r: properties



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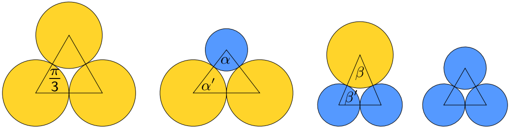
Corona $\to$ value of r: properties



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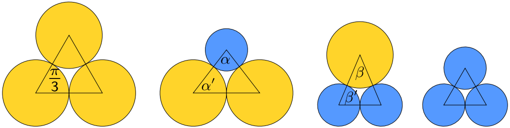
$\lim_{r\to 0} F_{ijk}(r) = i\,\pi + j\,\frac{\pi}{2} + k\,\frac{\pi}{3} \qquad > F_{ijk}(r) = 2\pi >$

Corona → value of r: properties



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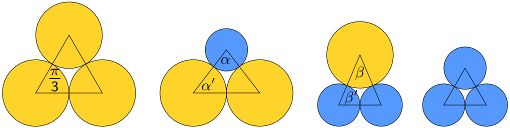
Corona $\to$ value of r: properties



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r-corona $\Longrightarrow$ equation on α, β': $\quad i\,\alpha + j\,\beta' + k\,\frac{\pi}{3} = 2\pi$

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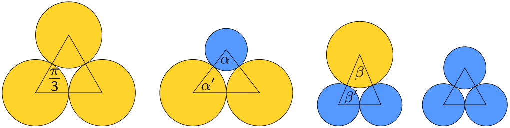
$\lim_{r\to 0} F_{ijk}(r) = i\,\pi + j\,\frac{\pi}{2} + k\,\frac{\pi}{3} \quad > F_{ijk}(r) = 2\pi > \quad F_{ijk}(1) = (i+j+k)\frac{\pi}{3}$

$\Rightarrow$ there is an $r \in [0,1]$ such that $F_{ijk} = 2\pi$ iff $6i + 3j + 2k > 12$ and $i + j + k < 6$

$\Rightarrow$ finite number of (i,j,k) with a solution

Corona $\to$ value of r: properties



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r-corona $\Longrightarrow$ equation on α, β': $i\,\alpha + j\,\beta' + k\,\frac{\pi}{3} = 2\pi$

1-corona $\Longrightarrow$ equation on α', β: $l\,\alpha' + m\,\beta + n\,\frac{\pi}{3} = 2\pi$

trivial solution: $i{=}j{=}l{=}m{=}0, \ k{=}n{=}6$ — phase separation no $1r$ contact, cannot use both disks

$\Rightarrow$ at least one of equations should have a non-trivial solution

how many solutions (another method)

$F_{ijk} = i\,\alpha + j\,\beta' + k\,\frac{\pi}{3}$ is decreasing on $r \Rightarrow \forall(i,j,k)$, at most one value of r such that $F_{ijk}(r) = 2\pi$

$\Rightarrow \forall(i,j,k), \ \forall r \in (0,1) \ F_{ijk}(0) > F_{ijk}(r) > F_{ijk}(1)$

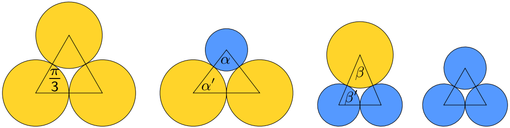
$\lim_{r\to 0} F_{ijk}(r) = i\,\pi + j\,\frac{\pi}{2} + k\,\frac{\pi}{3} \quad > F_{ijk}(r) = 2\pi > \quad F_{ijk}(1) = (i+j+k)\frac{\pi}{3}$

$\Rightarrow$ there is an $r \in [0,1]$ such that $F_{ijk} = 2\pi$ iff $6i + 3j + 2k > 12$ and $i + j + k < 6$

$\Rightarrow$ finite number of (i,j,k) with a solution

j is even; if $j = 0$ then $i = 0$ or $k = 0$

Corona $\to$ value of r: properties



$\cos(\alpha') = \frac{1}{1+r}$, $\alpha = \pi - 2\alpha'$ $\qquad$ $\cos(\beta') = \frac{r}{1+r}$, $\beta = \pi - 2\beta'$

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trivial solution: $i{=}j{=}l{=}m{=}0$, $k{=}n{=}6$ — phase separation no 1r contact, cannot use both disks

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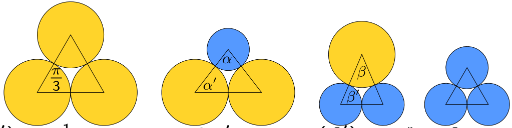
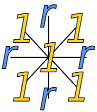
$\Rightarrow$ finite number of (i,j,k) with a solution

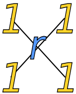
j is even; if $j = 0$ then $i = 0$ or $k = 0$ $\qquad$ what are the remaining (i,j,k)? **(exercise)**

Simple example in detail



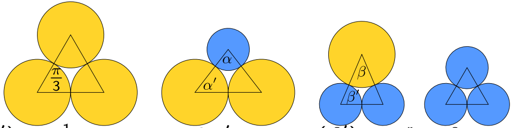
$\cos(\alpha') = \frac{1}{1+r},\ \alpha = \pi - 2\alpha'$ $\cos(\beta') = \frac{r}{1+r},\ \beta = \pi - 2\beta'$

Example: l=8, m=n=0 1-corona 1r1r1r1r 

i = 4, j=k=0 r-corona 1111 

Find triangulated binary packings

Simple example in detail



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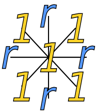
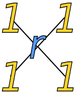
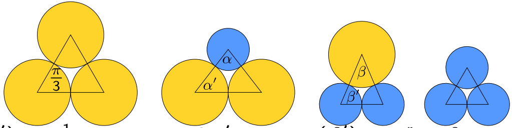
Example: l=8, m=n=0 1-corona $1r1r1r1r$  $i = 4, j$=k=0 r-corona 1111 

$$8\alpha' = 2\pi \qquad 4\alpha = 2\pi$$

$$\alpha' = \frac{\pi}{4} \qquad \alpha = \frac{\pi}{2}$$

Simple example in detail



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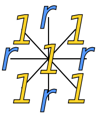

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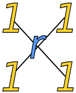

$$8\alpha' = 2\pi \qquad 4\alpha = 2\pi$$

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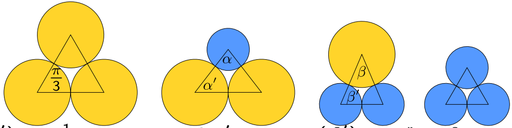
Simple example in detail



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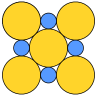

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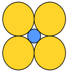

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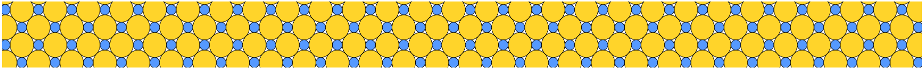
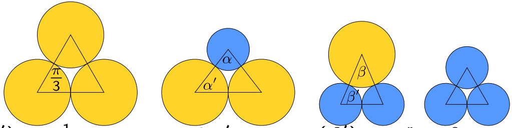


Find triangulated binary packings

Simple example in detail



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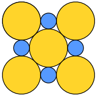
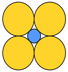
Example: $l{=}8, m{=}n{=}0$ 1-corona $1r1r1r1r$  $i = 4, j{=}k{=}0$ r-corona 1111 

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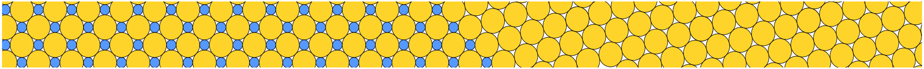

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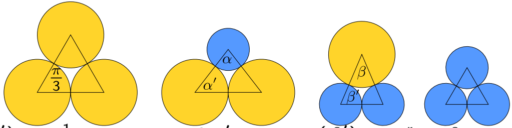

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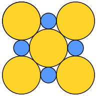
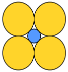
Example: $l{=}8, m{=}n{=}0$ 1-corona $1r1r1r1r$  $i = 4, j{=}k{=}0$ r-corona 1111 

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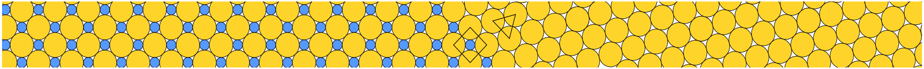

Another example in detail

$\cos(\alpha') = \frac{1}{1+r},\ \alpha = \pi - 2\alpha' \Rightarrow$ $\cos(\alpha) = -\cos(2\alpha') = 1 - 2\cos^2(\alpha') = 1 - \frac{2}{(1+r)^2}$

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Example: l=12, m=6, n=0 1-corona 1rr1rr1rr1rr1rr1rr i=1, j=2, k=1 r-corona 11rr

$$12\alpha' + 6\beta = 2\pi$$

$$\alpha + 2\beta' + \frac{\pi}{3} = 2\pi$$

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$$12\alpha' + 6\beta = 2\pi \qquad\qquad \alpha + 2\beta' + \frac{\pi}{3} = 2\pi$$

$$\cos^{12}\alpha'\cos^6\beta - 66\cos^{10}\alpha'\cos^6\beta\sin^2\alpha' + 495\cos^8\alpha'\cos^6\beta\sin^4\alpha' - 924\cos^6\alpha'\cos^6\beta\sin^6\alpha' + 495\cos^4\alpha'\cos^6\beta\sin^8\alpha' - 66\cos^2\alpha'\cos^6\beta\sin^{10}\alpha' + \cos^6\beta\sin^{12}\alpha' - 72\cos^{11}\alpha'\cos^5\beta\sin\alpha'\sin\beta + 1320\cos^9\alpha'\cos^5\beta\sin^3\alpha'\sin\beta$$
$$-4752\cos^7\alpha'\cos^5\beta\sin^5\alpha'\sin\beta + 4752\cos^5\alpha'\cos^5\beta\sin^7\alpha'\sin\beta - 1320\cos^3\alpha'\cos^5\beta\sin^9\alpha'\sin\beta + 72\cos\alpha'\cos^5\beta\sin^{11}\alpha'\sin\beta - 15\cos^{12}\alpha'\cos^4\beta\sin^2\beta + 990\cos^{10}\alpha'\cos^4\beta\sin^2\alpha'\sin^2\beta - 7425\cos^8\alpha'\cos^4\beta\sin^4\alpha'\sin^2\beta$$
$$+13860\cos^6\alpha'\cos^4\beta\sin^6\alpha'\sin^2\beta - 7425\cos^4\alpha'\cos^4\beta\sin^8\alpha'\sin^2\beta + 990\cos^2\alpha'\cos^4\beta\sin^{10}\alpha'\sin^2\beta - 15\cos^4\beta\sin^{12}\alpha'\sin^2\beta + 240\cos^{11}\alpha'\cos^3\beta\sin\alpha'\sin^3\beta - 4400\cos^9\alpha'\cos^3\beta\sin^3\alpha'\sin^3\beta + 15840\cos^7\alpha'\cos^3\beta\sin^5\alpha'\sin^3\beta$$
$$-15840\cos^5\alpha'\cos^3\beta\sin^7\alpha'\sin^3\beta + 4400\cos^3\alpha'\cos^3\beta\sin^9\alpha'\sin^3\beta - 240\cos\alpha'\cos^3\beta\sin^{11}\alpha'\sin^3\beta + 15\cos^{12}\alpha'\cos^2\beta\sin^4\beta - 990\cos^{10}\alpha'\cos^2\beta\sin^2\alpha'\sin^4\beta + 7425\cos^8\alpha'\cos^2\beta\sin^4\alpha'\sin^4\beta - 13860\cos^6\alpha'\cos^2\beta\sin^6\alpha'\sin^4\beta$$
$$+7425\cos^4\alpha'\cos^2\beta\sin^8\alpha'\sin^4\beta - 990\cos^2\alpha'\cos^2\beta\sin^{10}\alpha'\sin^4\beta + 15\cos^2\beta\sin^{12}\alpha'\sin^4\beta - 72\cos^{11}\alpha'\cos\beta\sin\alpha'\sin^5\beta + 1320\cos^9\alpha'\cos\beta\sin^3\alpha'\sin^5\beta - 4752\cos^7\alpha'\cos\beta\sin^5\alpha'\sin^5\beta + 4752\cos^5\alpha'\cos\beta\sin^7\alpha'\sin^5\beta$$
$$-1320\cos^3\alpha'\cos\beta\sin^9\alpha'\sin^5\beta + 72\cos\alpha'\cos\beta\sin^{11}\alpha'\sin^5\beta - \cos^{12}\alpha'\sin^6\beta + 66\cos^{10}\alpha'\sin^2\alpha'\sin^6\beta - 495\cos^8\alpha'\sin^4\alpha'\sin^6\beta + 924\cos^6\alpha'\sin^6\alpha'\sin^6\beta - 495\cos^4\alpha'\sin^8\alpha'\sin^6\beta + 66\cos^2\alpha'\sin^{10}\alpha'\sin^6\beta - \sin^{12}\alpha'\sin^6\beta = 1$$

$$-\frac{1}{2}\sqrt{3}\cos^2\beta'\sin\alpha - \sqrt{3}\cos\alpha\cos\beta'\sin\beta' + \frac{1}{2}\sqrt{3}\sin\alpha\sin^2\beta' + \frac{1}{2}\cos\alpha\cos^2\beta' - \cos\beta'\sin\alpha\sin\beta' - \frac{1}{2}\cos\alpha\sin^2\beta' = 1$$

Another example in detail

$\cos(\alpha') = \frac{1}{1+r}, \ \alpha = \pi - 2\alpha' \Rightarrow$ $\cos(\alpha) = -\cos(2\alpha') = 1 - 2\cos^2(\alpha') = 1 - \frac{2}{(1+r)^2}$

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$$12\alpha' + 6\beta = 2\pi \qquad \alpha + 2\beta' + \frac{\pi}{3} = 2\pi$$

$$\begin{aligned}
&-144\,r^{23} - 72\,X_1^{12}r^{10} - 912\,r^{22} + 720\,X_1^{12}r^{9} + 7696\,r^{21} + 240\,X_1^{14}r^{6} - 2520\,X_1^{12}r^{8} + 1320\,X_1^{10}r^{10} + 23295\,r^{20} - 1440\,X_1^{14}r^{5} + 2880\,X_1^{12}r^{7} - 13200\,X_1^{10}r^{9} - 255804\,r^{19} - 72\,X_1^{16}r^{2} + 2160\,X_1^{14}r^{4} - 2240\,X_1^{12}r^{6} + 46200\,X_1^{10}r^{8} - 4752\,X_1^{8}r^{10} - 740481\,r^{18} + 144\,X_1^{16}r + 960\,X_1^{14}r^{3} + 21504\,X_1^{12}r^{5}\\
&-52800\,X_1^{10}r^{7} + 47520\,X_1^{8}r^{9} + 1760142\,r^{17} + 72\,X_1^{16} - 840\,X_1^{14}r^{2} - 41760\,X_1^{12}r^{4} - 23760\,X_1^{10}r^{6} - 166320\,X_1^{8}r^{8} + 4752\,X_1^{6}r^{10} + 4410750\,r^{16} - 4080\,X_1^{14}r - 14720\,X_1^{12}r^{3} - 5280\,X_1^{10}r^{5} + 190080\,X_1^{8}r^{7} - 47520\,X_1^{6}r^{9} - 11877912\,r^{15} - 1560\,X_1^{14} + 37368\,X_1^{12}r^{2} + 182160\,X_1^{10}r^{4} + 126720\,X_1^{8}r^{6}\\
&+166320\,X_1^{6}r^{8} - 1320\,X_1^{4}r^{10} - 24927699\,r^{14} + 36624\,X_1^{12}r + 10560\,X_1^{10}r^{3} - 228096\,X_1^{8}r^{5} - 190080\,X_1^{6}r^{7} + 13200\,X_1^{4}r^{9} + 23801502\,r^{13} + 9224\,X_1^{12} - 184008\,X_1^{10}r^{2} - 285120\,X_1^{8}r^{4} - 138160\,X_1^{6}r^{6} - 46200\,X_1^{4}r^{8} + 72\,X_1^{2}r^{10} + 50622250\,r^{12} - 117744\,X_1^{10}r + 126720\,X_1^{8}r^{3} + 296736\,X_1^{6}r^{5}\\
&+52800\,X_1^{4}r^{7} - 720\,X_1^{2}r^{9} - 27866280\,r^{11} - 21912\,X_1^{10} + 307560\,X_1^{8}r^{2} + 182160\,X_1^{6}r^{4} + 39360\,X_1^{4}r^{6} + 2520\,X_1^{2}r^{8} - 59072955\,r^{10} + 145200\,X_1^{8}r - 172480\,X_1^{6}r^{3} - 88320\,X_1^{4}r^{5} - 2880\,X_1^{2}r^{7} - 3626886\,r^{9} + 21912\,X_1^{8} - 205848\,X_1^{6}r^{2} - 41760\,X_1^{4}r^{4} - 2160\,X_1^{2}r^{6} + 19689366\,r^{8}\\
&-74064\,X_1^{6}r + 51840\,X_1^{4}r^{3} + 4896\,X_1^{2}r^{5} + 4358088\,r^{7} - 9224\,X_1^{6} + 48360\,X_1^{4}r^{2} + 2160\,X_1^{2}r^{4} - 2048621\,r^{6} + 14640\,X_1^{4}r - 2880\,X_1^{2}r^{3} - 524646\,r^{5} + 1560\,X_1^{4} - 2520\,X_1^{2}r^{2} + 50787\,r^{4} - 720\,X_1^{2}r + 8196\,r^{3} - 72\,X_1^{2} + 108\,r^{2} - 144\,r
\end{aligned}$$

$$-2\sqrt{3}X_1r^3 - r^4 - 5\sqrt{3}X_1r^2 - 10\,r^3 + 4\sqrt{3}X_1r - 22\,r^2 + \sqrt{3}X_1 - 8\,r - 1$$

Another example in detail

$\cos(\alpha') = \frac{1}{1+r},\ \alpha = \pi - 2\alpha' \Rightarrow$ $\cos(\alpha) = -\cos(2\alpha') = 1 - 2\cos^2(\alpha') = 1 - \frac{2}{(1+r)^2}$

$\sin^2(\alpha') = \frac{2r+r^2}{(1+r)^2}$ $\sin(\alpha') = \frac{X_1}{1+r},\ X_1^2 = 2r + r^2$ $\sin^2\alpha = \frac{4(r^2+2r)}{(1+r)^4}$

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Example: $l{=}12, m{=}6, n{=}0$ 1-corona $1rr1rr1rr1rr1rr1rr$ $i{=}1, j{=}2, k{=}1$ r-corona $11rr$

$$12\alpha' + 6\beta = 2\pi \qquad \alpha + 2\beta' + \frac{\pi}{3} = 2\pi$$

$-144\,r^{23} - 984\,r^{22} + 7552\,r^{21} + 26655\,r^{20} - 249564\,r^{19} - 797105\,r^{18} + 1656078\,r^{17} + 4852206\,r^{16} - 11042680\,r^{15} - 26531163\,r^{14} + 20520174\,r^{13} + 52701610\,r^{12}$

$-22259880\,r^{11} - 58054275\,r^{10} - 6445302\,r^{9} + 16576710\,r^{8} + 4352616\,r^{7} - 1088797\,r^{6} - 375462\,r^{5} - 56541\,r^{4} - 6556\,r^{3} + 4836\,r^{2} - 288\,r$

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$$X_1^2 = 2r + r^2 \quad \sqrt{3}^2 = 3$$

Polynomial system of equations with variables $r \in [0, 1]$ and $X_1 > 0$

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Polynomial system of equations with variables $r \in [0,1]$ and $X_1 > 0$

$r = 5 - 2\sqrt{6} \approx 0.10102$

Triangulated packings

Kennedy, 2006

(Packings by disks of radii 1 and r) There are 10 values of r allowing an r-corona: $b_0, \ldots, b_9$:

	$i\ j\ k$	r-corona	exact	decimal
b_0	500	11111	$(1 - \sin(\pi/5))/\sin(\pi/5)$	0.7013
b_1	320	$1111r$	$r^4 - 10r^2 - 8r + 9 = 0$	0.6376
b_2	221	$111rr$	$P(r) = 0$	0.5452
b_3	140	$1r1r1$	$8r^3 + 3r\sqrt{2} - 2r - 1 = 0$	0.5333
b_4	400	1111	$2 - \sqrt{1}$	0.4142
b_5	122	$1rrr1$	$\left[2\sqrt{3} + 1 - \sqrt{2}\sqrt{1+3}\right]/3$	0.3861
b_6	041	$1rr1r$	$\sin(\pi/12)/(1 - \sin(\pi/12))$	0.3492
b_7	220	$111r$	$(\sqrt{17} - 3)/4$	0.2808
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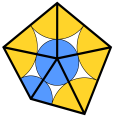

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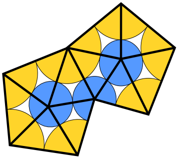

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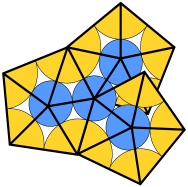

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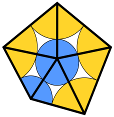

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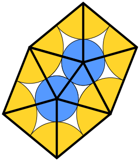

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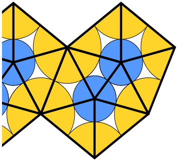

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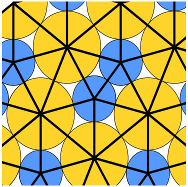

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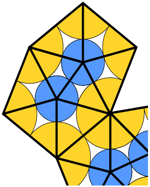

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b_4	400	1111	$2 - \sqrt{1}$	0.4142
b_5	122	$1rrr1$	$\left[2\sqrt{3} + 1 - \sqrt{2}\sqrt{1+3}\right]/3$	0.3861
b_6	041	$1rr1r$	$\sin(\pi/12)/(1 - \sin(\pi/12))$	0.3492
b_7	220	$111r$	$(\sqrt{17} - 3)/4$	0.2808
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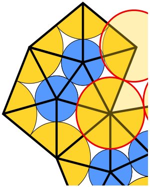

Triangulated packings

Kennedy, 2006

(Packings by disks of radii 1 and r) There are 10 values of r allowing an r-corona: $b_0, \ldots, b_9$:

	$i\ j\ k$	r-corona	exact	decimal
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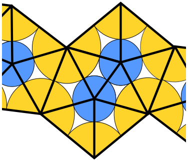

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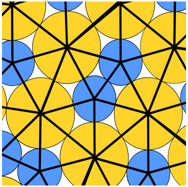

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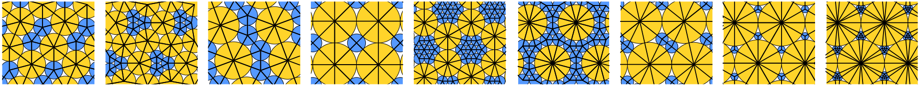
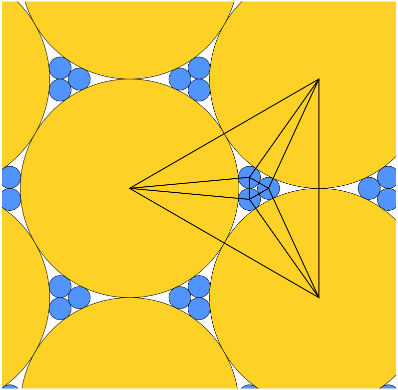
$b_1, \ldots, b_9$ allow triangulated packings of the plane:



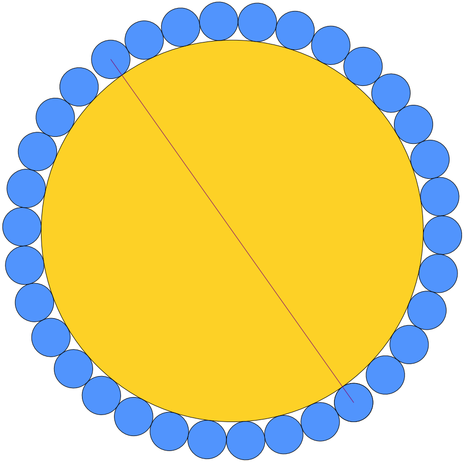
$r\approx0.63$ $\delta^*\approx91.1\%$ | $r\approx0.54$ $\delta^*\approx91.1\%$ | $r\approx0.53$ $\delta^*\approx91.4\%$ | $r\approx0.41$ $\delta^*\approx92\%$ | $r\approx0.38$ $\delta^*\approx92\%$ | $r\approx0.34$ $\delta^*\approx92.5\%$ | $r\approx0.28$ $\delta^*\approx93.2\%$ | $r\approx0.15$ $\delta^*\approx95\%$ | $r\approx0.1$ $\delta^*\approx96\%$

Corollary

If there is an r-corona of disks of radii $1, r$ then $r \in \{b_0, \ldots, b_9\}$ so $r \geq b_9 = 5 - 2\sqrt{6}$.



$r \approx 0.101021$



Then 1-corona has at most 34 disks.

9/12–16/12

1. What is the number (from 1 to 164) of the packing depicted on slide 1 using the numbering from Fernique, Hashemi, Sizova 2021? How did you find it?

2. * Find 5 pairs (i, j) $1 \leq i < j \leq 9$ such that $(1, b_i, b_j)$ admist a triangulated packing using all three disks. Find 4 pairs (i, j) $1 \leq i \neq j \leq 9$ such that $(1, b_i, b_i \cdot b_j)$ admist a triangulated packing using all three disks. Provide triangulated packings for each pair.

LaTeX-generated pdfs, txt, anything except handwriting to be submitted by email to: daria.pchelina@ens-lyon.fr

Deadline: beginning of the lecture in one week (16/12, 15h45)

Find triangulated ternary packings

Theorem (Fernique, Hashemi, Sizova 2019)

Disks of radii 1, r and s: there are 164 pairs (r, s) allowing triangulated packings.

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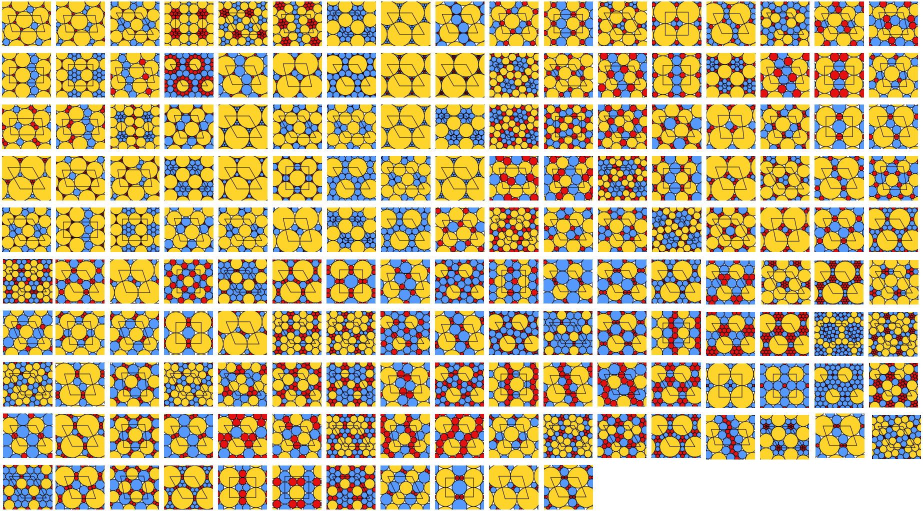
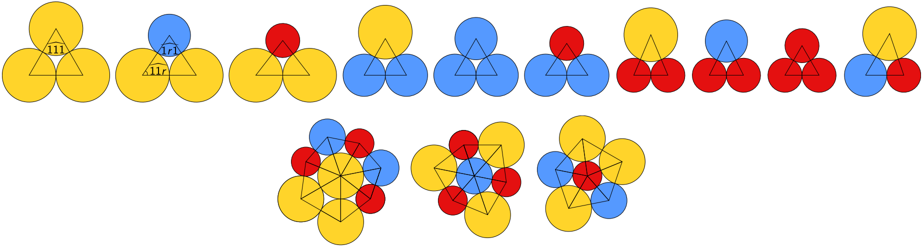


s-coronas



s-corona equation: $S_{\vec{k}}(r,s) = k_1\,\widehat{1s1} + k_2\,\widehat{1sr} + k_3\,\widehat{1ss} + k_4\,\widehat{rsr} + k_5\,\widehat{rss} + k_6\,\widehat{sss}$.

find all $\vec{k}$ having a solution (r,s) $1>r>s>0$ of $S_{\vec{k}}(r,s) = 2\pi$

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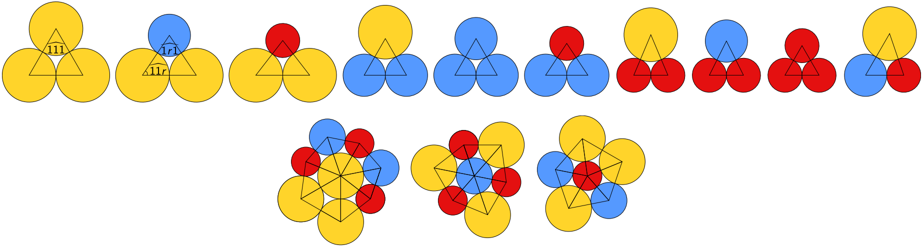

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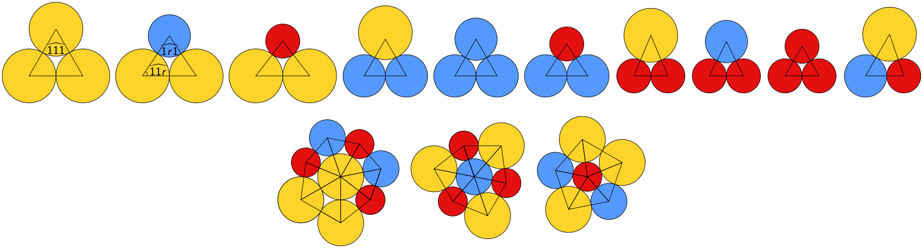

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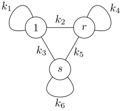
existance of a symbolic corona with these angles



$\to 56\ \overrightarrow{k}$

$\overrightarrow{k}$ should correspond to a cycle in the graph

s-coronas $\to$ polynomials

$\cos \widehat{1s1} = 1 - \frac{2}{(1+s)^2}, \ldots, \cos \widehat{sss} = \frac{\pi}{3}$ $\qquad$ $\sin^2 \widehat{1s1} = \frac{4s(s+2)}{(s+1)^4}, \ldots, \sin^2 \widehat{sss} = \frac{3}{4}$

$\sin \widehat{1s1} = \frac{X_1}{(s+1)^2}, \ldots, \sin \widehat{sss} = \frac{X_6}{2}$ $\qquad$ $X_1^2 = 4s(s+2), \ldots, X_6^2 = 3$

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for 10 s-coronas without r-disk, do not depend on r: $s = b_0, \dots, b_9$

degrees of polynomials of remaining coronas:

$1r1r$	2	$1r1s$	2	$1rsr$	2	$111r$	3	$11r$	3
$1rr$	3	$1rrr$	3	$11rr$	4	$1rss$	4	$11r1s$	6
$11rs$	6	$11rsr$	6	$1rr1s$	6	$1rrs$	6	$1rrsr$	6
$1111r$	7	$11r1r$	7	$1rs1s$	7	$11srs$	8	$1r1ss$	8
$1rssr$	8	$1rsss$	8	$1srrs$	10	$1srss$	10	$1r1rr$	10
$1rrrr$	11	$1rsrs$	11	$1r1rs$	11	$111rr$	12	$11rrr$	12
$111rs$	18	$1rrrs$	18	$11rss$	24	$1rrss$	24	$11rrs$	28

there is an s-corona $\vec{k} \Rightarrow$ the system has a solution $0 < s < r < 1$

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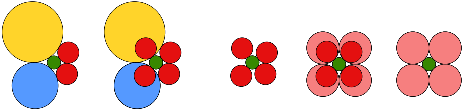

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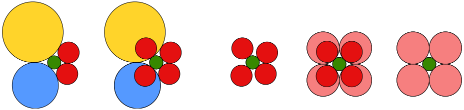

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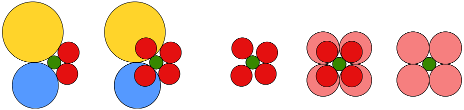

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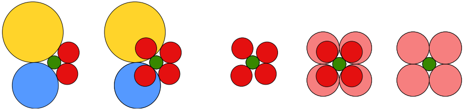

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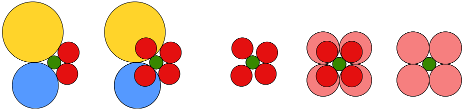

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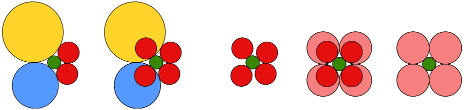

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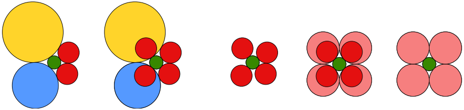

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precise bound on number of compatible r-coronas for each s-corona in function on what we get when deflate $1 \to r$

s — corona	rrrrr	rrrrs	rrrss	rrsrs	rrrr	rrsss	rsrss	rrrs	rrr	rrss
b_i approximation	0.701	0.637	0.545	0.533	0.414	0.386	0.349	0.280	0.154	0.101
upper bound on number of $r-coronas$	84	94	130	143	197	241	272	386	889	**1654**

pairs of coronas

$1 \to r$ deflation equivalence classes:

rrrrr	rrrrs	rrrss	rrsrs	rrrr	rrsss	rsrss	rrrs	rrr	rrss
11111	1111s	111ss	11s1s	1111	11sss	1s1ss	111s	111	11ss
1111r	111rs	11rss	11srs	111r	1rsss	1srss	11rs	11r	1rss
111rr	11r1s	1r1ss	1rs1s	11rr			1r1s	1rr	
11r1r	11rrs	1rrss	1rsrs	1r1r			1rrs		
11rrr	1r1rs	r1rss	rrs1s	1rrr			r1rs		
1r1rr	1rr1s								
1rrrr	1rrrs								
	r11rs								
	r1rrs								
8	10	6	6	6	3	3	6	4	3

Number of pairs of coronas:
$(84, 94, 130, 143, 197, 241, 272, 386, 889) \cdot (8, 10, 6, 6, 6, 3, 3, 6, 4, 3) = 16805$

huge polynomials: s-corona $11rrs$, r-corona $11rrs^12$ polynomials of degree 28 and 416 (1.4Mo in txt)

Gröbner basis, resultants

too many solutions: filter them with interval arithmetic

exact filtering (check the equations)

find packings: all preiodic, by hand

Find 4-disk triangulated packings?

Disks of radii $1 > r > s > t > 0$

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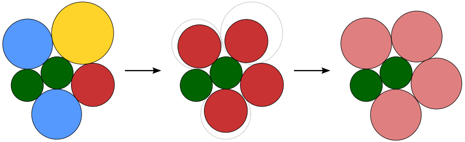

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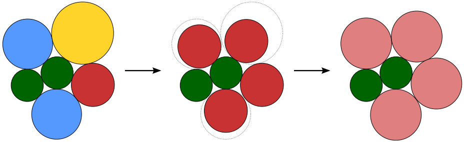

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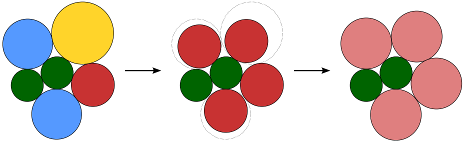
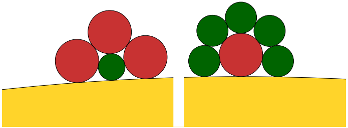


- r-corona — try to bound $\frac{s}{r}$ from below given existence of an s-corona? **no**

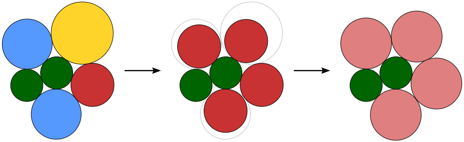


Find 4-disk triangulated packings?

Disks of radii $1 > r > s > t > 0$

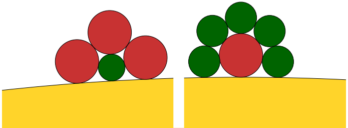
we need 3 coronas to obtain a polynomial system of equations determining (r, s, t)

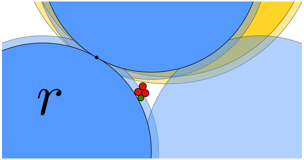
- t-corona — smallest disk, so < 5 disks in any non-trivial corona
- s-corona — bound $\frac{t}{s} > 5 - 2\sqrt{6}$ as for 3-disk $\Rightarrow$ finite number of s-coronas



- r-corona — try to bound $\frac{s}{r}$ from below given existence of an s-corona? **no**

  solution: look further — suppose $\frac{s}{r}$ very small, prove that no possible packing — $\frac{s}{r} > 0.054$





n-disk triangulated packings?

$\forall n$ there is finitely many $(r_1, r_2, \dots, r_n)$ such that $1 = r_1 > r_2 > \cdots > r_n > 0$ allowing triangulated packings using all of these disk radii Messerschmidt 2022

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$n > 4$ challenges:

need $n - 1$ coronas to find a solution:

- r_n-corona smallest disk, so < 5 disks in any non-trivial corona

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very big coronas: even if we can bound $\frac{r_n}{r_{n-1}} > \epsilon_{n-1}, \cdots \frac{r_3}{r_2} > \epsilon_2$, we only get

$\frac{r_n}{r_2} > \prod_{i=2}^{n-1} \epsilon_i \to 0 \Rightarrow$ huge number of disks in r_2-corona

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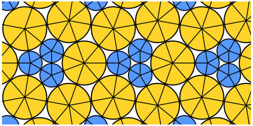
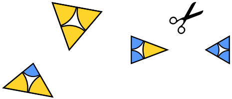
more variables and higher degrees of polynomials $\to$ hard to find exact solutions

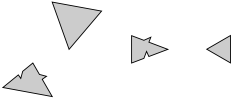
given a solution with coronas, automatize search for packing (search for a periodic tiling)

Packings and tilings

triangulated packings $\sim$ tilings by triangles with local rules







density = weighted proportion of tiles

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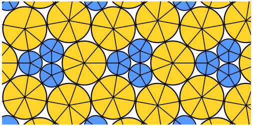
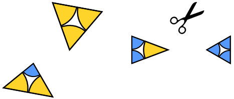
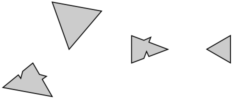






density = weighted proportion of tiles

Domino Problem

Given a set of Wang tiles, is there a valid tiling of the plane?

$\forall$ tileset with valid tilings, one is periodic $\Rightarrow$ **decidable**

(Wang algorithm: search for a period)

More disks more questions

Packings and tilings

triangulated packings ∼ tilings by triangles with local rules

density = weighted proportion of tiles

Domino Problem

Given a set of Wang tiles, is there a valid tiling of the plane?

∀ tileset with valid tilings, one is periodic ⇒ **decidable**

(Wang algorithm: search for a period)

∃ tileset with valid tilings which are all non-periodic ⇒ **undecidable**

Packings and tilings

triangulated packings $\sim$ tilings by triangles with local rules

density = weighted proportion of tiles

Triangulated Packing Problem

Given k disk radii $\overbrace{r_1, \cdots, r_k}^{\text{algebraic numbers represented by polynomials and intervals}}$, is there a triangulated packing of density $\overbrace{> \frac{\pi}{2\sqrt{3}}}^{\text{excludes hexagonal packing}}$

$\forall\ r_1, \cdots, r_k$ with triangulated packings, one is periodic $\Rightarrow$ **decidable**

(Wang algorithm: search for a period)

$\exists\ r_1, \cdots, r_k$ whose triangulated packings are all non-periodic $\Rightarrow$ **undecidable?**

Packings and tilings

triangulated packings $\sim$ tilings by triangles with local rules

density = weighted proportion of tiles

Dense Packing Problem

Given k disk radii $\overbrace{r_1, \cdots, r_k}^{\text{algebraic numbers represented by polynomials and intervals}}$, is there a packing of density $\overbrace{> \frac{\pi}{2\sqrt{3}}}^{\text{excludes hexagonal packing}}$

$\forall\ r_1, \cdots, r_k$ with dense packings, one is periodic $\Rightarrow$ **decidable**

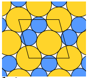
(interval arithmetic and subdivision until needed precision)

$\exists\ r_1, \cdots, r_k$ whose dense packings are all non-periodic $\Rightarrow$ **not possible!**

Binary triangulated packings

Kennedy, 2006

(Packings by discs of radii 1 and r) There are 9 values of r allowing triangulated packings: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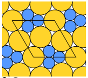
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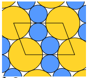
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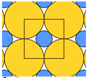
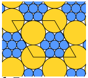

b4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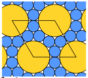
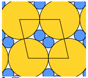
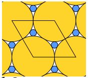
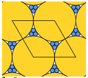

b6


b7


b8


b9

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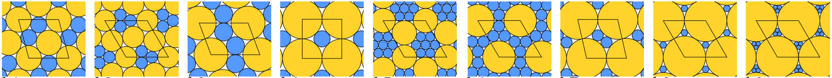

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Theorem (Heppes 2000, 2003, Kennedy 2005, Bedaride and Fernique 2022)

Each of these packings is optimal (densest) for discs of radii 1 and r.

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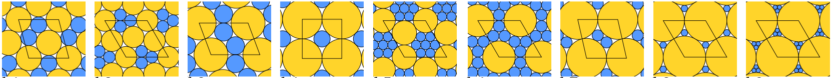

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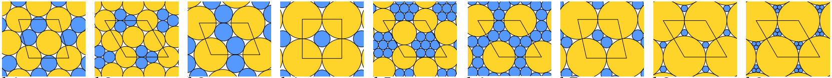
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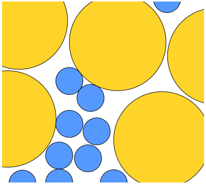
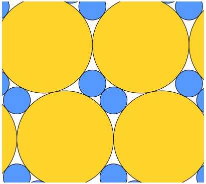


P of density $\delta(P)$



P^* of density δ^*

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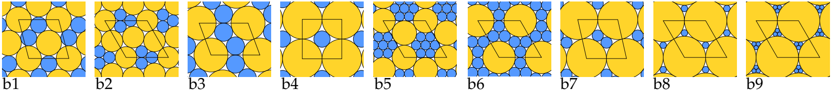

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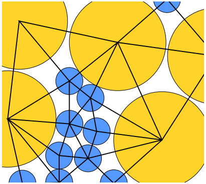

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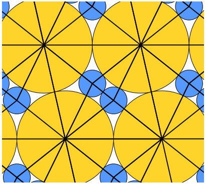

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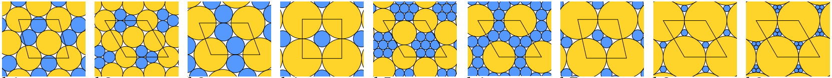

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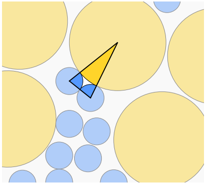
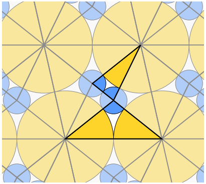


P of density $\delta(P)$

Triangles in P^* have different densities:

$\delta($  $) < \delta^* < \delta($  $)$

Hopeless to bound the density by δ^* in each triangle...



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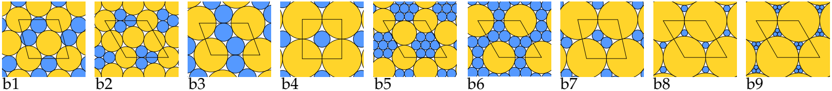

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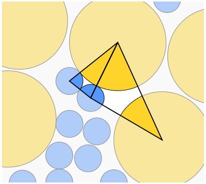
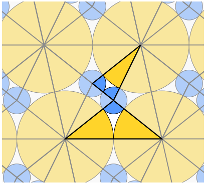


P of density $\delta(P) \leq \delta'(P)$

redistributed density $\delta' \geq \delta$:

dense triangles
share their density
with empty neighbors



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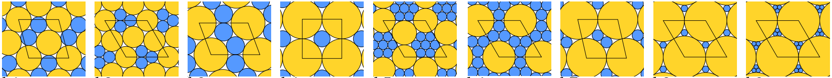

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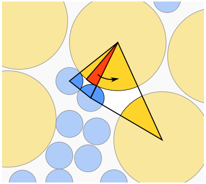

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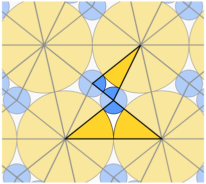

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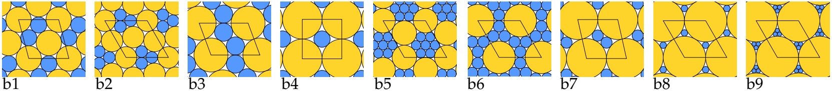

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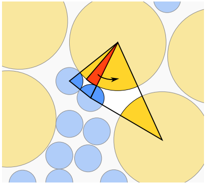

P of density $\delta(P) \leq \delta'(P)$
$\forall \Delta$, $\delta'(\Delta) \leq \delta^*$

$$\delta(P) \leq \delta'(P) \leq \delta^*$$

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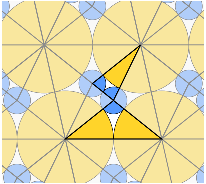

P^* of density δ^*

Spin systems → disk packings

Kennedy 2005

spin system: graph, each vertex has a "spin" taking values in a small set
two spins interact iff connected by an edge

total energy = sum of local interaction energies on edges: $\sum_{(i,j)\text{ adjacent}} E(\sigma_i, \sigma_j)$

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$E(\sigma_i, \sigma_j)$ minimized when $\sigma_i = \sigma_j$: $E(\sigma_i, \sigma_j) = -\sigma_i \cdot \sigma_j$

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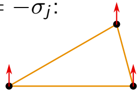

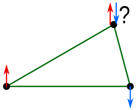

frustrated spin system: all local interactions can not be minimized at the same time

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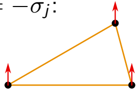
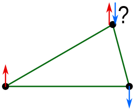
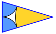
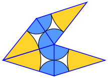




frustrated spin system: all local interactions can not be minimized at the same time

frustration in general: when local optimal pieces cannot be assembled consistently into a global optimum

frustation in multi-disk packings: the locally densest triangle  cannot tile the plane:



Solution to frustration: m-potentials

spin system: lattice vertices V, spin set $S = \{+1, -1\}$ **configuration**: $\sigma = \{\sigma_i\}_{i \in V} \in V^S$

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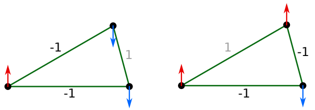
frustration: no configuration in a triangle satisfies all local minimization conditions
(i,j,k) is a triangle, $\forall \sigma \; E_{(i,j)}(\sigma) + E_{(i,k)}(\sigma) + E_{(k,j)}(\sigma) \geq -1 > -3 = 3 \min E_{(i,j)}$



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Φ is **equivalent** to E if $\forall$ finite subset $F \subset V$ $\Phi_{V \setminus F} = E_{V \setminus F}$

has the same ground states $\sum_{i,j,k \in V \setminus F} \Phi_{(i,j,k)} = \sum_{i,j \in V \setminus F} E_{(i,j)}$

example: $\Phi_{(i,j,k)}(\sigma) = \frac{1}{2}(\sigma_i\sigma_j + \sigma_i\sigma_k + \sigma_k\sigma_j)$ if (i,j,k) is a triangle equivalent to E

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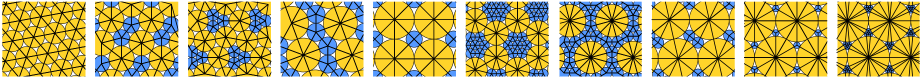
$\rightarrow$ find ground state configurations for E

Triangulated = optimal?

Optimal triangulated packings:



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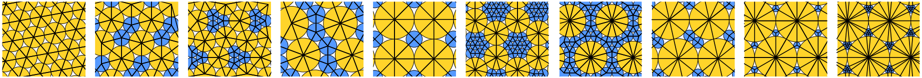

Conjecture (Connelly 2018)

If a finite set of discs allows saturated triangulated packings then one of them is optimal.

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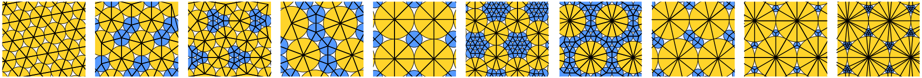

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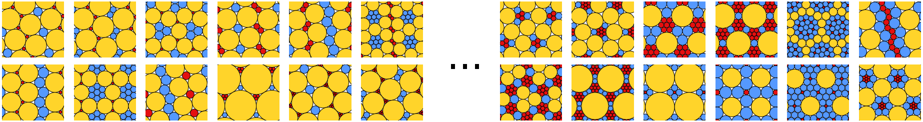
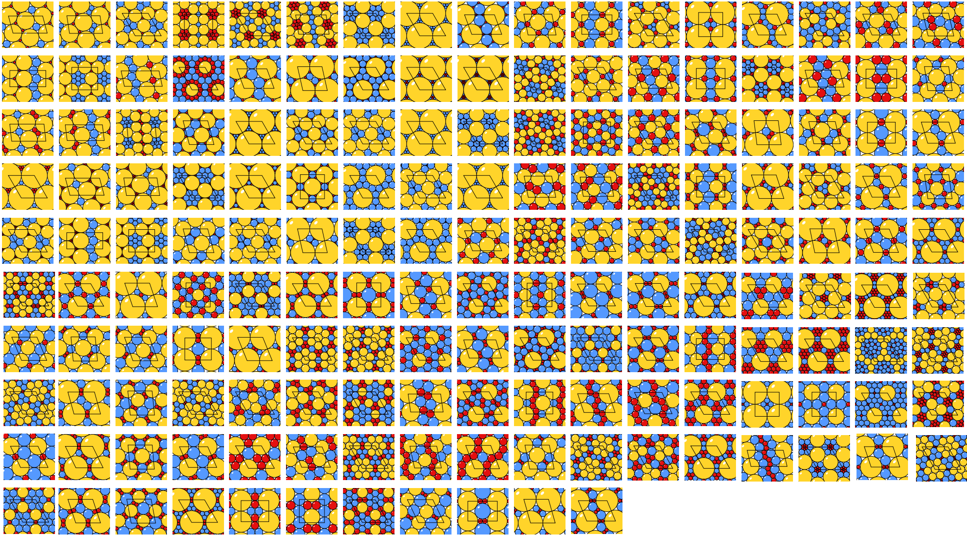
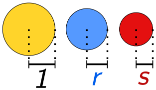
Theorem (Fernique, Hashemi, Sizova 2019)

Discs of radii 1, r and s: there are 164 pairs (r, s) allowing triangulated packings.

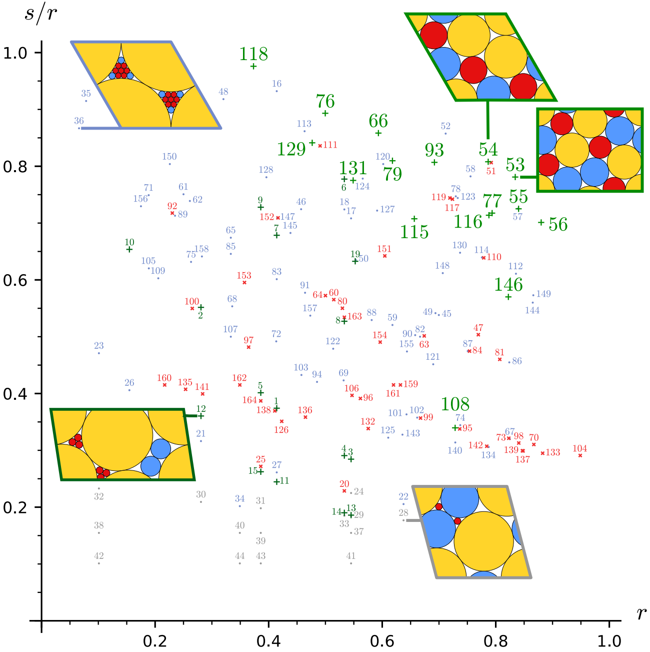






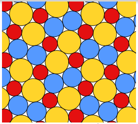
164 (r, s) allowing triangulated packings:

- 15 cases: non saturated
- 16+16 cases: a **ternary** or **binary** triangulated packing is densest
- 45 cases: a binary **non** triangulated packing is denser

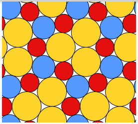


Theorem (Fernique, P 2023)

Each of the following packings is optimal for discs of radii 1, r and s:



$r\approx0.8$ $s\approx0.7$ $\delta^*\approx90.9\%$



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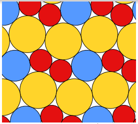

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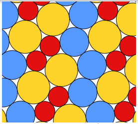

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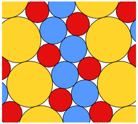
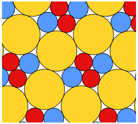
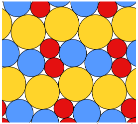


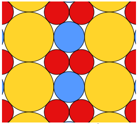
$r\approx0.6$ $s\approx0.5$ $\delta^*\approx91.2\%$



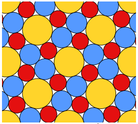
$r\approx0.5$ $s\approx0.4$ $\delta^*\approx91.5\%$



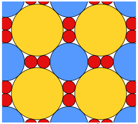
$r\approx0.8$ $s\approx0.6$ $\delta^*\approx91\%$



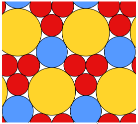
$r\approx0.6$ $s\approx0.5$ $\delta^*\approx91.3\%$



$r\approx0.7$ $s\approx0.5$ $\delta^*\approx91.1\%$



$r\approx0.7$ $s\approx0.2$ $\delta^*\approx93.2\%$



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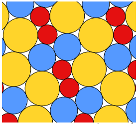

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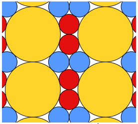
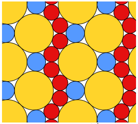
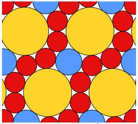


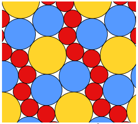
$r\approx0.4$ $s\approx0.4$ $\delta^*\approx92.2\%$



$r\approx0.5$ $s\approx0.4$ $\delta^*\approx91.8\%$



$r\approx0.5$ $s\approx0.4$ $\delta^*\approx91.7\%$



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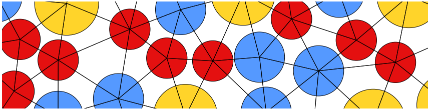
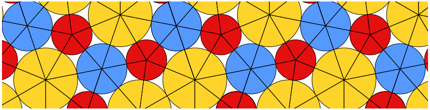
Emptiness instead of density



saturated packing P with the same discs
density δ, FM-triangulation $\mathcal{T}$



saturated triangulated packing P^*
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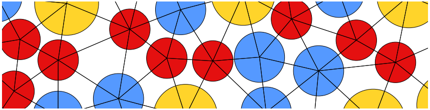

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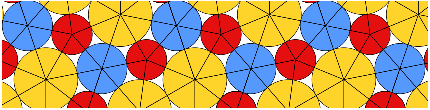

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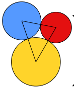
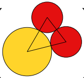
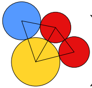
Density function is not additive: $\delta\Big($  $\Big) + \delta\Big($  $\Big) \neq \delta\Big($  $\Big)$ $\longrightarrow$

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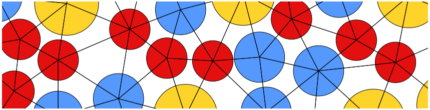

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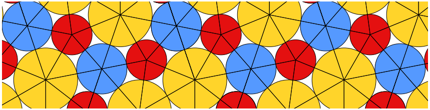

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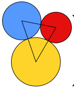
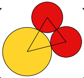
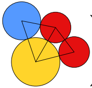
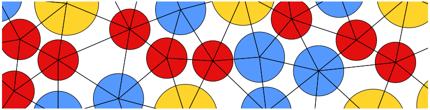
Density function is not additive: $\delta\Big($  $\Big) + \delta\Big($  $\Big) \neq \delta\Big($  $\Big)$ $\longrightarrow$

Emptiness of a triangle $\Delta \in \mathcal{T}$: $E(\Delta) = \delta^* \times area(\Delta) - area(\Delta \cap P)$

$E(\Delta) > 0$ iff the density of Δ is less than δ^*
$E(\Delta) < 0$ iff the density of Δ is greater than δ^*

Additive!

Emptiness instead of density



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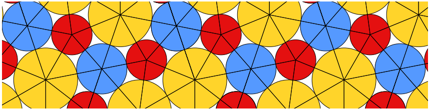
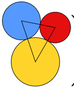
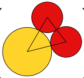
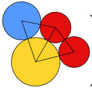


saturated triangulated packing P^*
density δ^*, FM-triangulation $\mathcal{T}^*$

Density function is not additive: $\delta\Bigg($  $\Bigg) + \delta\Bigg($  $\Bigg) \neq \delta\Bigg($  $\Bigg)$ $\longrightarrow$

Emptiness of a triangle $\Delta \in \mathcal{T}$: $E(\Delta) = \delta^* \times area(\Delta) - area(\Delta \cap P)$

$E(\Delta) > 0$ iff the density of Δ is less than δ^*
$E(\Delta) < 0$ iff the density of Δ is greater than δ^*

Additive!

$$\delta^* \geq \delta \;\Leftrightarrow\; \sum_{\Delta \in \mathcal{T}} E(\Delta) \geq 0$$

Potential is a redistribution of emptiness

We construct a **potential** $U(\Delta) := \overbrace{\dot{U}_\Delta^A + \dot{U}_\Delta^B + \dot{U}_\Delta^C}^{\text{vertices}}$ such that

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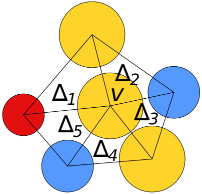

v-corona C_v

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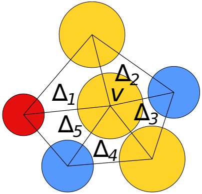

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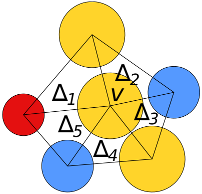

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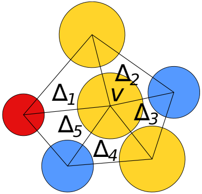


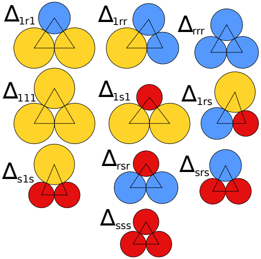
v-corona C_v

If such U exists then $\delta^* \geq \delta$

Construct it in way that $(\bullet)$ holds and then prove (Δ)

Choosing U to assure ($\bullet$)

Δ_{xyz}	tight triangle: tangent discs of radii x, y, z
$\widehat{xyz}$	angle of Δ_{xyz} in the center of the y-disc
V_{xyz}	potential of Δ_{xyz} in the center of the y-disc



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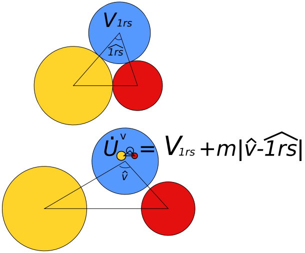
potential of a triangle Δ in v:

$$\dot{U}_{\Delta}^{v} := V_{xyz} + m|\hat{v} - \widehat{xyz}|$$

measures how "far" Δ is from being tight



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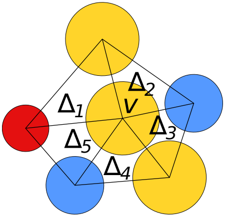

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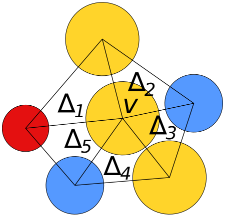

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angle values do not matter $\Rightarrow$ sequence of disc radii $S(C_v)$

FM-triangulation $\Rightarrow$ bounded $|S(C_v)|$

finite number of linear inequalities on m
$\Rightarrow$ computer search

Verifying (Δ) with recursive subdivision

Defining U, we make it as small as possible keeping it positive around any vertrex (•)

How to check $U(\Delta) \leq E(\Delta)$ on each possible triangle Δ? (there is a continuum of them)

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  instead of verifying $U(\Delta_{a,b,c}) \leq E(\Delta_{a,b,c})$ for all $(a,b,c) \in [\underline{a},\overline{a}] \times [\underline{b},\overline{b}] \times [\underline{c},\overline{c}]$,

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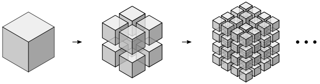
- If $[\underline{U},\overline{U}]$ and $[\underline{E},\overline{E}]$ intersect, recursive subdivision: 

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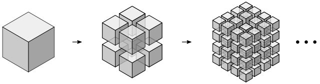

QED

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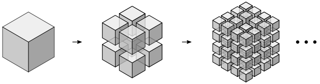
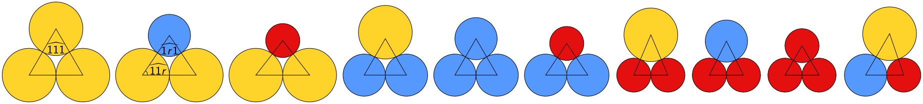
- If $[\underline{U}, \overline{U}]$ and $[\underline{E}, \overline{E}]$ intersect, recursive subdivision: 

never stops if $U(\Delta) = E(\Delta)$

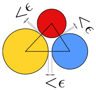
QED ?

Local optima



On tight triangles, $U(\Delta_{xyz}) := E(\Delta_{xyz}) \rightarrow$ impossible to use interval method around them

ϵ-triangles T_ϵ – triangles close to tight $\Rightarrow$ potential close to emptiness



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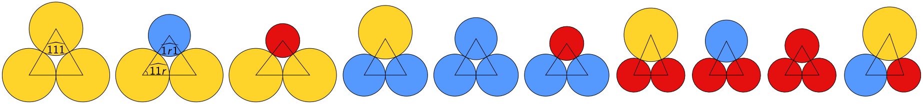

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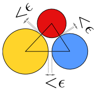

interval arithmetic + recursive subdivision on derivatives on side lengths x_i to check that:

$$\max_{T_\epsilon} \frac{\partial U}{\partial x_i} \Delta x_i < \min_{T_\epsilon} \frac{\partial E}{\partial x_i} \Delta x_i,$$

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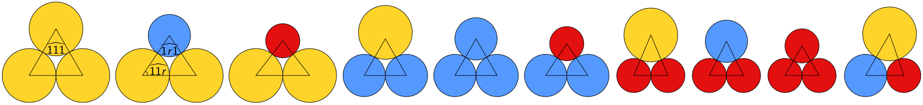

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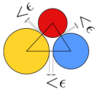

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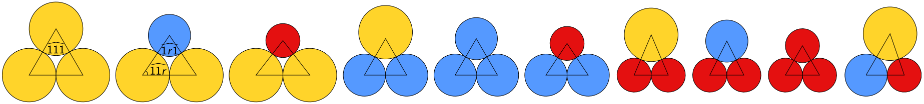
$\Rightarrow \Delta_{xyz}$ is the maximum of $U - E$ on T_ϵ

$\Rightarrow$ for all triangles Δ from T_ϵ, $U(\Delta) \leq E(\Delta)$

Local optima



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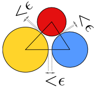

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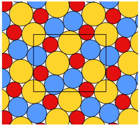
The proof worked for these cases: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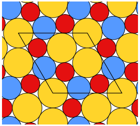
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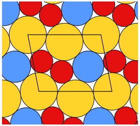
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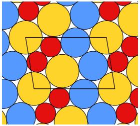
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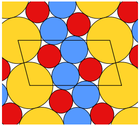
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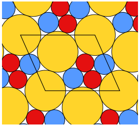
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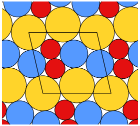
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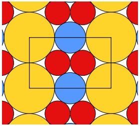
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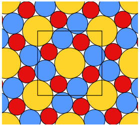
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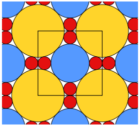
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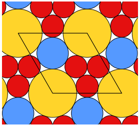
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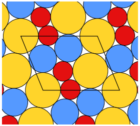
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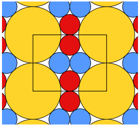
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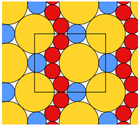
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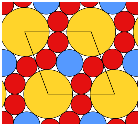
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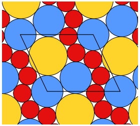
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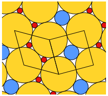
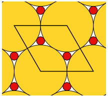
And these:

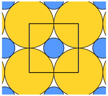
$\delta^* \approx 93\%$



$\delta^* \approx 95\%$



$\delta^* \approx 92\%$



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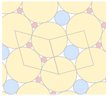
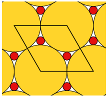
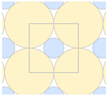
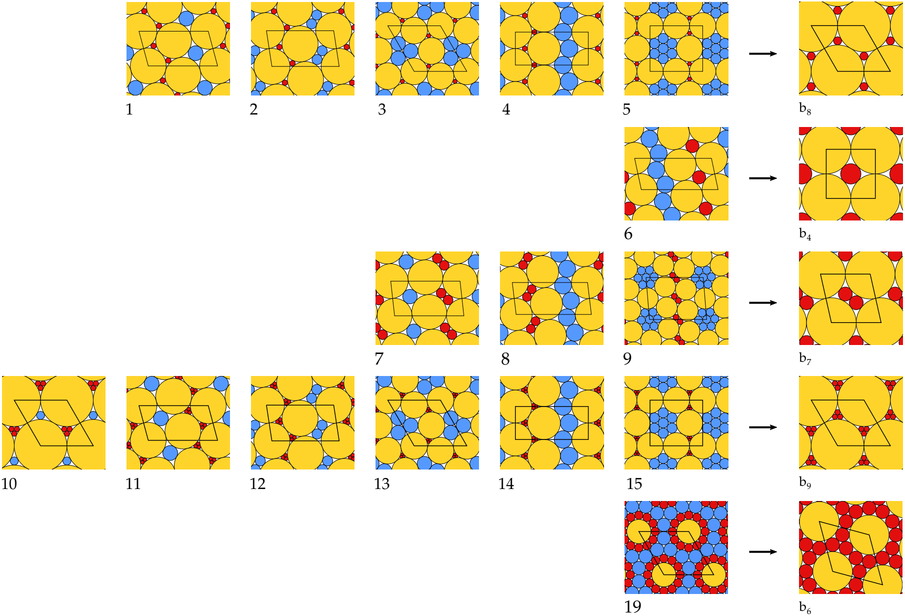
$\delta^* \approx 93\%$ 

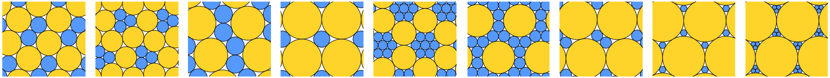
$\delta^* \approx 95\%$ 

$\delta^* \approx 92\%$ 

And these:

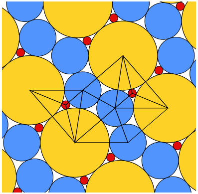


45 counter examples: *flip-and-flow* method



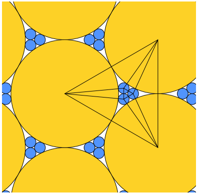
When the ratio of two discs is close enough to the ratio in a dense binary packing, we can pack these discs in a similar (non triangulated) manner and still get high density

triangulated ternary packing



$\delta \leq 0.931369$ $s \approx 0.121445$

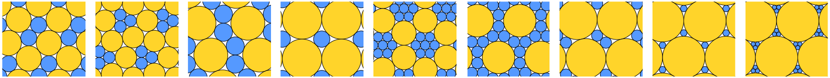
dense binary packing



$r \approx 0.101021$

Optimal triangulated packings

45 counter examples: *flip-and-flow* method



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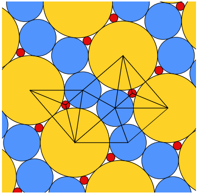
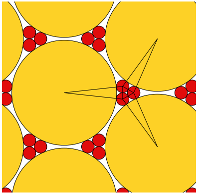
triangulated ternary packing



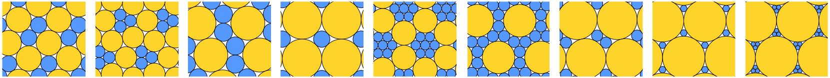
$\delta \leq 0.931369$ $s \approx 0.121445$

dense non-triangulated packing



$\delta \geq 0.937371$ $s \approx 0.121445$

45 counter examples: *flip-and-flow* method



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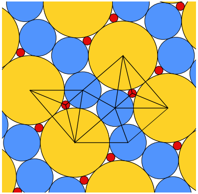
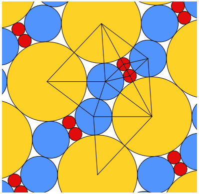
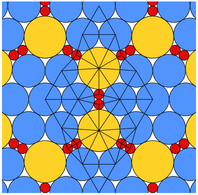
triangulated ternary packing



$\delta \leq 0.931369$ $s \approx 0.121445$

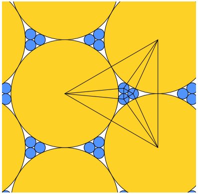


$\delta \leq 0.924522$ $s \approx 0.166169$

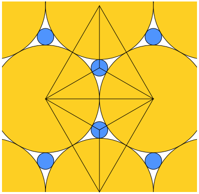


$\delta \leq 0.917352$ $s \approx 0.240205$

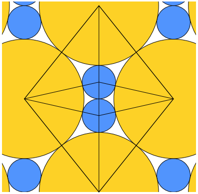
dense binary packing



$r \approx 0.101021$



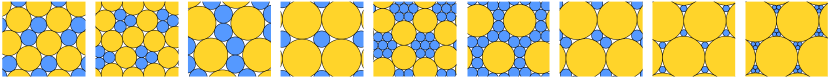
$\delta \approx 0.950308$ $r \approx 0.154701$



$\delta \approx 0.931901$ $r \approx 0.280776$

Optimal triangulated packings

45 counter examples: *flip-and-flow* method



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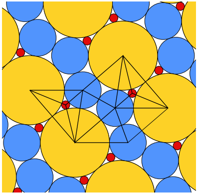
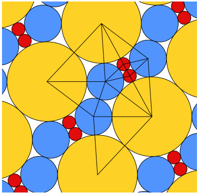
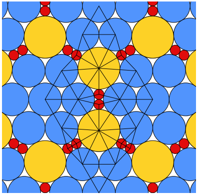
triangulated ternary packing



$\delta \leq 0.931369$ $s \approx 0.121445$

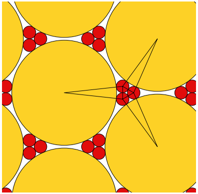


$\delta \leq 0.924522$ $s \approx 0.166169$

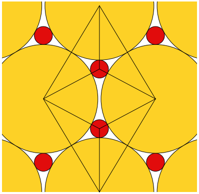


$\delta \leq 0.917352$ $s \approx 0.240205$

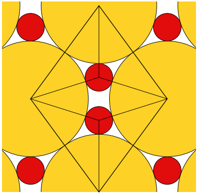
dense non-triangulated packing



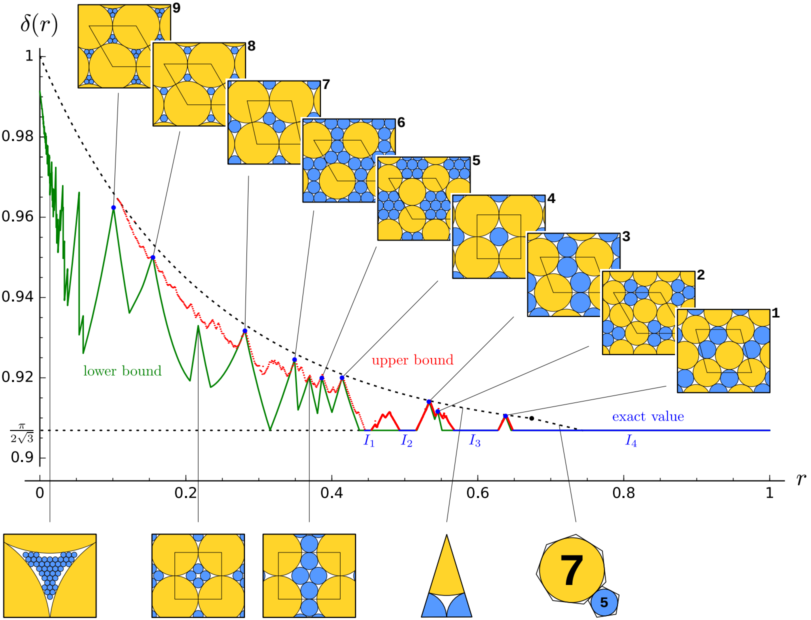
$\delta \geq 0.937371$ $s \approx 0.121445$



$\delta \geq 0.939305$ $s \approx 0.166169$



$\delta \geq 0.918420$ $s \approx 0.240205$


δ(r)
1
0.98
0.96
0.94
0.92
π/(2√3)
0.9
0
0.2
0.4
0.6
0.8
1
r
9
8
7
6
5
4
3
2
1
lower bound
upper bound
exact value
I1
I2
I3
I4
7
5

What's next?

When the proof does not work, need to look further?

Possibility to derive bounds on optimal density for 3-disk packings

Homework II

11/12–18/12

The **Appolonian-type packing of level 1** is the hexagonal packing of unit disks.

Given the set $\mathbb{A}_n$ of Appolonian-type packings of level n, the set $\mathbb{A}_{n+1}$ is constructed by inserting disks in packings from the previus level as follows:

For each packing $P \in \mathbb{A}_n$, let $T(P)$ denote the set of triangles in its FM-triangulation,
let R denote the radius of the largest support circle of a triangle from $T(P)$,
let $SC(P)$ be the packing P where we insert disks of radius R until it is saturated.

If there is at least one equilateral triangle in $T(P)$ (formed by three identical disks) let r be the disk radius of the largest of them, called t.
Packing $ET(P)$ is obtained from the packing P by insertion of triplets of disks of radius $(5 - 2\sqrt{6})r$ in each triangle t in the only possible way:  .

We define $\mathbb{A}_{n+1}$ as the set of all packings $SC(P)$ and $ET(P)$ for $P \in \mathbb{A}_n$:

$$\mathbb{A}_{n+1} = \{SC(P) | P \in \mathbb{A}_n\} \cup \{ET(P) | P \text{ has equlateral FM-triangles and } P \in \mathbb{A}_n\}$$

1. Prove that Appolonian-type packings are triangulated. What is the densest Appolonian-type packing of level 3?
2. * What is the densest Appolonian-type packing of level n? What are the disk radii present in packings from $\mathbb{A}_n$?

LaTeX-generated pdfs, txt, anything except handwriting to be submitted by email to: daria.pchelina@ens-lyon.fr

Deadline: beginning of the lecture in one week (18/12, 10h15)