Exhaustive search of convex pentagons which tile the plane

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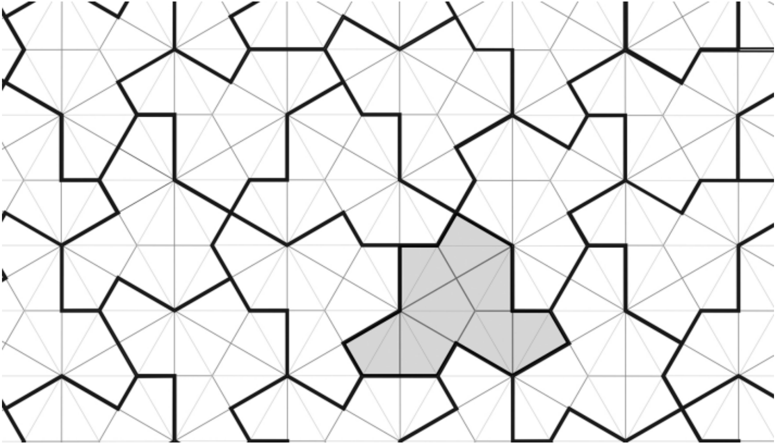
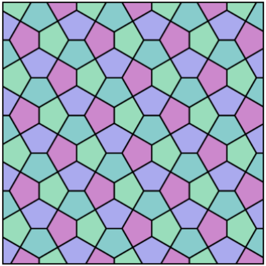
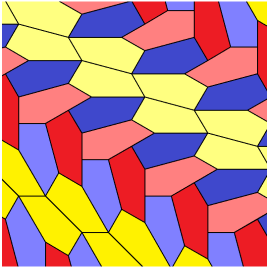
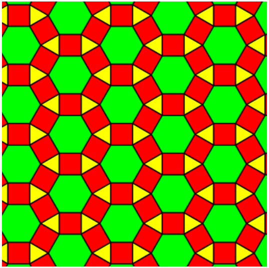
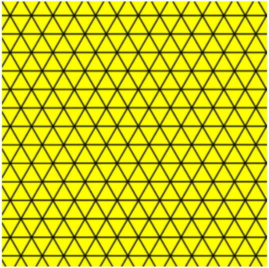
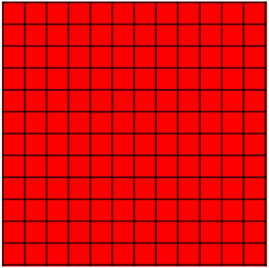
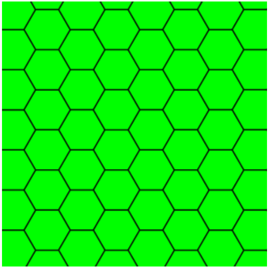
The “hat” (Smith, Myers, Kaplan, Goodman-Strauss, 2023)



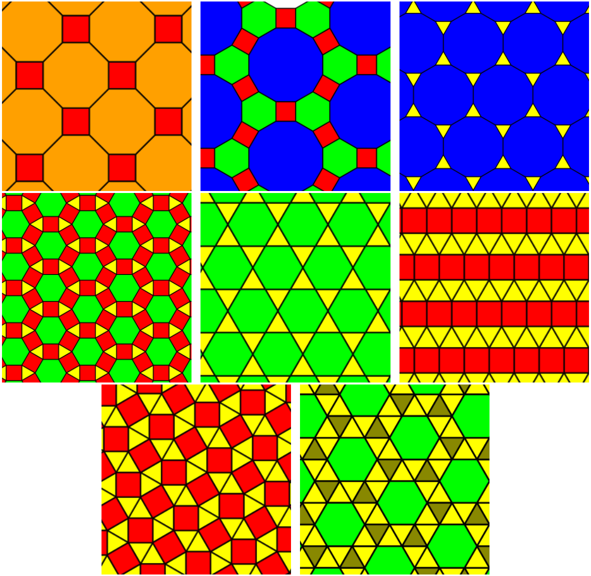
Tiling: covering of the plane using copies of one or more **tiles**, with no overlaps and no gaps.

Tilings with a regular polygon

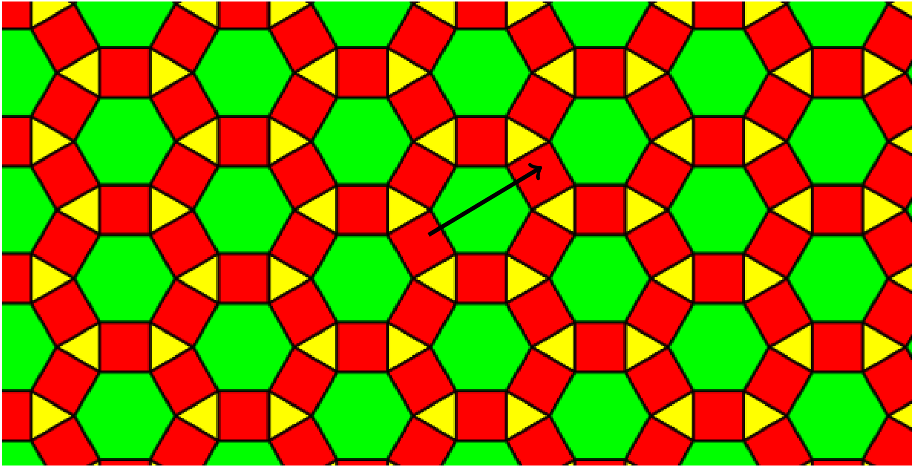




Semi-regular tilings



Aperiodic Tiling

Periodic tiling: there is a translation which does not change the tiling



previous tilings are periodic

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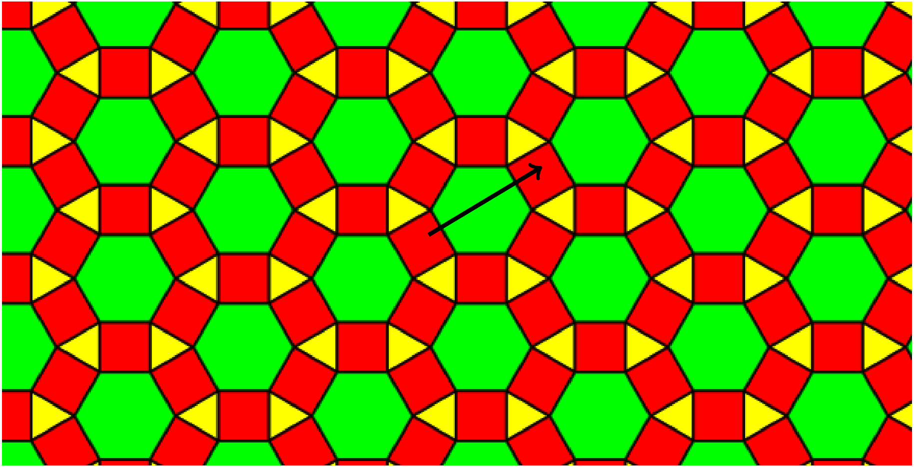

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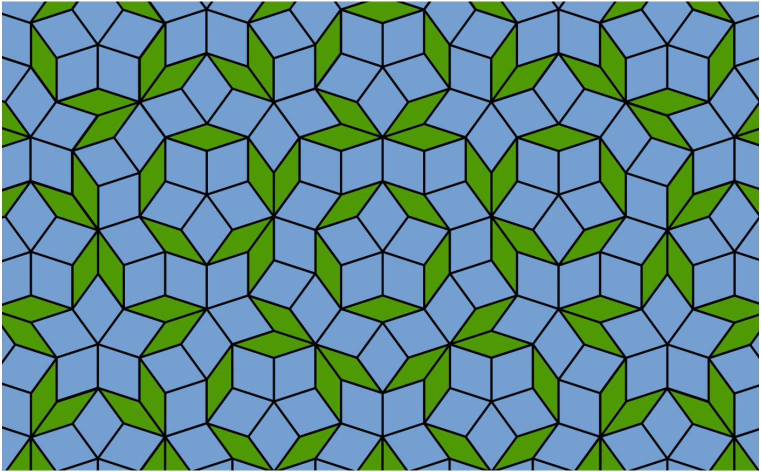
Aperiodic tiling: There is not such translation

Penrose Tiling: a well known aperiodic tiling



Penrose Tiling IRL

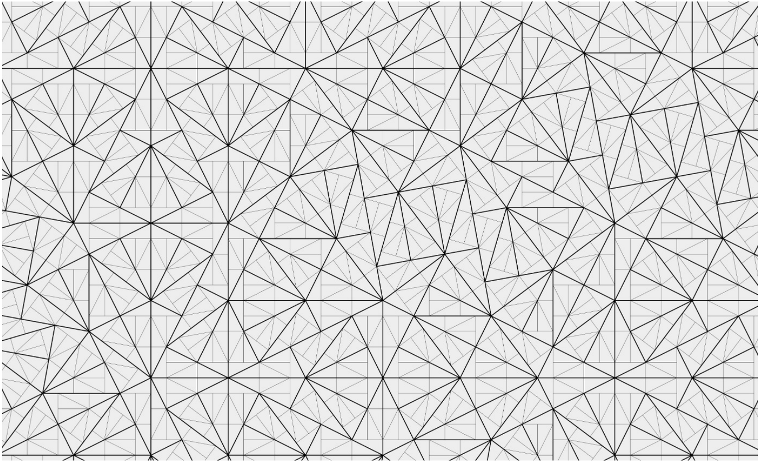
Roger Penrose, Institut Mitchell, Texas A&M University

Penrose Tiling IRL

By Thomas Fernique and Evgeny Poloskin

http://images.math.cnrs.fr/Un-parquet-de-Penrose.html

Pinwheel tiling



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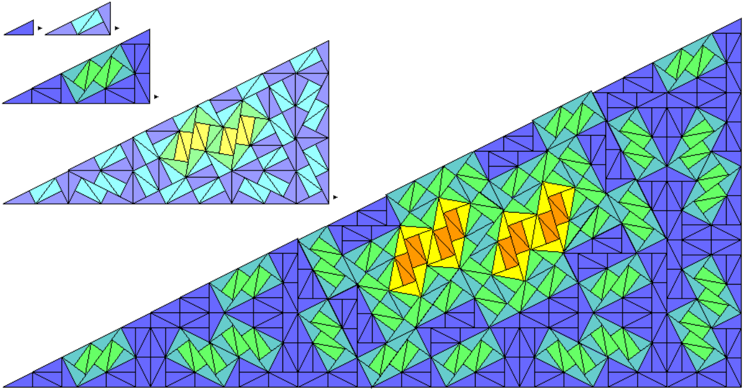
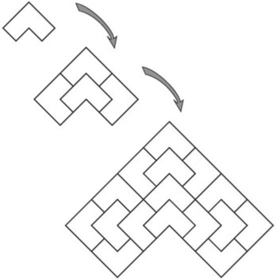
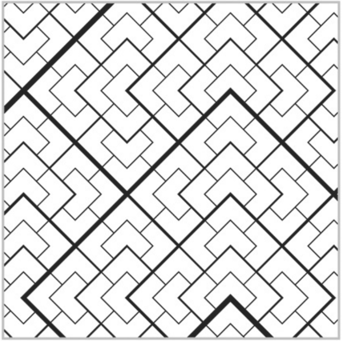


Pinwheel tiling

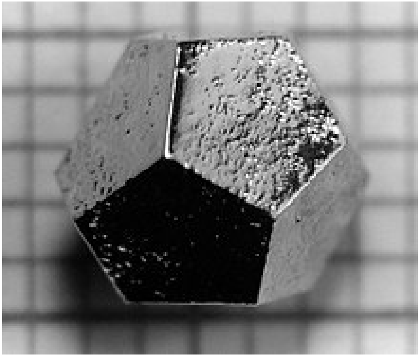
Federation Square (Melbourne, Australia)

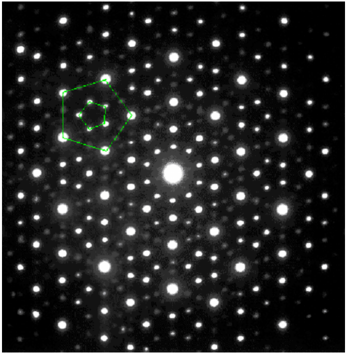
Chair Tiling





Aperiodicity in the nature





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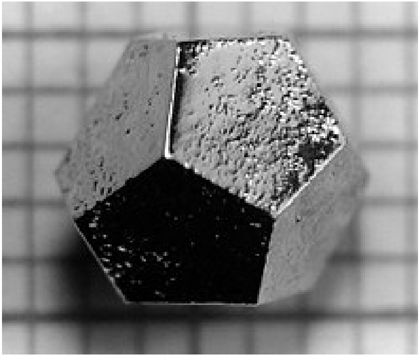

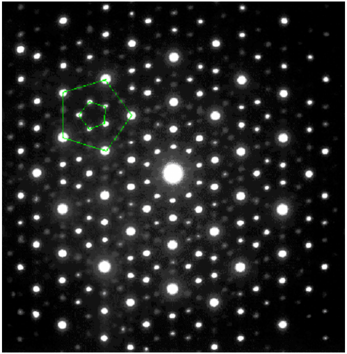

Quasiperiodic-crystal: crystal with non periodic structure

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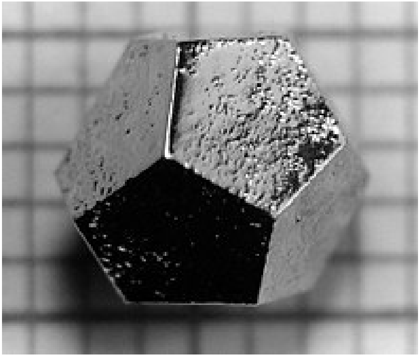 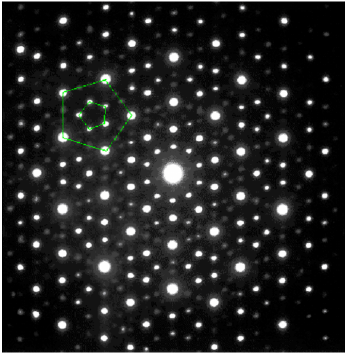

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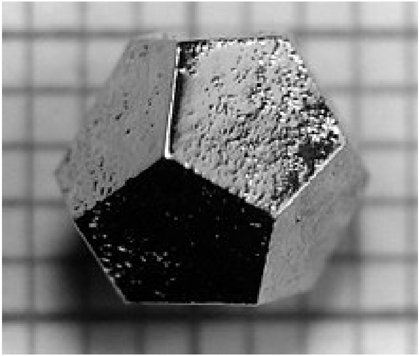

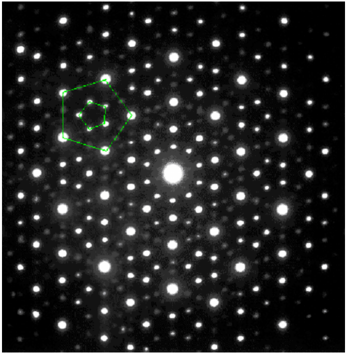

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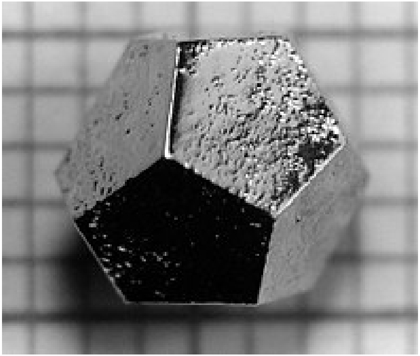

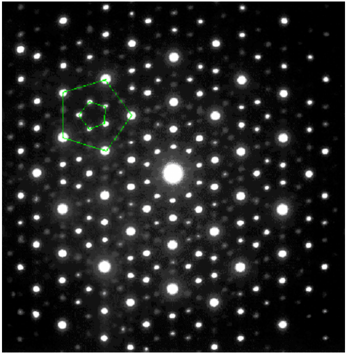

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Natural quasiperiodic-crystal discovered in 2009 in Koryak Mountains.

Force the aperiodicity ?

Suppose you want to find a set of local rules such that the only crystal/floor/... you can construct is aperiodic

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How to " force" a tiling to be aperiodic ?

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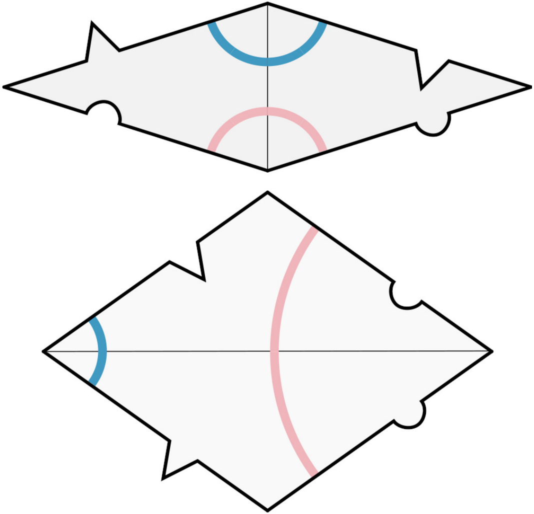
How to " force" a tiling to be aperiodic ?

A tile-set is **aperiodic** is it tiles the plane, and all tilings are aperiodic

Of course, we are interested in simple aperiodic tile-sets.

Penrose with decorations



Aperiodic tiling with one tile ?

Thare are aperiodic tile-sets with two tiles (e.g.: Penrose, Ammann–Beenker...)

Is there an aperiodic tile-set with <u>one</u> tile ?

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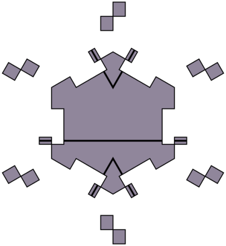
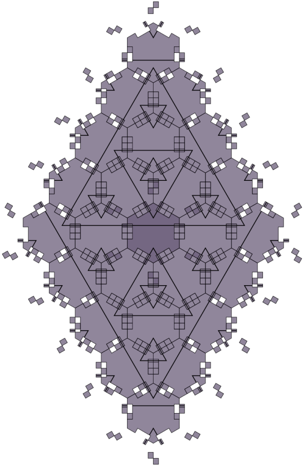
⇒ “Ein-stein” problem (from the German, “one stone”)

Taylor-Socolar tile (2011)





Aperiodic tiling with one connected tile ?

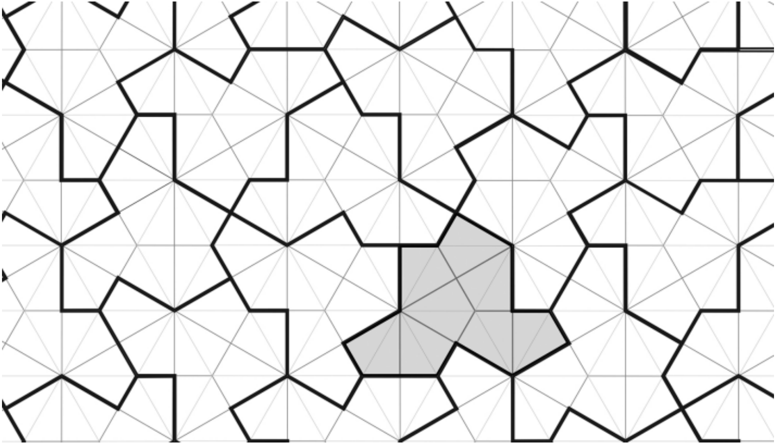
Is there an aperiodic tile-set with one connected tile ?

Aperiodic tiling with one connected tile ?

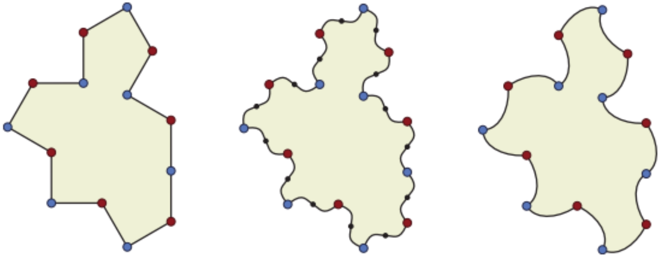
Is there an aperiodic tile-set with <u>one</u> <u>connected</u> tile ?

In 2023 : Yes, the "Hat", Discovered by David Smith.

The hat (Smith, Myers, Kaplan, Goodman-Strauss, 2023)



The spectre (Smith, Myers, Kaplan, Goodman-Strauss, 2023)



Finding polygons that tile the plane

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Is there an aperiodic tile-set with one convex tile ?

More generaly:

Wich convex shape tiles the plane ?

Tile the plane with convex polygons

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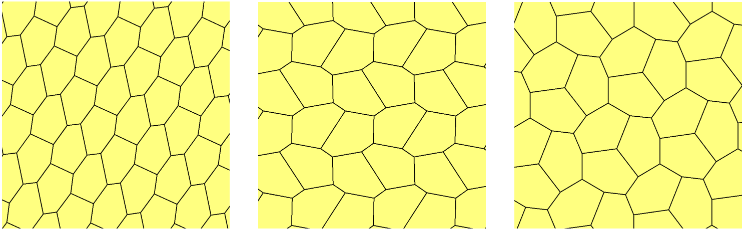

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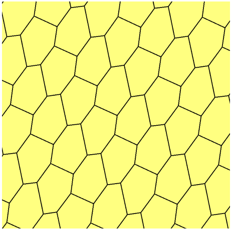
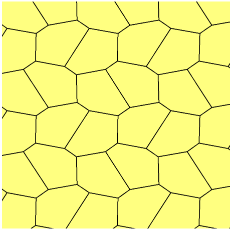
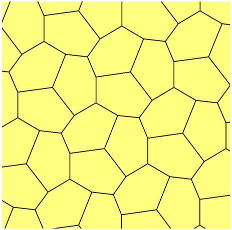

- Open question : Pentagons ?

Pentagons: history

From 1918 to 2015, 15 types of convex pentagons tiling the plane where found:

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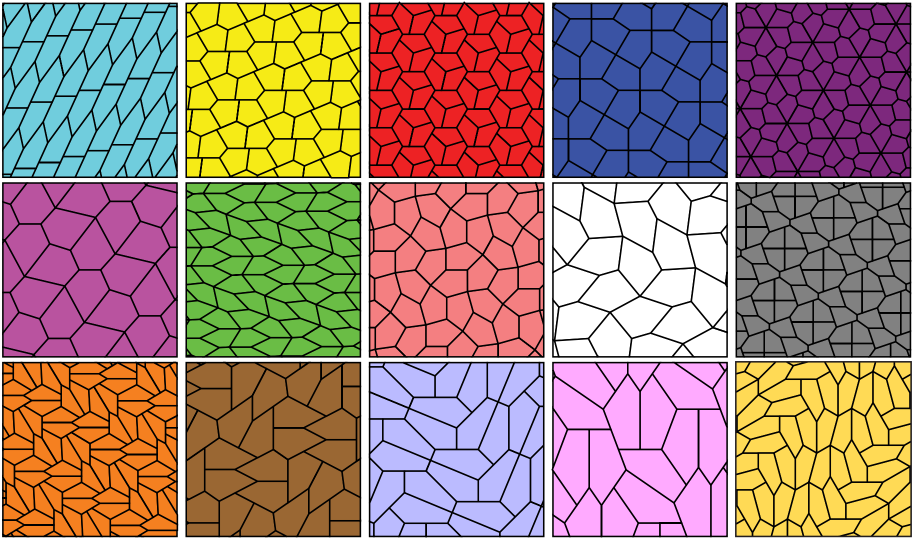
- Mann, McLoud & Von Derau (2015): Type 15.



(Wikipedia)

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- a set C_a of linear conditions on angles (form: $v \cdot \alpha = 2\pi$ with $v \in \mathbb{N}^5$)
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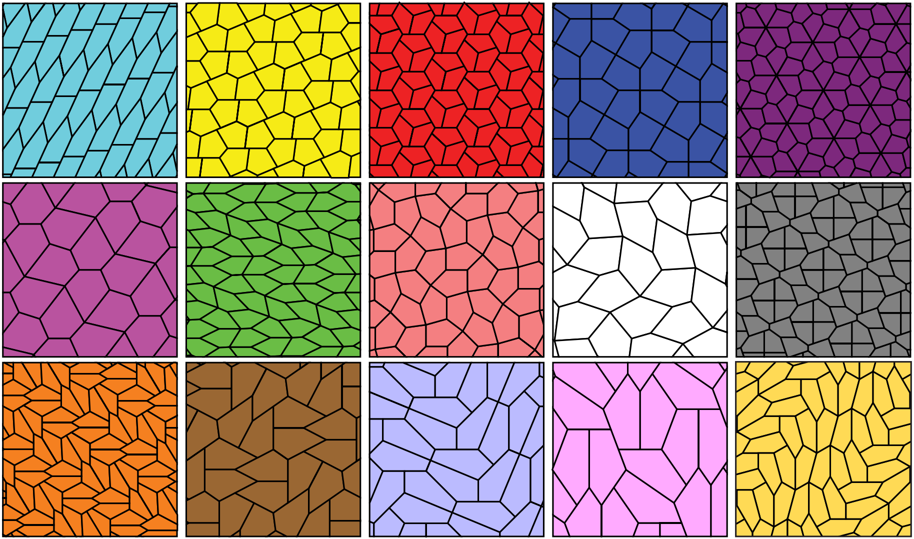
Examples:

- Type 1: $\alpha_1 + \alpha_2 = \pi$
- Type 2: $\alpha_1 + \alpha_3 = \pi$ and $\ell_1 = \ell_3$
- Type 4: $\alpha_3 = \alpha_5 = \pi/2$, $\ell_2 = \ell_3$ and $\ell_4 = \ell_5$
- ...



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Sketch

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Let $\mathcal{P}$ be a convex pentagon which tiles the plane.

- Part 1: There exist a tiling by $\mathcal{P}$ such that each vertex category has positive density
- The set of vertex category (i.e. conditions implied by angles) must be "good"
- Part 2: There are only 371 good sets to consider
- Part 3: For each good set : we do an exhaustive search
- Result: we found only the 15 known families (and some special cases).

Part 1: positive density tiling and good sets

Let $\mathcal{P}$ be a convex pentagon

- the vertices are $s_1, \ldots s_5$, in clockwise order
- the angles are respectively $\alpha_1 \times \pi, \ldots, \alpha_5 \times \pi$

$$\forall 1 \leq i \leq 5, \quad 0 < \alpha_i < 1$$

$$\sum_{i=1}^{5} \alpha_i = (1,1,1,1,1) \cdot \alpha = 3$$

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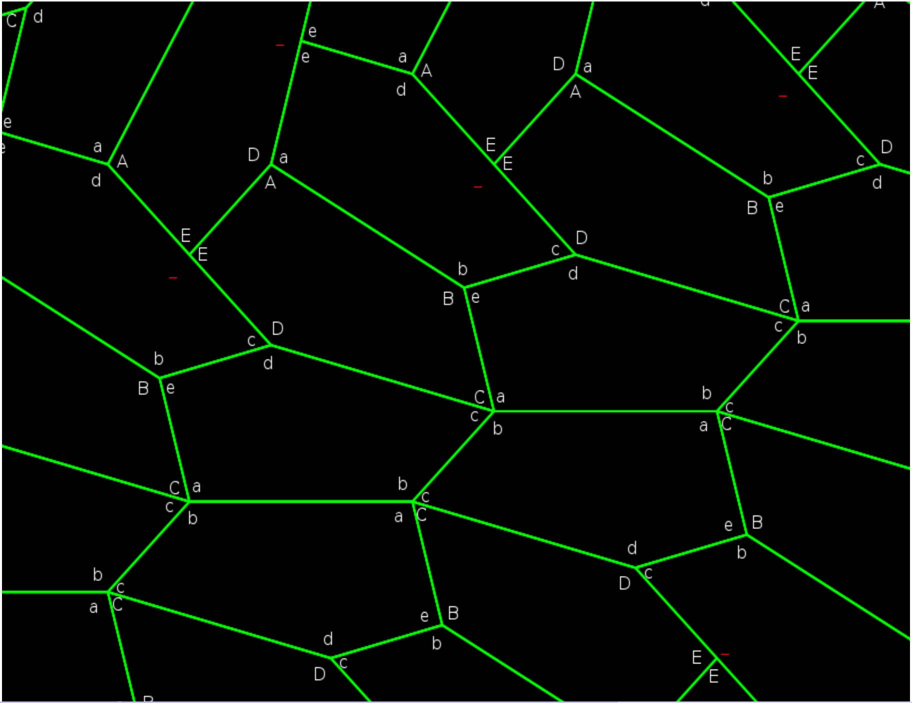
Let $\mathcal{T}$ be tiling of the plane by $\mathcal{P}$ (we allow rotation/transtation/mirror)

(Note: no hypothesis on periodicity / transitivity)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Vector category

Let s be a vertex of $\mathcal{T}$ (*i.e.* a vertex of one pentagon in $\mathcal{T}$)

The *vector category* of s, denoted $V(s)$, is the vector $v \in \mathbb{N}^5$ s.t. there are v_i angles s_i around s.

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Attention ! Two cases of vertices:

- "Half" : s is in the border of a tile P, but not a vertex of P
- "Full" : s is a vertex of every tile around s

We have to "correct" the vector category of "half" vertices.
Here, for the sake of simplicity, we do not talk about half vertices...

A toy problem

Suppose that the density of each vector category is definite.

$$\text{density}(v) = \frac{\text{number of vertices } s \text{ with } V(s) = v}{\text{number of tiles}}$$

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and $\mathcal{W}$ is the following:
$v_a = (1, 1, 1, 0, 0)$
$v_b = (0, 0, 0, 2, 2)$
$v_c = (1, 1, 0, 1, 0)$

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Can we tile only with v_a and v_b ?

Positive density tilings

Definition (Positive density tiling)

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- There are sub-tilings of an arbitrarily large disk without a vertex v (take a grid of girth x: if there is a v in every cell $\rightarrow$ contradiction)
- By compactness one can construct a tiling without v
- (warning: be careful with half vertices and "fracture lines")

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- $\{v_1, v_2\}$ is good since $2 \times u \cdot v_1 + u \cdot v_2 = 0$

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$$\sum_{v \in \mathcal{W}} (u \cdot v) \times d_v = 0$$

Contradiction since:

$$\sum_{v \in \mathcal{W}} (u \cdot v) \times d_v \geq (u \cdot v^+) \times d_{v^+} > 0$$

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Let $\mathcal{X} \subseteq \mathbb{N}^5$

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Spoil: only finitely many...

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- Moreover, one can show that this algorithm always terminates.

We suppose w.l.o.g. that:

- $1 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \alpha_4 \geq \alpha_5 \geq 0$ ($\mathfrak{P}^{\geq}_{\mathcal{X}}$ instead of $\mathfrak{P}_{\mathcal{X}}$)
- $\mathcal{X}$ is *maximal*, *i.e.* every condition implied by conditions in $\mathcal{X}$ is in $\mathcal{X}$

procedure RECURSE($\mathcal{X}$)
    $\mathcal{X} \leftarrow \text{Compat}(\mathcal{X})$ (i.e. complete $\mathcal{X}$ to make it maximal)
    if $\mathfrak{P}^{\geq}_{\mathcal{X}} \cap]0,1[^5 = \emptyset$ **then return end if**
    if $\mathcal{X}$ is good **then**
        Add $\mathcal{X}$ to the list of good sets
    end if
    Let $u \in \mathbb{R}^5$ such that:

- $u \cdot (1,1,1,1,1) = 0$
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    $V \leftarrow \{v \in \mathbb{N}^5 : v \cdot u \geq 0 \qquad \}$
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$\textsc{Recurse}$ always terminates: finitely many good sets with $\mathfrak{P}^{\geq}_{\mathcal{X}} \cap]0,1[^5 \neq \emptyset$.

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- or no convex pentagon exists with these conditions

Backtracking: general idea

The object on which we work and backtrack is a pair (G, Q):

- G is a embedded planar graph which represent the partial tiling ("Tiling graph")
- Q is a set of conditions we know on the lengths of the pentagon: *i.e.* a linear program (LP) on $\ell_1 \dots \ell_5$

We add linear conditions on sides "on the fly"

Tiling graph

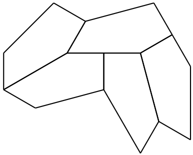
Tiling graph : embedded planar graph with labels on angles and edges

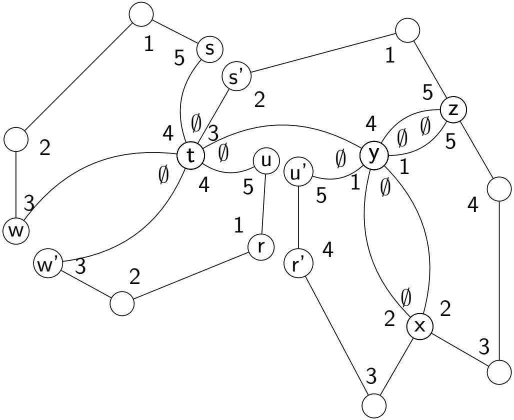
Two types of faces: usual and special

- usual: corresponds to a pentagon in the tiling. The degree is 5, and the angles are marked from 1 to 5 (in CW or CCW)
- special: corresponds to frontier between tiles, or an unknown area of the plane. Angles are marked with $\emptyset$, π or ?

A special face is *complete* if there no "?"

Tiling graph: example





Example of a tiling graph (Type 15). Unmarked angles are labeled "?"

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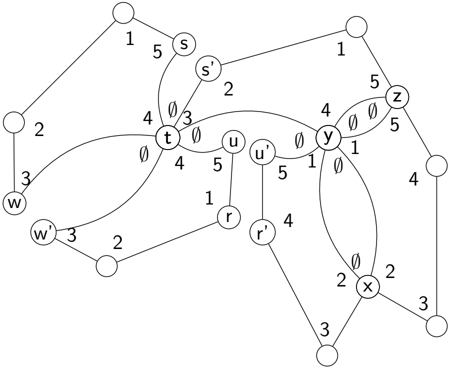
Branching on length suppositions: example



Q :

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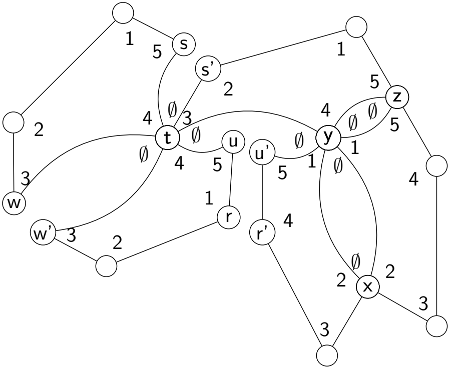

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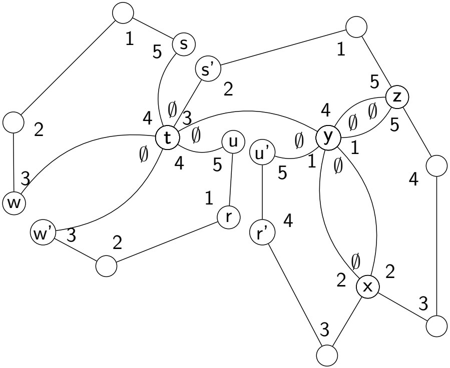

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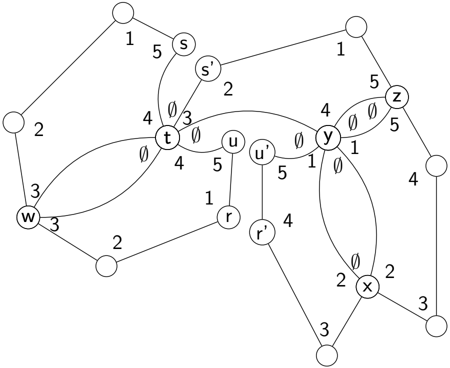

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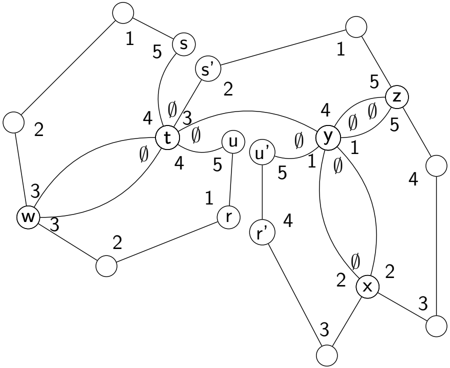

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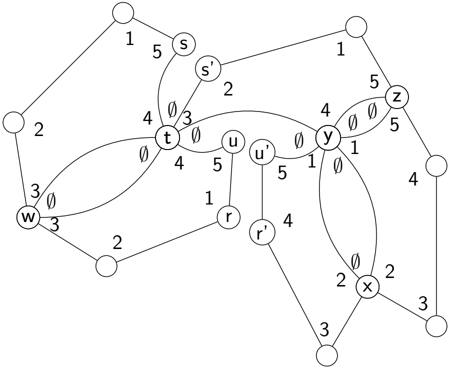

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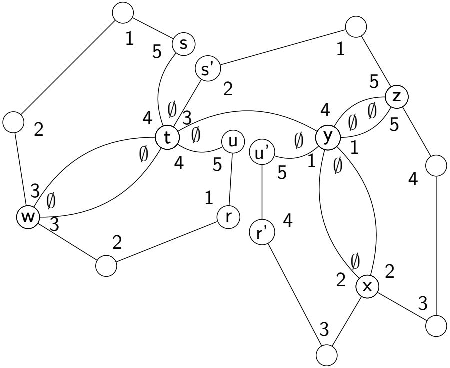

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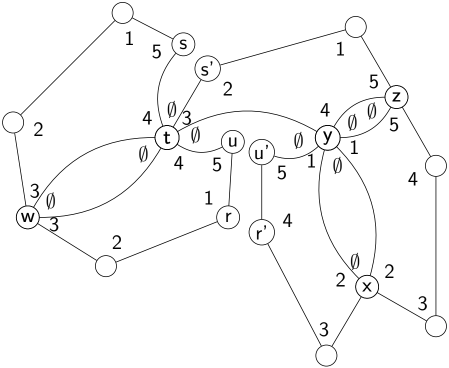

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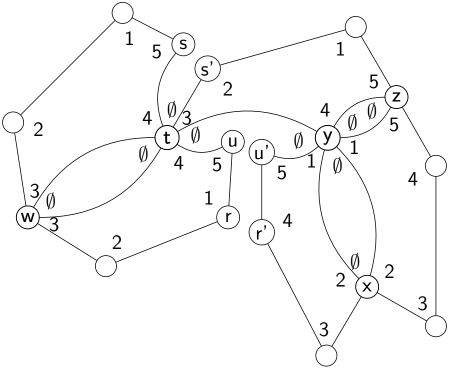

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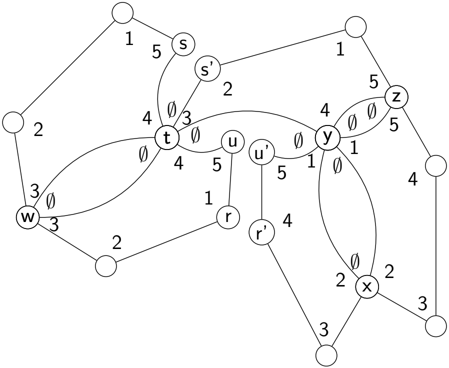

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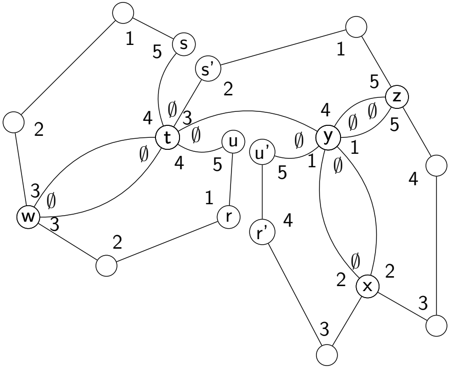

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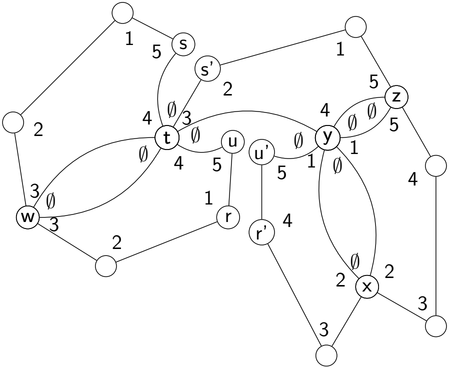

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first case : add $\ell_3 > \ell_4 + \ell_5$ to Q and branch

second case : add $\ell_3 = \ell_4 + \ell_5$ to Q and branch

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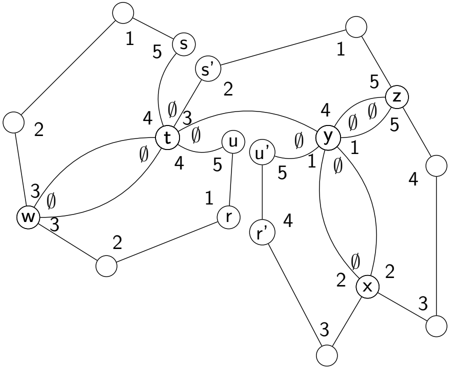

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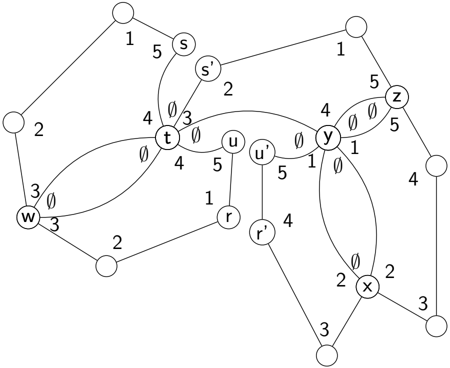

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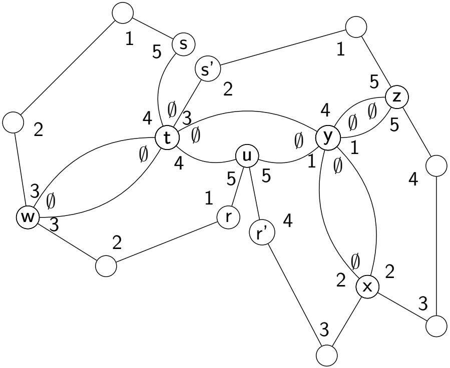

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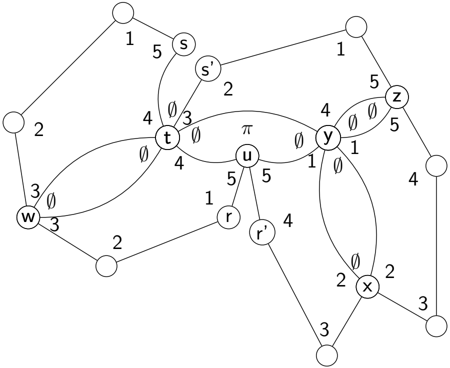

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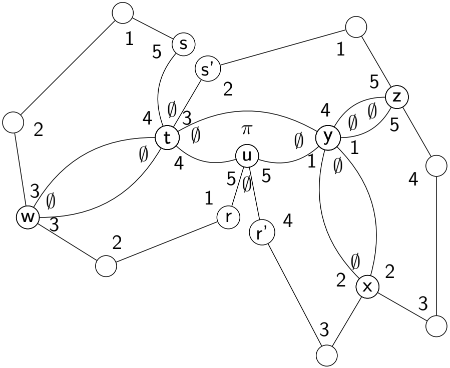

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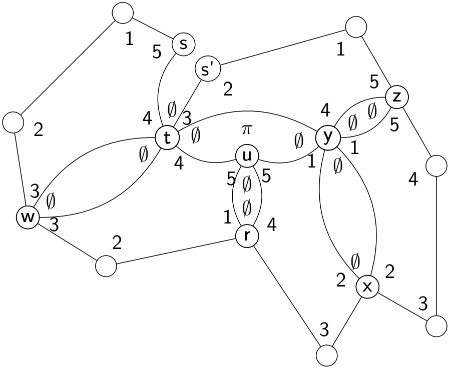

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in the run (r, u, r'), r and r' have the same position: we merge...

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We take a non-complete vertex w in the graph, and we try (branch on) every possibility to add a new face adjacent to w

Existence of the pentagon

Given the LP Q, we denote by $\mathfrak{Q}$ the set of solutions ℓ of Q with $\sum \ell = 1$
Let $s(\alpha)$ be the vector such that $s(\alpha)_i = (i-1) - \sum_{j=1}^{i-1} \alpha_i$.
One have:

$$\sum_i \ell_i \exp(s(\alpha)_i \times \pi \times \sqrt{-1}) = 0. \tag{1}$$

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We backtrack if there is no convex pentagon exists with the properties, that is if the following condition is not fulfilled:

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If $\dim(\mathfrak{P}) > 0$: we backtrack if we have a certificate (computations in $\mathbb{Q}$) that there a no solution. Problem: this cannot detect "degenerate case". So we manually add some degenerate case. (Types 20 to 24 in Table 1).

Conditions for which we backtrack (Table 1)

Type 1 (i=1)	$a+b+c=2\pi$		Type 2 (i=2)	$a+b+d=2\pi$	$C=E$
Type 3 (i=31)	$3e=2\pi$ $d+2e=2\pi$ $b+2e=2\pi$	$C+E=D$ $A=B$	Type 4 (i=6)	$a+b+d=2\pi$ $2e=\pi$	$D=E$ $B=C$
Type 5 (i=4)	$3e=2\pi$ $a+b+d=2\pi$	$D=E$ $B=C$	Type 6 (i=13)	$d+2e=2\pi$ $a+c+d=2\pi$	$C=D=E$ $A=B$
Type 7 (i=17)	$d+2e=2\pi$ $a+2c=2\pi$	$A=C=D=E$	Type 8 (i=14)	$d+2e=2\pi$ $2b+c=2\pi$	$A=B=C=D$
Type 9 (i=15)	$d+2e=2\pi$ $2a+c=2\pi$	$A=B=C=D$	Type 10 (i=69)	$2c+d=2\pi$ $b+c+e=2\pi$ $a+2b=2\pi$	$A+C=D=E$
Type 11 (i=67)	$c+2d=2\pi$ $b+d+e=2\pi$ $a+2b=2\pi$	$A=B=C+2E$	Type 12 (i=67)	$c+2d=2\pi$ $b+d+e=2\pi$ $a+2b=2\pi$	$A+C=B=2E$
Type 13 (i=63)	$b+2d=2\pi$ $a+b+d=2\pi$ $2e=\pi$	$A=2B=2C$	Type 14 (i=67)	$c+2d=2\pi$ $b+d+e=2\pi$ $a+2b=2\pi$	$A=B=2C=2E$
Type 15 (i=303)	$c+2d=2\pi$ $2b+e=2\pi$ $2a+d=2\pi$ $2e=\pi$	$B=D=E$ $C=2B$	Type 16 (i=72) $\subset$ T10	$b+c+e=2\pi$ $2b+d=2\pi$ $a+2c=2\pi$	$2A=D=E$ $A=C$
Type 17 (i=25) $\subset$ T2	$c+2e=2\pi$ $2b+d=2\pi$	$A=B=C=D=E$	Type 18 (i=73) $\subset$ T2	$d+2e=2\pi$ $c+2e=2\pi$ $b+d+e=2\pi$	$D=E$ $A=B$
Type 19 (i=23) $\subset$ T1	$c+2e=2\pi$ $b+2d=2\pi$	$A=B=C=D$	Type 20 (i=2) degen.	$a+b+d=2\pi$	$A=C+D$ $B=E$
Type 21 (i=12) degen.	$d+2e=2\pi$ $2a+b=2\pi$	$A=B$ $C=D$	Type 22 (i=27) degen.	$c+2e=2\pi$ $a+2d=2\pi$	$A=B=C=E$
Type 23 (i=64) degen.	$2b+d=2\pi$ $a+b+d=2\pi$ $2e=\pi$	$A=2C=2D$	Type 24 (i=69) degen.	$2c+d=2\pi$ $b+c+e=2\pi$ $a+2b=2\pi$	$2D=A+C$ $2E=A+C$

Part 3: Results

For every family, the exhaustive search is finite

That is: if a pentagon does not respect condition of Type i for a $i \in \{1, \ldots 24\}$, then it cannot tile the plane.

- Types 1 to 15 are the already known families
- Types 16 to 19 are special cases of known families
- Types 20 to 24 are "degenerate" ($\dim(\mathfrak{P}) > 0$): there are no convex pentagons which respects these conditions

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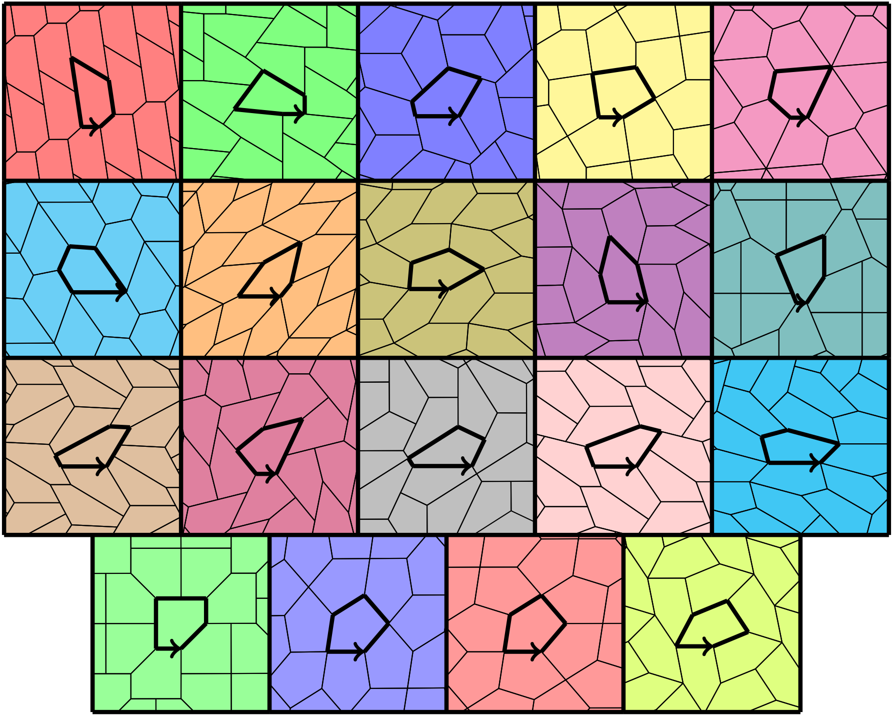
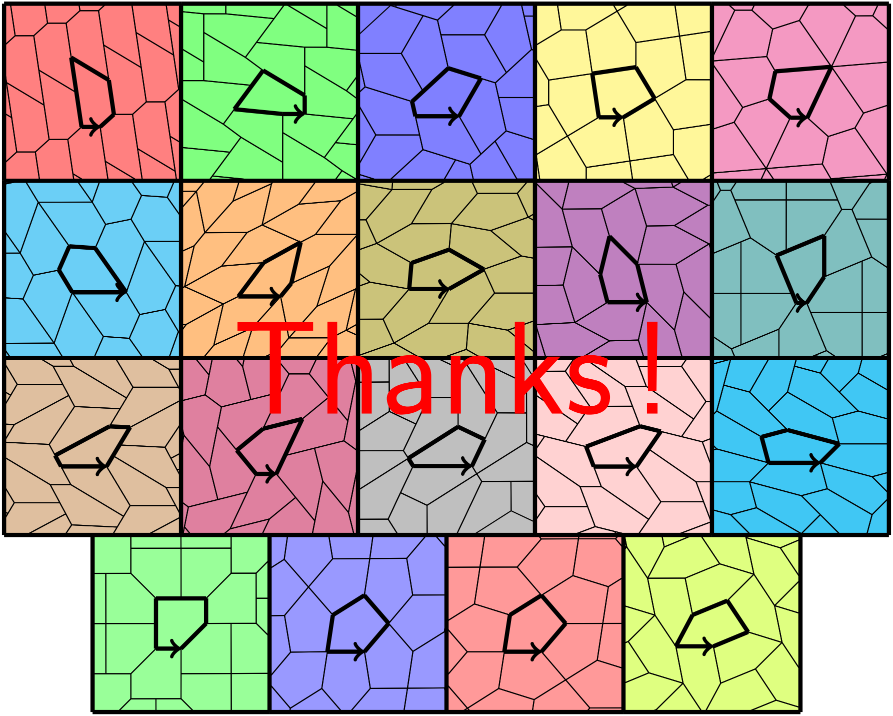
- But the techniques can be used on non convex tiles...
- and can be used on every polygon with n sides (n fixed)...
- But one have a combinatorial explosion:
- E.g.: 371 famillies for convex pentagons, and ~ 6000 families for non-convex pentagons




Thanks !