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The Quantum Ising Spin Glass in one Dimension

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Contents

1 Introduction

1.1 General Motivation

In this report, we consider a random Ising model in transverse field described by the Hamiltonian

$$H = -\sum_{<ij>} J_{ij}\sigma_i^z\sigma_j^z - \sum_i h_i\sigma_i^x \tag{1}$$

that we place on a d-dimensional lattice. Here σ_i designs the Pauli spin matrices and $< ij >$ means pairs of nearest neighbours in the d-dimensional lattice. The system is subject to some random influences: there exist site dependant magnetic fields h_i as well as random couplings J_{ij} between spins at sites i and j. The Ising spin glass corresponds to the case of couplings J_{ij} of arbitrary signs.

We face the classical Ising spin glass if all random fields are zero $\{h_i\} \equiv 0$, finding an Hamiltonian consisting of commutating operators.

One of the interesting properties of such spin glasses is that their ground state is "a frozen disordered one, rather than the kind of uniform or periodic pattern we are accustomed to finding in conventional magnets"[1].

A phenomenon often encountered with spin glasses is that of "frustation". Responsable for frustration in the above model are the negative couplings. They - in the search for the ground state of the system - lead to a competition between all the couplings "in the sense that no single configuration of the spins is uniquely favoured by all the interactions"[2]. One can check the presence of frustration in the following way: Choose some closed loop consisting of couplings between lattice sites. Count then the number of negative couplings involved. If this number is even there will be no frustration in the loop. But should this number be odd then you will encounter a conflict trying to put the spins at the intervening sites and to satisfy all the bonds, you have found a frustrated loop and one of the spins will get "frustrated".

Now appearently in the above model (1) frustration can only be found in dimension $d \geq 2$ because in the one-dimensional case one can not construct a closed loop. So in the first part of this work we will not be worrying about frustration. But then we introduce to our model (small) random couplings between next nearest neighbour couplings, too, and in the following also can find frustrated loops.

Such classical - and already very complicated - systems can be shown to have a very slow dynamics ("aging") due to high energy barriers.

Admitting now random transverse fields h_i we enter the regime of the "Quantum Ising Spin Glass", [5]. One of its additional features are appearing quantum fluctuations enabling the system to overcome energy barriers by tunneling effect.

Why should we now deal with disordered systems in *one* dimension when our real world is *three*-dimensional?

Sometimes our real world is very complex and not easy to understand. One of the materials that can be described in the terms of the Hamiltonian (1) in dimension $d = 3$ is the substance $LiHo_xY_{1-x}F_4$, [3], however its properties are far from being understood. In such cases it can be useful to examine first -

[1] [4], page 2
[2] ibid.

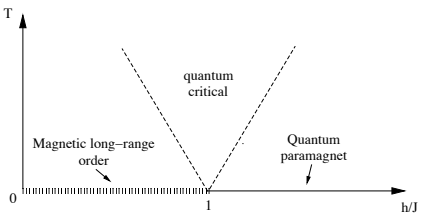

Figure 1: Schematic phase diagram of the quantum Ising spin chain.

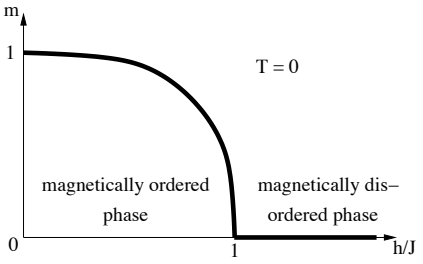

Figure 2: Schematic diagram of the magnetisation of the quantum Ising spin chain at zero temperature.

instead of the actual system - some simpler model that does not represent all "real" aspects very well. Its study can give an intuition of what is going on in reality. In spite of its appearent distance from reality sometimes this model even can share some of the properties of the true system. The examination of such models can help to develop mathematical methods that could find application in the analysis of the more complex case.

Furthermore, the magnetic system that we are going to analyse can find some reinterpretation in the general class of two-level-systems (amorphous solides at very low temperatures, noise in $1/f$).

So in this work we are going to deal with the one-dimensional case $d = 1$ of (1). In order to handle the present disorder we are going to use a renormalisation group approach in a functional space, developed in [1] and [2]. We find a new fixed point where positive and negative couplings are equally distributed. We then examine its stability and calculate some of its magnetic properties.

In the following we will shortly recall some results about the quantum Ising spin chain which is the pure, that is randomness-free model associated to our model of interest. We then explain the mentionned renormalisation procedure by reproducing the case of the nearest neighbour chain with only non negative couplings, treated by Fisher in [2]. Afterwards we are going to examine the more general case of the nearest neighbour chain with couplings of arbitrary signs and then concentrate our attention to the fixed point that we will find.

1.2 The Quantum Ising Spin Chain

The quantum Ising spin chain is the pure, randomness-free counterpart to the system of our interest (1) and for this reason we shortly recall some of its properties. It is given by the Hamiltonian

$$H = -J \sum_i \sigma_i^z \sigma_{i+1}^z - h \sum_i \sigma_i^x. \tag{2}$$

with some constants $J, h \geq 0$. The main ingredient of this model is a struggle between the spin couplings - that tend to order the system trying to set up all the spins in parallel σ^z-eigenstates - and the magnetic fields - with tendence to disorder the system by favouring spins that are in the low energy σ^x-eigenstate.

One can identify the order parameter of the system to be the magnetisation per site[3] $m :=< \sigma_i^z >$. We give a schematic phase diagram of the system in figure 1. It shows two phases, a magnetically ordered one with non vanishing magnetisation m and a magnetically disordered one with magnetisation $m = 0$. The ordered phase is uniquely realised at zero temperature $T = 0$ and for fields $h < J$.

For $h < J$ there exists a phase transition at $T = 0$ where thermal fluctuations of the system finally disappear. Furthermore at zero temperature there exists another phase transition induced by fluctuations of quantum mechanical nature. This transition is therefore called a quantum phase transition. One can go through it for example by varying the field h.

In figure 2 we also give an illustration of the magnetisation of the chain at zero temperature.

We point out that in this model the field constant h and the coupling constant J have in some sense symmetric roles: There exists a "duality transformation" from site to bond variables $\{\underline{\sigma}_i\} \rightarrow \{\underline{\tau}_i\}$ given by

$$\tau_i^x := \sigma_i^z \ \sigma_{i+1}^z, \quad \tau_i^z := \prod_{j \leq i} \sigma_j^x \tag{3}$$

that yields exactly the same system (2) but with h and J interchanged. From this observation one can - knowing that there exists a quantum phase transistion at $T = 0$ - easily determine the transistion point to be $h = J$ because the assumptions $h < J$ or $h > J$ lead - after performing the duality transformation - to a contradiction. We will retrieve this "duality" also in our disordered model (1).

2 The Renormalisation Group Approach of Dasgupta - Ma

One of the most important instruments in our examinations of the chain (1) is a renormalisation group approach developed by Dasgupta and Ma, [1], and later revived by Fisher, see e. g. [2]. They introduced a real space renormalisation scheme for concrete finite spin chains and used this one to propose a *functional* renormalisation procedure, that is a renormalisation procedure acting in a functional space, the space of the probability distributions for fields and couplings.

We explain in this section the renormalisation of a concrete spin chain and show in the next section - treating a special example - how one can then establish the functional renormalisation procedure.

The proposed renormalisation scheme is a technique designed for very inhomogenous spin chains, that means spin chains where the occuring random fields and couplings are all of very different values. Or - still expressed in another way - this corresponds to spin chains that follow very broad probability distributions for fields and couplings. This condition is crucial. We will see that it is well satisfied in the cases we are going to examine.

[3] We note $< A >$ the mean value of an observable A in the statistical ensemble.

Let us now describe the renormalisation scheme. We consider a very long but finite spin chain (1) that is some realisation of very broad probability distributions for fields and couplings. Now we are interested in the low energy behavior - the behavior at small temperatures or in small external magnetic fields, for instance - of this chain. We remark that excited states of the chain with an energy gap to the ground state much greater than the energy at which we observe the system will practically not be realised. That is why we are going to throw them out, projecting the system (1) on a low energy subspace of the original Hilbert space. In this way we reduce the degrees of freedom of our system.

In order to determine the high energy states of the chain we look for the highest field or coupling in the chain. We then have a closer look to the associated subsystem in the chain - the two strongly coupled spins or the one spin interacting with its high field - and throw out the easily findable high energy states of this subsystem. Depending on whether having found a field or coupling there are two processes to consider.

Having identified a high random field h_i we consider for a moment the associated spin not to be influenced by the rest of the chain but only by this high field. It then will be realised in one of the two eigenstates of the σ^x operator. Between these two states there is a large energy gap and if we consider the system at sufficiently small energies the eigenstate of higher energy will not occure. We throw it out and thus "freeze" this spin variable to the low energy eigenstate. We later refer to this procedure as the "decimation of a field".

Having found a coupling of high value we examine the subsystem of the two coupled spins and neglect for a short time other influences to these two spins. The two spins can then take one of the four states where every spin is in a σ^z-eigenstate and the orientation of the two spins one towards the other is either parallel or antiparallel. Depending on the sign of the found coupling either the two parallel or the two antiparallel states will be the ground states of the system. Between these two pairs of states there exists a large energy gap and if we - again - consider the system at sufficiently low energies the two excited states will not be realised. Therefore we throw them out and ban the two spins to act collectively, being always either parallel or antiparallel depending on the sign of the coupling. We call this process the "decimation of a coupling".

The interactions between the local subsystem - the two strongly bound spin variables or the one spin variable gouverned by its high transversal field - and the rest of the chain are then rewritten with the help of second order perturbation theory. This yields the actual renormalisation group transformation. Here the mentionned condition to face broad probability distributions for fields and couplings intervenes: In this case the fields and couplings in the environment of the identified highest one will be - with high probability - very different from it, that is they will be much smaller and so the perturbational treatment is justified.

Let us summarise this procedure once again: We try to establish a renormalisation scheme for spin chains. Each renormalisation step consists of two parts, these are

(i) First, one identifies the highest field or coupling in the chain. Next one determines the associated excited state(s) and throws it(them) out. Thus one restricts the space of states of the system to some low energy subspace

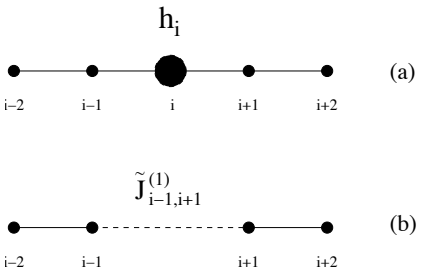

Figure 3: Decimation of a field at site i, transformation from (a) to (b). The lines represent the behavior of the couplings under the renormalisation procedure. Here a new effective nearest neighbour coupling (dashed line) between sites $i-1$ and $i+1$ is created.

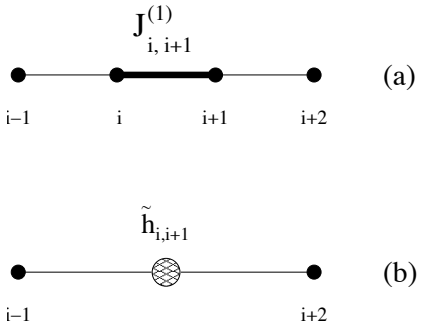

Figure 4: Decimation of a nearest neighbour coupling between sites i and $i+1$ yielding one new effective spin (drawn in a checkered pattern) in a new effective field.

of the original Hilbert space and decreases the degrees of freedom of the system.

(ii) One can then use perturbation theory to rewrite the original Hamiltonian in the low energy subspace of the space of states. After this, one can hope to be able to redefine the spin variables in the chain so that one recovers a new system described by an Hamiltonian of the form of the original one but with one spin variable less. Thus one has mapped the restriction of the Hilbert space from (i) to the elimination of some spin variable. In this way one has realised the idea of a renormalisation group transformation.

We anticipate the results of the following calculations. It turns out that the system (1) can be treated in this way: We illustrate the decimation of a field or of a coupling in figures 3 and 4. By appropriate redefinition of spin variables both decimation of a field or a coupling yield a new system that is described by a Hamiltonian of the structure of the original one (1) but with exactly one spin variable less. Spins that - resulting from a field decimation - are forced to one special state - the lower energy σ^x-eigenstate - are thrown out of the chain, constituting no degree of freedom anymore. Those spins bound strongly by the decimation of a coupling are seen to act as one new "effective spin".

We now place these ideas on firmer ground. We explain the essential but somewhat lengthy perturbational treatment in appendix A. In the next section we use the results of this perturbational calculation to explain how to establish then the functional renormalisation procedure.

3 The Random Ferromagnetic Quantum Spin Chain revisited

This case has been solved by Fisher[2]. We are going to present it in order to make clear the starting point for our future considerations and to introduce the

details of the renormalisation scheme in the functional space of the probability distributions.

3.1 Renormalisation in a functional space

We are facing the hamiltonian

$$H = -\sum_i \left(J^{(1)}_{i,i+1} \sigma^z_i \sigma^z_{i+1} + h_i \sigma^x_i \right), \qquad \{h_i\}, \{J^{(1)}_{i,i+1}\} \geq 0, \tag{4}$$

admitting - for the moment - only non negative fields and couplings. Later we show how the following considerations can be extended to the more general case. We denote in general $J^{(L)}_{i,i+L}$ a coupling between the sites i and $i+L$, so between sites of a distance L.

We are interested in finding possible phases and phase transitions of the system at low temperature

The idea is the following: We imagine to be given some probability distributions for fields and couplings and a very long chain which is a concrete realisation of these distributions. For sufficiently long chains one concrete realisation will contain (almost) the same amount of information as the probability distributions and all realisations will be equivalent for our intended purpose. We simplify that chain as proposed in the last section by decimating the highest field or coupling - using the results of the perturbational calculations of appendix A. Simultaneously we keep readjusting the associated probability distributions and thus find the functional renormalisation scheme. We explain how to find a set of differential equations that describe the renormalisation of the probability distributions and then indicate fixed points and phase transistions.

As a measure for the effectuated renormalisation steps we add a parameter $\Omega := \max_i \{h_i, J^{(1)}_{i,i+1}\}$ to the probability distributions, it yields the value of the highest left field or coupling and decreases during the renormalisation procedure.

The spin chain at renormalisation step Ω is described by the number of remaining spins (its length) $\mathcal{N}(\Omega)$ as well as $\mathcal{N}^{(h)}(h,\Omega)$, $\mathcal{N}^{(1)}(J^{(1)},\Omega)$ which are the number of random fields of value h and the number of nearest neighbour couplings $J^{(1)}$, respectively, present in the chain. The probability distributions for fields and couplings are then in the case of very long chains given by

$$\tilde{\mathcal{P}}^{(h)}(h,\Omega)\ \mathrm{d}h = \frac{\mathcal{N}^{(h)}(h,\Omega)}{\mathcal{N}(\Omega)}, \qquad \tilde{\mathcal{P}}^{(1)}(J,\Omega)\ \mathrm{d}J = \frac{\mathcal{N}^{(1)}(J,\Omega)}{\mathcal{N}(\Omega)}, \tag{5}$$

respectively.

Now we consider one renormalisation step and decimate the highest energy fields and couplings, that is all fields and couplings $J^{(1)}_i, h_i \in [\Omega - \mathrm{d}\Omega, \Omega]$, with some small $\mathrm{d}\Omega$. This shortens the spin chain and yields its new length

$$\mathcal{N}(\Omega - \mathrm{d}\Omega) = \mathcal{N}(\Omega) - \mathrm{d}\Omega\ \mathcal{N}(\Omega) \left[\tilde{\mathcal{P}}^{(h)}(h = \Omega, \Omega) + \tilde{\mathcal{P}}^{(1)}(J = \Omega, \Omega) \right] \tag{6}$$

The balance equations for $\mathcal{N}^{(h)}$ and $\mathcal{N}^{(1)}$ are found as follows: We specialise the perturbative treatment of appendix A to the current case (4) and see that in fact by the variable redefinitions as given in this appendix section we can establish a renormalisation group scheme. We find that decimation of a field

at site i or of a coupling between sites i and $i+1$ results in throwing out one spin and taking care of interactions between the decimated system parts and the rest of the system in writing effective couplings or fields

$$\tilde{J}^{(1)}_{i-1,i+1} = \frac{J^{(1)}_{i-1,i} J^{(1)}_{i,i+1}}{h_i}, \qquad \tilde{h}_{i,i+1} = \frac{h_i h_{i+1}}{J^{(1)}_{i,i+1}}, \tag{7}$$

respectively[4]. We have illustrated already the decimation procedure in figures 3 and 4. So the decimation of a field h_i at site i destructs the couplings $J^{(1)}_{i-1,i}$, $J^{(1)}_{i,i+1}$ and constructs out of them the new coupling $\tilde{J}^{(1)}_{i-1,i+1} = \frac{J^{(1)}_{i-1,i} J^{(1)}_{i,i+1}}{h_i}$. On the other hand the decimation of a coupling $J^{(1)}_{i,i+1}$ between sites i, $i+1$ destructs the two magnetic fields associated to these sites, h_i and h_{i+1}, and creates out of them the new effective field $\tilde{h}_{i,i+1} = \frac{h_i h_{i+1}}{J^{(1)}_{i,i+1}}$ to the new effective spin.

We find

$$\begin{aligned}
\mathcal{N}^{(h)}(h, \Omega - \mathrm{d}\Omega) &= \mathcal{N}^{(h)}(h, \Omega) + \mathrm{d}\Omega\, \mathcal{N}^{(1)}(J = \Omega, \Omega) \;\times \\
&\times \int_0^\Omega \mathrm{d}h_1\; \mathrm{d}h_2\; \tilde{\mathcal{P}}^{(h)}(h_1, \Omega)\; \tilde{\mathcal{P}}^{(h)}(h_2, \Omega)\; \left[\delta(h - \frac{h_1 h_2}{\Omega}) - \delta(h - h_1) - \delta(h - h_2)\right] \\
\mathcal{N}^{(1)}(J, \Omega - \mathrm{d}\Omega) &= \mathcal{N}^{(1)}(J, \Omega) + \mathrm{d}\Omega\, \mathcal{N}^{(h)}(h = \Omega, \Omega) \;\times \\
&\times \int_0^\Omega \mathrm{d}J_1\; \mathrm{d}J_2\; \tilde{\mathcal{P}}^{(1)}(J_1, \Omega)\; \tilde{\mathcal{P}}^{(1)}(J_2, \Omega)\; \left[\delta(J - \frac{J_1 J_2}{\Omega}) - \delta(J - J_1) - \delta(J - J_2)\right]
\end{aligned} \tag{8}$$

where the first occuring delta distribution in each equation takes account of the creation of a new field or coupling and the following two represent the destruction of the two original fields or couplings.

With the division of these balance equations by (6) and taking the limit $\mathrm{d}\Omega \to 0$ our considerations result in a set of two coupled differential equations for the two probabiliy distributions. Its solutions for given initial distributions describe the evolution of the probability distributions during the renormalisation procedure.

Before writing down that system we perform a variable transformation proposed by Fisher, [2], which is more adapted to the problem

$$\begin{aligned}
(h, \Omega) \quad &\to \quad \left(\beta := \ln \frac{\Omega}{h},\; \Gamma := \ln \frac{\Omega_0}{\Omega}\right), \\
(J, \Omega) \quad &\to \quad \left(\zeta := \ln \frac{\Omega}{J},\; \Gamma := \ln \frac{\Omega_0}{\Omega}\right),
\end{aligned} \tag{9}$$

with Ω_0 being the value of the highest initial field or coupling, thus the starting value for the renormalisation step parameter Ω. The logarithmic variables introduced make the relations (7) take a simple form. Remark that high fields or couplings now correspond to small β, ζ and we always have $\beta, \zeta \geq 0$.

[4]In these equations - and also in all following relations - we see that fields and couplings have perfectly symmetric roles. We refind the "duality" already encountered at the pure model, section 1.2. The explanation is that also the Hamiltonian (1) is transformed to itself by the duality transformation (3), but with the h's and J's interchanged.

The probability distributions in the new variables are derived from (8) by $\tilde{\mathcal{P}}^{(h)}(h,\Omega) =: \mathcal{P}^{(h)}(\beta,\Gamma)\ |\frac{\partial\beta(h,\Omega)}{\partial h}|$ and $\tilde{\mathcal{P}}^{(1)}(J,\Omega) =: \mathcal{P}^{(1)}(\zeta,\Gamma)\ |\frac{\partial\zeta(J,\Omega)}{\partial J}|$ so that we finally find

$$\begin{aligned}
\frac{\partial P^{(h)}(\beta,\Gamma)}{\partial\Gamma} &= \frac{\partial P^{(h)}(\beta,\Gamma)}{\partial\beta} + P^{(h)}(\beta,\Gamma)\left(P^{(h)}(0,\Gamma) - P^{(1)}(0,\Gamma)\right) + \\
&\quad + P^{(1)}(0,\Gamma)\int_0^\infty \mathrm{d}\beta_1\ \mathrm{d}\beta_2\ P^{(h)}(\beta_1,\Gamma)\ P^{(h)}(\beta_2,\Gamma)\ \delta(\beta-\beta_1-\beta_2), \\
\frac{\partial P^{(1)}(\zeta,\Gamma)}{\partial\Gamma} &= \frac{\partial P^{(1)}(\zeta,\Gamma)}{\partial\zeta} + P^{(1)}(\zeta,\Gamma)\left(P^{(1)}(0,\Gamma) - P^{(h)}(0,\Gamma)\right) + \\
&\quad + P^{(h)}(0,\Gamma)\int_0^\infty \mathrm{d}\zeta_1\ \mathrm{d}\zeta_2\ P^{(1)}(\zeta_1,\Gamma)\ P^{(1)}(\zeta_2,\Gamma)\ \delta(\zeta-\zeta_1-\zeta_2).
\end{aligned} \tag{10}$$

Note that the transformation of variables (9) lets enter a derivative with respect to β or ζ and so we now face a set of two coupled partial integro-differential equations that describe the renormalisation flow in the functional space of the probability distributions.

3.2 Fixed point distributions

Fisher studied these flow equations (10) and found that almost all initial distributions evolve towards one of two possible scenarios: either most of the random fields will become smaller than all coupling constants - this corresponds to an ordered case - or, inversely, the random fields will be dominant - corresponding to a disordered case. Furthermore, it exists exactly one physical fixed point in the probability distributions - related to the quantum phase transition between these two phases. At that fixed point fields and couplings have the same importance and because of the duality one can anticipate that the associated probability distributions will take exactly the same form. We define

$$\mathcal{P}_F(\alpha,\Gamma) := \frac{e^{-\alpha/\Gamma}}{\Gamma} \tag{11}$$

and then the fixed point is given by

$$\left(\mathcal{P}^{(h)}(\beta,\Gamma) = \mathcal{P}_F(\beta,\Gamma),\ \mathcal{P}^{(1)}(\zeta,\Gamma) = \mathcal{P}_F(\zeta,\Gamma),\quad \begin{array}{l}\text{no couplings between}\\ \text{sites of distances} \geq 2\end{array}\right). \tag{12}$$

The found fixed point probability distribution is extremely broad. In fact in the course of the renormalisation - with Γ increasing - it broadens even indefinitely. That is why Fisher called that fixed point an "Infinite Disorder Fixed Point". The extreme broadness of the fixed point probability distributions justifies a posteriori the effected perturbative treatment, at least near the fixed point: Having identified at some renormalisation step the highest energy field or coupling in the chain it is overwhelmingly probable that all other interactions of the concerned spins with their environment - via couplings and fields - are much smaller.

We can rewrite the fixed point (12) explicitely independent of Γ by passing to the correct scaling variables via

$$(\beta,\Gamma) \to (\theta := \frac{\beta}{\Gamma},\Gamma),\ (\zeta,\Gamma) \to (\eta := \frac{\zeta}{\Gamma},\Gamma). \tag{13}$$

The new probability distributions $p^{(h)}(\theta,\Gamma)$, $p^{(1)}(\eta,\Gamma)$ for fields and couplings in these variables let (12) take the form

$$\left(p^{(h)}(\theta,\Gamma)=e^{-\theta},\ p^{(1)}(\eta,\Gamma)=e^{-\eta},\quad \begin{array}{l}\text{no couplings between}\\ \text{sites of distances}\geq 2\end{array}\right). \tag{14}$$

Let us finish in commenting on the stability of the fixed point with respect to perturbations to field and nearest neighbour distributions. Linearising the flow equations (10) around the fixed point (12) Fisher found that there are only exactly two physical eigenperturbations. One of them is irrelevant, corresponding to deforming in a certain manner the two fixed point distributions (12) pointwise symmetrically. The other one is relevant and corresponds to a certain deformation of the fixed point distributions pointwise antisymmetrically, thus favouring one of the two parties - fields and couplings - in their struggle and destroying the equilibrium of forces. Every generic perturbation can then be rewritten as a projection to these two eigenperturbations and a fast decaying remainder.

3.3 Susceptibility

We are now interested in the magnetic properties of the spin chain exactly at the fixed point. Therefore we switch on a small magnetic field h yielding the system

$$H=-\sum_i\left(J^{(1)}_{i,i+1}\sigma^z_i\sigma^z_{i+1}+h_i\sigma^x_i+h\mu_i\sigma^z_i\right),\qquad \{h_i\},\{J^{(1)}_{i,i+1}\}\geq 0. \tag{15}$$

We associate magnetic moments μ_i with each site and set them initially to 1. We define the magnetisation operator $M:=\sum_i\mu_i\sigma^z_i$ and the (linear) susceptibility as a function of temperature

$$\chi(T):=\left[\ \left.\frac{\partial}{\partial h}\right|_{h=0}<M>_{H(\{h_i,J^{(1)}_{i,i+1}\},\ h)}\ \right]_{av}. \tag{16}$$

Here

$$<A>_H:=\mathrm{tr}\ A\varrho(H),\quad \varrho(H):=\frac{e^H}{\mathrm{tr}\ e^H} \tag{17}$$

denotes the mean value in the statistical ensemble with respect to the Hamiltonian H and

$$\left[A(\{h_i,J^{(1)}_{i,i+1}\})\right]_{av}:=\int\left(\prod_i \mathrm{d}h_i\ \tilde{\mathcal{P}}^{(h)}(h_i,\Omega)\right)\ \left(\prod_i \mathrm{d}J^{(1)}_{i,i+1}\ \tilde{\mathcal{P}}^{(1)}(J^{(1)}_{i,i+1},\Omega)\right)\ A(\{h_i,J^{(1)}_{i,i+1}\}) \tag{18}$$

denotes the average over the random fields and couplings.

Let us calculate $<M>_{H(\{h_i,J^{(1)}_{i,i+1}\},\ h)}$ for $h\ll T$ and sufficiently small temperatures T at the fixed point and therefore - following Fisher [2] - take profit once again of the introduced renormalisation procedure. The renormalisation of an Hamiltonian of the form (15) is clear from the considerations in A.1 and

A.2. Remark that by decimating a coupling the magnetic moment of the new effective spin is the sum of the original magnetic moments $\tilde{\mu}_{i,i+1} = \mu_i + \mu_{i+1}$. The idea is that - being at the fixed point with broad probability distributions - present fields or couplings will be either much smaller or much greater than T. So we carry out renormalisation down to an energy scale $\Omega = T$, neglect thus high energy degrees of freedom and then consider the remaining effective fields and couplings in the renormalised Hamiltonian - which are much smaller than T - to be 0. So we face a system of free effective spins

$$H_{\Omega=T} \approx -h \sum_{i=1}^{\mathcal{N}(\Omega=T)} \mu_i \sigma_i^z \tag{19}$$

with known magnetisation and find

$$< M >_{H(\{h_i, J^{(1)}_{i,i+1}\},\, h)}\Big|_{h \ll T} \approx < M >_{H_{\Omega=T}} = \sum_{i=1}^{\mathcal{N}(\Omega=T)} \mu_i \, \mathrm{th}\,(\beta h \mu_i)\,. \tag{20}$$

The random nature of the problem is now present nowhere but in the distribution of the magnetic moments associated with the effective spins. We describe how to perform averaging over the random influences. Therefore we write

$$\chi(T) = \beta \sum_{i=1}^{\mathcal{N}(\Omega=T)} \mu_i^2 = \beta \, \mathcal{N}(\Omega = T) \, \overline{\mu^2}(\Omega = T) \tag{21}$$

where we introduced the average value of the squared magnetic moments $\overline{\mu^2}(\Omega)$. Then the last equality is justified for sufficiently long renormalised chains.

Solving equation (6) at the fixed point (12), so for the known fixed point probability distributions, we calculate the number of effective spins at (logarithmic) renormalisation step $\Gamma = \ln(\Omega_0/\Omega)$ as

$$\mathcal{N}(\Gamma) \propto \Gamma^{-2}. \tag{22}$$

Fisher found - in a rather long analysis of the problem and not keeping trace of magnetic moments μ_i but bond and site lengths - that

$$\bar{\mu}(\Gamma) \sim \Gamma^{\phi}, \qquad \text{with } \phi = \frac{1+\sqrt{5}}{2}. \tag{23}$$

Supposing the probability distribution of the magnetic moments to satisfy a scaling form

$$p(\mu, \Gamma) \sim f(\frac{\mu}{\Gamma^{\phi}}), \qquad \mu \in \mathbb{N} \tag{24}$$

with some function f we get then

$$\overline{\mu^2}(\Gamma) = \sum_{\mu \in \mathbb{N}} \mu^2 \; p(\mu, \Gamma) \sim \Gamma^{2\phi} \sum_{\mu \in \mathbb{N}} \left(\frac{\mu}{\Gamma^{\phi}}\right)^2 f(\frac{\mu}{\Gamma^{\phi}}) \;\propto\; \Gamma^{2\phi} \quad . \tag{25}$$

Thus we finally find from equation (21) that

$$\chi(T) \sim \frac{\Gamma^{2\phi-2}}{\beta} \sim \frac{|\ln T|^{2\phi-2}}{T} \quad . \tag{26}$$

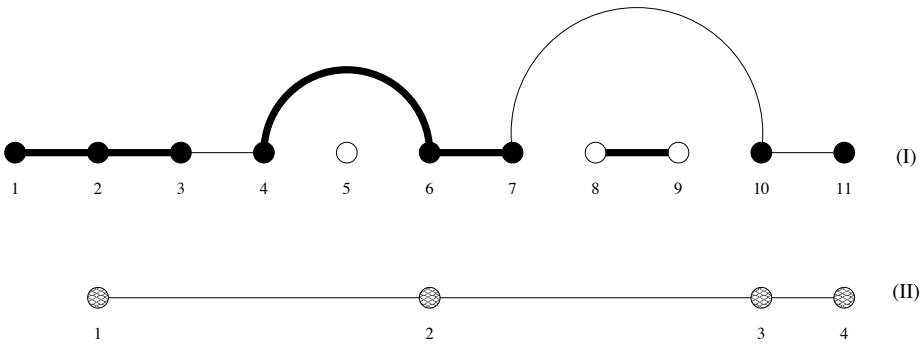

Figure 5: Repeated application of the renormalisation group transformation following the two steps on page 4. Illustration for a chain of initial length 11 and 7 RG-transformations. In (II) we show the result after having performed always the complete RG-transformation while in (I) we renounced to perform step (ii). - Filled circles symbolise active spins, unfilled are non active spins. Bold lines represent decimated couplings that hence bound spins strongly, normal lines stand for not yet decimated couplings. Circles in a checkered pattern mean effective spins.

We still want to think about why the renormalisation scheme is compatible with the calculation of magnetic properties of the chain.

We therefore have again a look at the renormalisation scheme, see points (i) and (ii) on page 4. The result of step (i) is to throw high energy configurations out of the space of states of the chain. Point (ii) has rather been crucial to realise then the idea of a renormalisation scheme by a redefinition of spin variables. We consider some spin chain in its initial configuration and are interested in its magnetisation $< M >_{H(\{h_i, J^{(1)}_{i,i+1}\},\ h)}$ in an external field $h \ll T$ for sufficiently small temperatures T.

The essential remark is the following: The trace intervening in the calculation of statistical mean values, (17), can be taken over every basis of the Hilbert space, for example $B := \{\otimes_i \{\uparrow_i, \downarrow_i\}\}$. However, vectors containing high energy configurations of the chain will not contribute much. The repeated application of the renormalisation group (RG) transformation can now be seen to do nothing else but to succesively restrict the space of states of the chain to some subspace spanned by vectors significantly contributing to mean values in the statistical ensemble, we explicitly renounce to perform step (ii) here. We illustrate this iterative process in figure 5.

Let us introduce some notions in the context of this "incomplete RG transformation"(that forgets step (ii)): We call a set of spins that are forced to act collectively a "cluster". We call a spin "active" if it has not yet been banned to the state $| \rightarrow >:= \frac{|\uparrow> + |\downarrow>}{\sqrt{2}}$. In the course of the decimation procedure we will encounter active and non active clusters, e.g. clusters of active and non active spins, respectively.

In this way we successively restrict B to vectors where field-dominated sites are in the ground state $|\rightarrow>$ and coupling-dominated spins are forced to be parallel and act collectively[5]. We throw out the high energy states associated to all fields and couplings present in the chain that are greater than the temperature T of interest and then consider all still present fields and couplings to be 0 - taking profit once again of the fact to face very broad probability distributions. Then we perform the "trace" in the calculation of statistical mean values only by involving the vectors remaining in B.

So we find a non interacting randomness free system of active and non active clusters of collectively acting spins.

The statistical mean value of the magnetisation $M = \sum_i \sigma_i^z$ of such a system is the sum of statistical mean magnetisations in the non interacting subsystems. We enumerate active and non active clusters by capital letters and define C_I as the set of spin sites belonging to cluster I.

The statistical mean magnetisation of an active cluster I of μ_I spins is easily calculated: The cluster can exist only in the two states where either all spins are up or down and it is thus

$$< M_I >:= \frac{\sum_{|\varphi>} < \varphi| \sum_{i\in C_I} \sigma_i^z e^{\beta h \sum_{i\in C_I} \sigma_i^z} |\varphi >}{\sum_{|\varphi>} < \varphi| e^{\beta h \sum_{i\in C_I} \sigma_i^z} |\varphi >} = \mu_I \ \mathrm{th}(\mu_I \beta h), \qquad (27)$$

so its contribution to susceptibility is $\chi_I = \frac{\partial}{\partial h}\big|_{h=0} \ < M_I >= \beta \mu_I^2$.

On the other hand non active clusters will not contribute to susceptibility. This is not that evident and we show it for the case of a non active cluster of one spin (what can easily be extended to the case of μ_I spins). Let us consider a spin i that has been frozen to the state $|\rightarrow>$ at renormalisation step $\omega \gg T$ and is thus described now by the Hamiltonian $H = -h\sigma^z - \omega\sigma^x$. Then the corresponding statistical mean magnetisation given by $< M_i >= \frac{\mathrm{tr}\ \sigma^z e^{-\beta H}}{\mathrm{tr}\ e^{-\beta H}}$ can be simply evaluated after performing a rotation of the system around the y-axis by an angle φ with $\cos\varphi = \frac{\beta h}{\sqrt{(\beta h)^2+(\beta\omega)^2}}$, $\sin\varphi = \frac{\beta\omega}{\sqrt{(\beta h)^2+(\beta\omega)^2}}$ yielding new spin variables $\underline{\tilde{\sigma}}$. It is

$$\begin{aligned} < M_i > &= \frac{\mathrm{tr}\ (-\sin\varphi\ \tilde{\sigma}^x + \cos\varphi\ \tilde{\sigma}^z) e^{\sqrt{(\beta h)^2+(\beta\omega)^2}\ \tilde{\sigma}^z}}{\mathrm{tr}\ e^{\sqrt{(\beta h)^2+(\beta\omega)^2}\ \tilde{\sigma}^z}} = \\ &= \frac{\mathrm{th}\sqrt{(\beta h)^2+(\beta\omega)^2}}{\sqrt{h^2+\omega^2}}. \end{aligned} \qquad (28)$$

Remark that $\lim_{\omega\to\infty} < M_i >= 0 \ \forall\ h$, so there is no magnetisation in the case $h \ll T \ll \omega$ that we consider and non active spins do not contribute to magnetisation. (By the way it is $\chi_i = \frac{\partial}{\partial h}\big|_{h=0} \ < M_i >= 0$ even for all finite $\omega \neq 0$, so the (linear) susceptibility vanishes even in finite cases).

Our considerations yield now the magnetisation

$$< M >_{H(\{h_i, J^{(1)}_{i,i+1}\},\ h)}\Big|_{h\ll T} \approx \sum_{\substack{\text{Active clusters } I \text{ at} \\ \text{temperature } T}} \mu_I \ \mathrm{th}\,(\beta h \mu_I)\,. \qquad (29)$$

Now remark that an active cluster of μ_I spins corresponds exactly to an "effective spin" of a magnetic moment μ_I in the terminology of the renormalisation scheme, as well as non active spins correspond to spins that have been

[5]Note that the vectors constructed in this manner are in general *not* eigenvectors of H.

directly thrown out by a field decimation during the renormalisation. So we can finally justify the calculation of magnetic properties in the renormalisation picture: Spins whose field has been decimated do not need to be considered. Active clusters and effective spins contribute in the same way, compare (29) and (20).

4 The Quantum Spin Glass Chain

Let us consider once again the spin chain of the last section, given by (4),

$$H = -\sum_i \left(J^{(1)}_{i,i+1} \sigma^z_i \sigma^z_{i+1} + h_i \sigma^x_i \right), \tag{30}$$

and this time not restrict us to non negative couplings.

We first describe how to overtake easily some of the results of the last section to this more general case. Nevertheless we then perform explicitely the renormalisation for this new case and will be able to characterise this system still more in detail.

4.1 Gauge transformation

So far we restricted the discussion to non negative fields and couplings. Let us now regard the case of a chain of fields and couplings of arbitrary signs and remark that we can always unitarily transform this system to the previous random ferromagnetic chain (4). The unitary operator

$$\mathcal{R}_l := e^{\frac{i}{2}\pi\sigma^x_l} = i\sigma^x_l \tag{31}$$

effects a rotation in the Hilbert space, transforming operators by $\tilde{\mathcal{O}} := \mathcal{R}_i^{-1}\, \mathcal{O}\, \mathcal{R}_i$, so that spin variables at site i become $\tilde{\sigma}^x_i = \sigma^x_i,\ \tilde{\sigma}^z_i = -\sigma^z_i$.

Given a special chain with positive and negative couplings $\{J^{(1)}_{i,i+1}\}$ we construct the set $A := \{i \mid J^{(1)}_{i-1,i} < 0\}$ containing the ending positions of the negative couplings in the chain. Then the operator

$$U := \prod_{l\in A} \left(\prod_{k\geq l} \mathcal{R}_k \right) \tag{32}$$

is a unitary transformation of the given chain to one with non negative couplings. Note that the transformation rotates for every negative coupling *all* spins on the right side of it. Let us still observe that there exists a unique sequence (ε_i) of values in $\{0,1\}$ which is determined by

$$U = \prod_l e^{\frac{i}{2}\pi\varepsilon_l\sigma^x_l}, \tag{33}$$

that is in terms of which U can be expressed particularly simply as a product of identity operators ($\varepsilon_l = 0$) and rotations ($\varepsilon_l = 1$) on the sites $\{l\}$. We find explicitely

$$\varepsilon_i = \begin{cases} 0 & \text{number of elements in } \{l \in A \mid l \leq i\} \text{ is even} \\ 1 & \text{number of elements in } \{l \in A \mid l \leq i\} \text{ is odd} \end{cases}, \tag{34}$$

so the spin variable at the site i will be rotated by the transformation if there is an odd number of negative couplings in front of it - and remain unchanged in the other case.

Analogously, by appropriate rotations around the z-axis we can always withdraw to the case $\{h_i\} \geq 0$.

Now let us take profit of this gauge transformation in the examination of the spin glass chain (30). First - because of the unitary character of this gauge transformation - we remark that we can associate to every spin glass chain a ferromagnetic chain with the same energy spectrum. Furthermore, in applying first these transformations we can then perform the renormalisation procedure of subsection 3.1 for the new, more general case (30), recover the fixed point (12) and get at least some information about the renormalisation of the absolute value of couplings.

4.2 Renormalisation of the Spin Glass Chain

With the help of the gauge transformation (32) we know already much about the system (30). However some of the properties of interest - e.g. magnetic properties or the behavior of the separated probability distributions of positive and negative couplings - are somewhat distorted by the gauge transformation. That is why we now carry out the renormalisation for the more general case (30). We withdraw to the case $\{h_i\} \geq 0$.

As in section 3 we use the described renormalisation procedure to establish differential flow equations for the probability distributions of fields and couplings, the essential points remain the same. Perturbation theory tells us that decimations of fields and couplings are still gouverned by (7). The only difference is that we have to account for negative couplings. We thus count couplings by $\mathcal{N}^{(1)}(J,\Omega)$ and note the probability distribution for couplings $\tilde{\mathcal{P}}^{(1)}(J,\Omega)$ where now $J \in [-\Omega,\Omega]$. Furthermore, also negative couplings $J^{(1)}_{i,i+1} < 0$ will be subject to decimation at renormalisation step Ω if $|J^{(1)}_{i,i+1}| \in [\Omega - \mathrm{d}\Omega, \Omega]$.

Thus, the length of the spin chain, equation (6) in the last section, behaves as

$$\begin{aligned} \mathcal{N}(\Omega - \mathrm{d}\Omega) &= \mathcal{N}(\Omega) - \mathrm{d}\Omega\, \mathcal{N}(\Omega)\Big[\tilde{\mathcal{P}}^{(h)}(h=\Omega,\Omega) + \\ &\quad + \tilde{\mathcal{P}}^{(1)}(J=\Omega,\Omega) + \tilde{\mathcal{P}}^{(1)}(J=-\Omega,\Omega)\Big]\,, \end{aligned} \tag{35}$$

and the balance equations for the number of fields and couplings, equations (8) before, now take the form

$$\begin{aligned} \mathcal{N}^{(h)}(h,\Omega-\mathrm{d}\Omega) &= \mathcal{N}^{(h)}(h,\Omega) + \mathrm{d}\Omega\, \Big[\mathcal{N}^{(1)}(J=\Omega,\Omega) + \mathcal{N}^{(1)}(J=-\Omega,\Omega)\Big] \times \\ &\quad \times \int_0^\Omega \mathrm{d}h_1\, \mathrm{d}h_2\, \tilde{\mathcal{P}}^{(h)}(h_1,\Omega)\, \tilde{\mathcal{P}}^{(h)}(h_2,\Omega)\, \Big[\delta(h - \frac{h_1 h_2}{\Omega}) - \delta(h-h_1) - \delta(h-h_2)\Big] \\ \mathcal{N}^{(1)}(J,\Omega-\mathrm{d}\Omega) &= \mathcal{N}^{(1)}(J,\Omega) + \mathrm{d}\Omega\, \mathcal{N}^{(h)}(h=\Omega,\Omega) \times \\ &\quad \times \int_{-\Omega}^\Omega \mathrm{d}J_1\, \mathrm{d}J_2\, \tilde{\mathcal{P}}^{(1)}(J_1,\Omega)\, \tilde{\mathcal{P}}^{(1)}(J_2,\Omega)\, \Big[\delta(J - \frac{J_1 J_2}{\Omega}) - \delta(J-J_1) - \delta(J-J_2)\Big]. \end{aligned} \tag{36}$$

In order to pass to logarithmic variables, (9), we split up $\mathcal{N}^{(1)}(J,\Omega)$ where $J \in [-\Omega,\Omega]$ into balances for positive and negative couplings, $\mathcal{N}^{(1+)}(J,\Omega)$ and $\mathcal{N}^{(1-)}(J,\Omega)$, respectively, where $J \in [0,\Omega]$. We therefore define

$$\mathcal{N}^{(1)}(J,\Omega) =: \Theta(J)\,\mathcal{N}^{(1+)}(J,\Omega) + \Theta(-J)\,\mathcal{N}^{(1-)}(-J,\Omega) \tag{37}$$

with the Heaviside step function $\Theta(x)$. Thus we get separate balance equations for positive and negative couplings. Next we define the probability distributions for positive and negative couplings by $\tilde{\mathcal{P}}^{(i)}(J,\Omega)\ \mathrm{d}J := \mathcal{N}^{(i)}(J,\Omega)/\mathcal{N}(\Omega),\ i \in \{1+,1-\}$ and then perform the variable transformation to logarithmic variables (9). This yields a system of 3 differential equations for the probability distributions $\mathcal{P}^{(h)}(\beta,\Gamma)$, $\mathcal{P}^{(1+)}(\zeta,\Gamma)$ and $\mathcal{P}^{(1-)}(\zeta,\Gamma)$.

As a last step, we still pass to new functions $\mathcal{P}^{(1)} := \mathcal{P}^{(1+)} + \mathcal{P}^{(1-)}$ and $\mathcal{D} := \mathcal{P}^{(1+)} - \mathcal{P}^{(1-)}$ in order to decouple the system of equations and recover - as one could have expected - for $\mathcal{P}^{(h)}$ and $\mathcal{P}^{(1)}$ the system (10) of section 3. As to $\mathcal{D}$, we find that it is determined by the same equation as $\mathcal{P}^{(1)}$,

$$\begin{aligned}\frac{\partial\mathcal{D}(\zeta,\Gamma)}{\partial\Gamma} &= \frac{\partial\mathcal{D}(\zeta,\Gamma)}{\partial\zeta} + \mathcal{D}(\zeta,\Gamma)\left(\mathcal{D}(0,\Gamma) - P^{(h)}(0,\Gamma)\right) + \\ &\quad + P^{(h)}(0,\Gamma)\int_0^\infty \mathrm{d}\zeta_1\ \mathrm{d}\zeta_2\ \mathcal{D}(\zeta_1,\Gamma)\ \mathcal{D}(\zeta_2,\Gamma)\ \delta(\zeta-\zeta_1-\zeta_2),\end{aligned} \tag{38}$$

but with the important difference that we now allow positive and negative values for $\mathcal{D}$.

4.3 Fixed point distributions

We are looking for fixed points in these three distributions $\mathcal{P}^{(h)}$, $\mathcal{P}^{(1)}$, $\mathcal{D}$. Note that Fisher found the only physical fixed point to equations (10) as (12), so in our case we have necessarily $\mathcal{P}^{(h)} = \mathcal{P}^{(1)} = \mathcal{P}_F$. It is now interesting to ask how at this point $\mathcal{P}^{(h)} = \mathcal{P}^{(1)} = \mathcal{P}_F$ positive and negative couplings can be "fixedly" distributed. The answer is given by the fixed points of (38). Without effort we discover two solutions as $\mathcal{D} = \mathcal{P}_F$ and $\mathcal{D} \equiv 0$ yielding the two fixed points

$$\left(\mathcal{P}^{(h)} = \mathcal{P}_F,\ \mathcal{P}^{(1+)} = \mathcal{P}_F,\ \mathcal{P}^{(1-)} \equiv 0,\quad \begin{array}{l}\text{no couplings between}\\ \text{sites of distances} \geq 2\end{array}\right), \tag{39}$$

which was already found in section 3, and

$$\left(\mathcal{P}^{(h)} = \mathcal{P}_F,\ \mathcal{P}^{(1+)} = \mathcal{P}^{(1-)} = \frac{1}{2}\,\mathcal{P}_F,\quad \begin{array}{l}\text{no couplings between}\\ \text{sites of distances} \geq 2\end{array}\right), \tag{40}$$

respectively[6].

This new fixed point (40) corresponds to an equal repartition of random positive and negative couplings. We loosely call it the spin glass fixed point as well as we call the first one, (39), the ferromagnetic fixed point.

We are now going to investigate the stability of these fixed points with respect to small perturbations that change the proportion of positive and negative couplings, so perturbations that are somewhat "in the direction" of $\mathcal{D}$. We keep

[6]We do not know if there exist other well-behaved fixed points.

$\mathcal{P}^{(h)} = \mathcal{P}^{(1)} = \mathcal{P}_F$ fixed and linearize (38) for some perturbation $\varepsilon(\zeta, \Gamma)$ around $\mathcal{D} = \mathcal{P}_F$ and $\mathcal{D} \equiv 0$.

In the second case, $\mathcal{D} \equiv 0$, the evolution of ε is then given by

$$(\partial_\Gamma - \partial_\zeta)\ \varepsilon(\zeta, \Gamma) = -\frac{1}{\Gamma}\ \varepsilon(\zeta, \Gamma). \tag{41}$$

This equation can be solved by introducing light cone variables $\Gamma \pm \zeta$ yielding

$$\varepsilon(\zeta, \Gamma) = \frac{f(\zeta + \Gamma)}{\Gamma} \tag{42}$$

for some arbitrary differentiable function f. We consider the evolution of the norm of ε given by $\|\varepsilon\|_\zeta(\Gamma) = \frac{1}{\Gamma} \int_0^\infty \mathrm{d}\zeta\ |f(\zeta + \Gamma)| = \frac{1}{\Gamma} \int_\Gamma^\infty \mathrm{d}\zeta\ |f(\zeta)|$. The existence of the integral for all $\Gamma \geq \Gamma_0$ is clear from the initial condition $\|\varepsilon\|_\zeta(\Gamma_0) < \infty$ and we then see that both factors of the last member decrease monotonic in Γ so that the perturbation $\varepsilon(\zeta, \Gamma_0)$ renormalises to zero.

Linearizing around the fixed point (39), $\mathcal{D} = \mathcal{P}_F$, we get

$$(\partial_\Gamma - \partial_\zeta)\ \varepsilon(\zeta, \Gamma) = \varepsilon(0, \Gamma) + \frac{2}{\Gamma} \int_0^\infty \mathrm{d}\zeta_1\ \mathrm{d}\zeta_2\ \varepsilon(\zeta_1, \Gamma)\ \mathcal{P}_F(\zeta_2, \Gamma)\ \delta(\zeta - \zeta_1 - \zeta_2). \tag{43}$$

We integrate both members over ζ in the interval $[0, \infty[$, assume naturally $\varepsilon(\zeta \to \infty, \Gamma) \equiv 0$ and set $\alpha(\Gamma) := \int_0^\infty \mathrm{d}\zeta\ \varepsilon(\zeta, \Gamma)$. Thus we find

$$(\partial_\Gamma - \frac{2}{\Gamma})\ \alpha(\Gamma) = 0 \tag{44}$$

with the solution

$$\alpha(\Gamma) = \alpha(\Gamma_0)\ \left(\frac{\Gamma}{\Gamma_0}\right)^2 \tag{45}$$

that increases monotonic in Γ and shows that during renormalisation $\varepsilon(\zeta, \Gamma)$ becomes at least in one region of couplings greater and greater, so the perturbation gets more and more important.

The results (42) and (45) show that the ferromagnetic fixed point is unstable with respect to the introduction of negative couplings while the spin glass fixed point is attractive.

We remark that the stability examinations of Fisher with respect to small perturbations to the probability distributions $\mathcal{P}^{(h)}(\beta, \Gamma)$ and $\mathcal{P}^{(1)}(\zeta, \Gamma)$ - originally carried out for the fixed point (12) - apply in this more general case, too. Thus in these "directions" in the functional space, $\mathcal{P}^{(h)}$ and $\mathcal{P}^{(1)}$, there exist exactly two physical eigenperturbations, one of them is relevant and the other one is irrelevant.

4.4 Magnetic properties of the new fixed point

Let us characterise the fixed point (40) by calculating some of its magnetic properties. Therefore we proceed in the same manner as for the chain with exclusively positive couplings, subsection 3.3, and exploit the found results - making use of the gauge transformation from subsection 4.1.

We first examine in a closer way the gauge transformation (33) for (initial) chains following the fixed point distributions (40). The concrete gauge transformation is entirely given by the sequence ε_i, (34). Now we are at a fixed point

with the same probability $= 1/2$ to meet a positive or a negative coupling. We find - considering the spin variable at some special site i - the probability for $\varepsilon_i = 0$ to be the sum of the probabilities of having an even number of negative couplings in front of it, $\sum_{l=0,\ l \text{ even}}^{i-1} \begin{pmatrix} i-1 \\ l \end{pmatrix} \left(\frac{1}{2}\right)^{i-1} = \frac{1}{2}$. So spins will be rotated by the gauge transformation with probability 1/2, what is the same distribution as that for the signs of the couplings.

The basic idea in subsection 3.3 has been to determine with the help of the renormalisation procedure high energy configurations of the chain and to neglect them in the calculation of mean values in the statistical ensemble. In fact, in this case there is not so much that changes. We treat the problem in the same way and overtake notions introduced in this subsection. We remark that the gauge transformation - because of its unitarity - lets invariant especially the matrice elements of the (original) Hamiltonian. So the negligible high energy configurations of the chain with positive and negative couplings can be identified in the following way: Interested in the behavior of the chain at temperature T, we gauge transform our system (30) to the one with exclusively non negative couplings, then apply the renormalisation procedure as described in subsection 3.3 up to the step $\Omega = T$ and thus find the only vectors contributing significantly to the trace involved in the statistical mean. We retransform these vectors with the gauge transformation to our original system and thus reencounter a system of free active and non active clusters. This time the collectively acting spins in an active cluster are not necessarily forced to be parallel but can also be - as a result of the occuring negative couplings - antiparallel.

Let us now apply this strategy and deal with the magnetisation in a small field $h \ll T$ for sufficiently small temperatures T. We combine gauge transformation and renormalisation as described above and find a system of free active and non active clusters. We again can neglect non active clusters. But we now consider a special active cluster I with n_I spins and have to rethink its contribution to magnetisation

$$< M_I >:= \frac{\sum_{|\varphi>} < \varphi| \sum_{i \in C_I} \sigma_i^z e^{\beta h \sum_{i \in C_I} \sigma_i^z} |\varphi >}{\sum_{|\varphi>} < \varphi| e^{\beta h \sum_{i \in C_I} \sigma_i^z} |\varphi >} \ . \tag{46}$$

Again the cluster can only exist in one of two states, however now some of its spins - those forced to be antiparallel instead of parallel - will compensate their contribution to magnetisation. We find from (46)

$$< M_I >= \mu_I \ \mathrm{th}(\mu_I \beta h), \tag{47}$$

where μ_I, $0 \leq \mu_I \leq n_I$, is the number of "uncompensated" spins in the cluster, so the difference between the number of spins in the cluster that are forced to be in one direction and those pointing in the opposite one. Then it follows

$$< M >= \sum_{\substack{\text{active} \\ \text{clusters } I}} < M_I >= \sum_{\substack{\text{active} \\ \text{clusters } I}} \mu_I \ \mathrm{th}(\mu_I \beta h) \tag{48}$$

as the total magnetisation.

How can we determine μ_I? We already have shown that in an active cluster spins will be up or down with probability 1/2. In order to determine the number

of uncompensated spins μ_I in a cluster of n_I spins we are facing a problem similar to a symmetric random walk of n_I steps with final position μ_I but where at the end we identify negative final positions to the positive ones, $\mu_I \geq 0$. In fact we will be interested only in the even moments of this process, for these this last detail is unimportant and we can use the known results about random walks. We denote $P_n(\mu)$ the probability that a cluster of n spins has μ uncompensated spins and denote the α-th moment of this process $\overline{\mu^\alpha}(n) := \sum_{\mu \geq 0} \mu^\alpha \; P_n(\mu)$. Then we find

$$\begin{aligned} \overline{\mu^2}(n) &:= \sum_{\mu \geq 0} \mu^2 \; P_n(\mu) \propto n \; , \\ \overline{\mu^4}(n) &:= \sum_{\mu \geq 0} \mu^4 \; P_n(\mu) \propto n^2 \; . \end{aligned} \tag{49}$$

Finally we know - from subsection 3.3, argueing once again with the gauge transformation - that the number of active clusters at renormalisation step Γ is still $\mathcal{N}(\Gamma) \propto \Gamma^{-2}$, see (22), and that the probability distribution for cluster lengths n follows $p(n,\Gamma) \sim f(\frac{n}{\Gamma^\phi}),\; n \in \mathbb{N}$ with $\phi = \frac{1+\sqrt{5}}{2}$, see (24).

With these results we can attack the calculation of the linear susceptibility $\chi(T)$, see definition (16), and find

$$\begin{aligned} \chi(T) \;&\overset{(48)}{=}\; \partial_h|_{h=0} \sum_{\substack{\text{active} \\ \text{clusters } I}} \mu_I \; \text{th}(\mu_I \beta h) = \beta \sum_{I=1}^{\mathcal{N}(\Gamma)} \mu_I^2 \overset{\text{large } \mathcal{N}(\Gamma)}{\approx} \beta \mathcal{N}(\Gamma) \; \overline{\mu^2} = \\ &= \; \beta \mathcal{N}(\Gamma) \sum_{\mu \geq 0} \mu^2 \sum_{n \geq 0} p(n,\Gamma) \; P_n(\mu) \overset{(49)}{\propto} \beta \mathcal{N}(\Gamma) \sum_{n \geq 0} n \; p(n,\Gamma) \overset{(24)}{\sim} \\ &\sim \; \beta \mathcal{N}(\Gamma) \; \Gamma^\phi \sim \beta \; \Gamma^{\phi-2} \sim \frac{|\ln T|^{\phi-2}}{T} \; . \end{aligned} \tag{50}$$

Note the change of the exponent with respect to the ferromagnetic case (26).

Next we define the non linear susceptibility

$$\chi_{nl}(T) := \left[\left. \frac{\partial^3}{\partial h^3} \right|_{h=0} < M >_{H(\{h_i, J^{(1)}_{i,i+1}\},\; h)} \right]_{av} \; . \tag{51}$$

which we calculate analogously to be

$$\begin{aligned} \chi_{nl}(T) \;&\overset{(48)}{=}\; \partial_h^3|_{h=0} \sum_{\substack{\text{active} \\ \text{clusters } I}} \mu_I \; \text{th}(\mu_I \beta h) = -2\beta^3 \sum_{I=1}^{\mathcal{N}(\Gamma)} \mu_I^4 \overset{\text{large } \mathcal{N}(\Gamma)}{\approx} -2\beta^3 \mathcal{N}(\Gamma) \; \overline{\mu^4} = \\ &= \; -2\beta^3 \mathcal{N}(\Gamma) \sum_{\mu \geq 0} \mu^4 \sum_{n \geq 0} p(n,\Gamma) \; P_n(\mu) \overset{(49)}{\propto} \beta^3 \mathcal{N}(\Gamma) \sum_{n \geq 0} n^2 \; p(n,\Gamma) \overset{(24)}{\sim} \\ &\sim \; \beta^3 \mathcal{N}(\Gamma) \; \Gamma^{2\phi} \sim \beta^3 \; \Gamma^{2\phi-2} \sim \frac{|\ln T|^{2\phi-2}}{T^3} \; . \end{aligned} \tag{52}$$

In the case of a classical spin glass the appropriate order parameter is that of Edwards-Anderson. It can be defined as

$$q_{EA}(T,h) := \left[< \sigma_i^z >^2_{H(\{h_i, J^{(1)}_{i,i+1}\}, h)} \right]_{av} . \tag{53}$$

Let us now calculate its behavior in the quantum case.

We remark that spins i in non active clusters do not contribute once again. Now imagine that the site i belongs to an active cluster C_I with μ_I "uncompensated" spins, then it is

$$< \sigma_i^z >^2 \big|_{i \in C_I} = \mathrm{th}^2(\beta \mu_I h) \overset{\beta \mu_I h \ll 1}{\approx} (\beta \mu_I h)^2. \tag{54}$$

The probability for having an active spin at site i is proportional to the number of active spins in the chain $\propto \Gamma^{\phi-2}$ and so we have

$$q_{EA}(T,h) \sim h^2 \ \beta^2 \Gamma^{\phi-2-2\phi} \ \sim h^2 \ \frac{|\ln T|^{-\phi-2}}{T^2} \tag{55}$$

5 Effects of couplings between next nearest neighbours

We are going to consider the spin chain near the fixed point (12), perturbed by small next nearest neighbour couplings. In this way we introduce now the possibility of frustration to our one dimensional system.

We examine the case of $\{J^{(1)}_{i,i+1}\} \geq 0$ but with $J^{(2)}_{i,i+2}$ of arbitrary signs

$$H = -\sum_i \left(J^{(1)}_{i,i+1} \sigma_i^z \sigma_{i+1}^z + J^{(2)}_{i,i+2} \sigma_i^z \sigma_{i+2}^z + h_i \sigma_i^x \right), \qquad \{J^{(1)}_{i,i+1}\} \geq 0 \tag{56}$$

Let us assume that the next nearest neighbour couplings will always be very small in the following sense:

(i) a $J^{(2)}_{i,i+2} \geq 0$ will never constitute the highest energy in the system and therefore never be decimated

(ii) in sums, the $J^{(2)}_{i,i+2}$ are negligible with respect to $J^{(1)}_{i,i+1}$ with high probability

(iii) creation of third neighbour couplings out of second neighbour couplings can be neglected. (57)

We show that initial probability distributions verifying these conditions (57) are renormalised to distributions that satisfy (57) even better and better. We conclude that the fixed point (12) is stable with respect to small perturbations caused by next nearest neighbour couplings.

5.1 Establishment of the flow equations for the probability distributions

Let us apply the developed decimation procedure to this case making use of (57). We give some illustration by the figures 6 and 7.

The decimation of a field h_i produces first of all the changes of couplings

$$\begin{aligned} \tilde{J}^{(1)}_{i-1,i+1} &= \frac{J^{(1)}_{i-1,i} J^{(1)}_{i,i+1}}{h_i}, \\ \tilde{J}^{(2)}_{i-2,i+1} &= \frac{J^{(2)}_{i-2,i} J^{(1)}_{i,i+1}}{h_i}, \quad \tilde{J}^{(2)}_{i-1,i+2} = \frac{J^{(1)}_{i-1,i} J^{(2)}_{i,i+2}}{h_i}, \end{aligned} \tag{58}$$

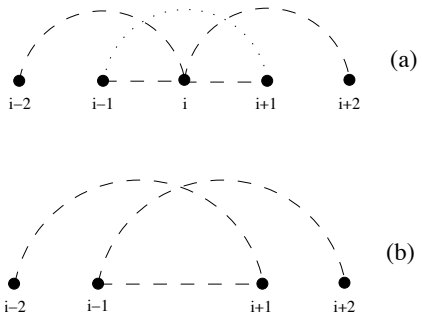

Figure 6: Decimation of a field at site i. The lines represent the behavior of the couplings under the renormalisation procedure: Dashed lines symbolize couplings that are modified by the renormalisation procedure, the dotted one is simply destructed.

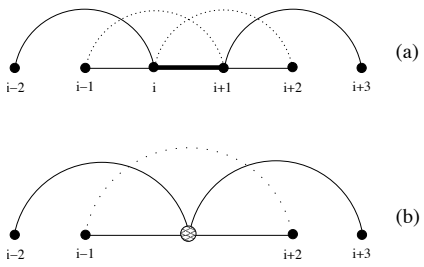

Figure 7: Decimation of a nearest neighbour coupling between sites i and $i+1$. Solid couplings are overtaken without change, the dotted couplings in the original chain are thrown out without any other effect, the dotted next nearest neighbour coupling in the resulting chain is of value zero.

see appendix A.

The decimation of a $J^{(1)}_{i,i+1}$ takes some of the original couplings - without change - as the couplings to the new spin variable. The field of the new spin is

$$\tilde{h}_{i,i+1} = \frac{h_i h_{i+1}}{J^{(1)}_{i,i+1}} \tag{59}$$

and the sites $i-1$ and $i+2$ will remain uncoupled, so

$$\tilde{J}^{(2)}_{i-1,i+2} = 0. \tag{60}$$

We remark that we just extend the case examined by Fisher, equations (7).

We proceed in the same way as in section 3 and adopt the notations introduced there. We refind the balance equations (8) for fields and nearest neighbour couplings as well as the chain length balance (6), because next nearest neighbour couplings are never decimated. Using equations (58)-(60) and taking care of destructed couplings (see figures 6 and 7) we find[7]

$$\begin{aligned}
\mathcal{N}^{(2)}(J,\Omega-\mathrm{d}\Omega) &= \mathcal{N}^{(2)}(J,\Omega) + \mathrm{d}\Omega\ \mathcal{N}^{(h)}(h=\Omega,\Omega)\ \times \\
&\times \int_0^\Omega \mathrm{d}J_1 \int_{-\Omega}^{\Omega} \mathrm{d}J_2\ \mathrm{d}J_3\ \tilde{\mathcal{P}}^{(1)}(J_1,\Omega)\ \tilde{\mathcal{P}}^{(2)}(J_2,\Omega)\ \tilde{\mathcal{P}}^{(2)}(J_3,\Omega) \times \\
&\times \left[2\delta(J-\frac{J_1J_2}{\Omega}) - 2\delta(J-J_2) - \delta(J-J_3)\right] + \\
&+\mathrm{d}\Omega\ \mathcal{N}^{(1)}(J=\Omega,\Omega) \int_{-\Omega}^{\Omega} \mathrm{d}J_1\ \mathrm{d}J_2\ \tilde{\mathcal{P}}^{(2)}(J_1,\Omega)\ \tilde{\mathcal{P}}^{(2)}(J_2,\Omega)\ \times \\
&\times \left[-2\delta(J-J_1) + \alpha\ \delta(J-\tilde{\Lambda}\Omega) + (1-\alpha)\ \delta(J+\tilde{\Lambda}\Omega)\right]
\end{aligned} \tag{61}$$

[7]In order to keep the probability distributions normalised we have to care for every decimated or created coupling, even if it is of value zero.

For being able to pass later to logarithmic variables we take care of (60) not by creating a zero coupling but by creating a small coupling of absolute value $\tilde{\Lambda}\Omega$ where $\tilde{\Lambda}$ is an arbitrary small constant fixed at the beginning of the renormalisation procedure. We also introduced a constant $\alpha \in [0,1]$ to decide about the sign of this coupling and will find later that this constant α is unimportant.

Now we split up (61) in balances for positive and negative next nearest neighbour couplings by defining $\mathcal{N}^{(2)}(J,\Omega) =: \Theta(J)\,\mathcal{N}^{(2+)}(J,\Omega)+\Theta(-J)\,\mathcal{N}^{(2-)}(-J,\Omega)$, we write - as in the last section - the differential equations for the probability distributions and perform the change of variables (9), yielding in particular the evolution of the distributions of positive and negative next nearest neighbour couplings, $\mathcal{P}^{(2+)}(\zeta,\Gamma)$ and $\mathcal{P}^{(2-)}(\zeta,\Gamma)$. Finally, being interested in the evolution of the absolute values of next nearest neighbour couplings but not in their signs we define $\mathcal{P}^{(2)}(\zeta,\Gamma) := \mathcal{P}^{(2+)}(\zeta,\Gamma) + \mathcal{P}^{(2-)}(\zeta,\Gamma)$ and obtain a system of three differential equations describing the renormalisation of the probability distributions. As expected, we find that the probability distributions for fields and nearest neighbour couplings, $\mathcal{P}^{(h)}$ and $\mathcal{P}^{(1)}$, follow the system of equations (10) already found in subsection 3.1. The evolution of the distribution of next nearest neighbour couplings is determined by

$$\begin{aligned}\frac{\partial P^{(2)}(\zeta,\Gamma)}{\partial\Gamma} &= \frac{\partial P^{(2)}(\zeta,\Gamma)}{\partial\zeta} - P^{(2)}(\zeta,\Gamma)\left(2P^{(h)}(0,\Gamma)+P^{(1)}(0,\Gamma)\right)+\\ &+2P^{(h)}(0,\Gamma)\int_0^\infty \mathrm{d}\zeta_1\;\mathrm{d}\zeta_2\;P^{(1)}(\zeta_1,\Gamma)\;P^{(2)}(\zeta_2,\Gamma)\;\delta(\zeta-\zeta_1-\zeta_2)+\\ &+P^{(1)}(0,\Gamma)\;\delta(\zeta-\Lambda)\end{aligned} \tag{62}$$

with $\Lambda = \ln(1/\tilde{\Lambda})$. Note that α disappeared from (62).

5.2 Examination of the stability

We are now interested in the evolution during renormalisation of initial probability distributions $\mathcal{P}^{(h)}(\zeta,\Gamma_0) = \mathcal{P}^{(1)}(\zeta,\Gamma_0) = \mathcal{P}_F(\zeta,\Gamma_0)$ and some $\mathcal{P}^{(2)}(\zeta,\Gamma_0)$ verifying (57).

Due to the assumptions (57), fields and nearest neighbour couplings renormalise as (10) without taking note of the existence of next nearest neighbour couplings. Thus, the associated distributions will keep their form $\mathcal{P}_F$.
Inserting the special form of $\mathcal{P}^{(h)}$, $\mathcal{P}^{(1)}$ in (62) and Laplace transforming $\mathcal{P}^{(2)}(\zeta,\Gamma)$ in ζ to $p(z,\Gamma)$ we obtain

$$(\partial_\Gamma - z + \frac{3}{\Gamma} - \frac{2}{\Gamma^2 z+\Gamma})\;p(z,\Gamma) = -\mathcal{P}^{(2)}(\zeta=0,\Gamma) + \frac{e^{-z\Lambda}}{\Gamma}. \tag{63}$$

Before solving this equation let us think about information directly accessible from the Laplace transformed p. Introducing a norm for integrable functions on $[\,0,\infty[$ by $\|Q(\zeta,\Gamma)\|_\zeta := \int_0^\infty \mathrm{d}\zeta\;|Q(\zeta,\Gamma)|$ we remark that for functions $Q(\zeta,\Gamma)$ that are everywhere non negative this is the same as their Laplace transformed $q(z,\Gamma)$ evaluated at $z=0$. Furthermore, derivatives with respect to z of $q(z,\Gamma)$ taken at $z=0$ yield the moments of $Q(\zeta,\Gamma)$.

By construction we have $\|\mathcal{P}^{(2)}(\zeta,\Gamma)\|_\zeta = 1\;\forall\;\Gamma$ and thus $p(z=0,\Gamma)\equiv 1$.

Evaluating (63) at $z = 0$ we find[8]

$$\mathcal{P}^{(2)}(\zeta = 0, \Gamma) \equiv 0. \tag{64}$$

The solution of (63) knowing (64) can be written as the sum of the solution to the homogenous equation and a special solution to the inhomogenous equation $p(z, \Gamma) = p_h(z, \Gamma) + p_s(z, \Gamma)$ with

$$\begin{aligned} p_h(z,\Gamma) &= q(z)\ \frac{\Gamma_0}{\Gamma}\frac{(z\Gamma_0+1)^2}{(z\Gamma+1)^2}\ e^{z(\Gamma-\Gamma_0)} \\ p_s(z,\Gamma) &= \frac{e^{z(\Gamma-\Lambda)}}{\Gamma\ (z\Gamma+1)^2}\Bigg[\frac{e^{-z\Gamma_0}}{z}\Big((z\Gamma_0+1)(z\Gamma_0+3)+2\Big) + \\ &\qquad -\frac{e^{-z\Gamma}}{z}\Big((z\Gamma+1)(z\Gamma+3)+2\Big)\Bigg] \end{aligned} \tag{65}$$

for an initial value Γ_0 of Γ and the initial condition $p(z, \Gamma_0) = q(z)$.

Let us have a closer look to this function describing the evolution of the perturbation in question. First we remark that the linearity of the Laplace transform and its inverse yields the solution $\mathcal{P}^{(2)} = \mathcal{P}_h^{(2)} + \mathcal{P}_s^{(2)}$ of (62) as the sum of the inversely Laplace transformed $p_h,\ p_s$. We note that the initial conditions $\mathcal{P}^{(2)}(\zeta, \Gamma_0)$ only occur in $\mathcal{P}_h^{(2)}$. We calculate the norm of the two parts of the solution

$$\|\mathcal{P}_h^{(2)}\|_\zeta = \lim_{z\to 0} p_h(z,\Gamma) = \frac{\Gamma_0}{\Gamma}, \quad \|\mathcal{P}_s^{(2)}\|_\zeta = \lim_{z\to 0} p_s(z,\Gamma) = 1 - \frac{\Gamma_0}{\Gamma} \tag{66}$$

and find that during the renormalisation $\mathcal{P}_h^{(2)}$ becomes more and more unimportant. Thus, the system "forgets" its initial conditions and flows to a general state gouverned by $\mathcal{P}_s^{(2)}$.

Next we show consistency of our calculations by verifying (57) at every renormalisation step, at least at every sufficiently late renormalisation step accepting to make faults during the initial renormalisation period. Point (i) is clear from (64). As to condition (ii) we calculate the mean value of the couplings

$$< \zeta_\Gamma^{(2)} >_Q := \int_0^\infty \mathrm{d}\zeta\ \zeta\ Q(\zeta,\Gamma) = -\lim_{z\to 0}\ \partial_z q(z,\Gamma), \tag{67}$$

We find

$$\begin{aligned} < \zeta_\Gamma^{(2)} >_{\mathcal{P}_h^{(2)}} &= \Gamma_0 + \frac{< \zeta_{\Gamma_0}^{(2)} >_{\mathcal{P}_h^{(2)}} - \Gamma_0}{\Gamma} \ \overset{\Gamma\gg\Gamma_0}{\approx}\ \Gamma_0, \\ < \zeta_\Gamma^{(2)} >_{\mathcal{P}_s^{(2)}} &= \left(1-\frac{\Gamma_0}{\Gamma}\right)\left(\Lambda + \frac{\Gamma-\Gamma_0}{2}\right) \ \overset{\Gamma\gg\Gamma_0}{\approx}\ \Lambda + \frac{\Gamma}{2}, \end{aligned} \tag{68}$$

what we have to compare to the mean value of the nearest neighbour couplings

$$< \zeta_\Gamma^{(1)} >_{\mathcal{P}^{(1)}\equiv\mathcal{P}_F} = \Gamma. \tag{69}$$

[8]This result confirms that the solutions $\mathcal{P}^{(2)}$ to be found are compatible with point (i) of (57). Thus, we are not working towards a contradiction.

Therefore, we consider some finite or infinite chain in a concrete realisation. We first decide to renormalise up to some *finite* step[9] $\Gamma_f > \Gamma_0$. We now choose - *before* starting renormalisation - some $\Lambda \gg \Gamma_f$ and find hence with (68), (69)

$$< \zeta_\Gamma^{(2)} >_{\mathcal{P}^{(2)}} \;\gg\; < \zeta_\Gamma^{(1)} >_{\mathcal{P}^{(1)}} \;\forall\, \Gamma \;\leq\; \Gamma_f. \tag{70}$$

In this sense we now understand the limit of an infinite number of renormalisation steps (for infinite chains) as the limit $\Gamma_f \to \infty$ and know (70) verified for all Γ_f. More mathematically, the physics tells us to take the limit $\Lambda \to \infty$ before $\Gamma \to \infty$. The relation (70) is a first support for (ii), but for broad probability distributions not sufficient.

That is why we now calculate the probability that at renormalisation step Γ a drawn next nearest neighbour coupling $\zeta^{(2)}$ is of higher energy than a drawn nearest neighbour coupling $\zeta^{(1)}$. We get

$$\text{Probability}_\Gamma(\ \text{“}\zeta^{(2)} < \zeta^{(1)}\text{”}\ ) = \int_0^\infty \mathrm{d}\zeta_2\ \mathcal{P}^{(2)}(\zeta_2,\Gamma)\ \int_{\zeta_2}^\infty \mathrm{d}\zeta_1\ \mathcal{P}^{(1)}(\zeta_1,\Gamma) =$$

$$= p(z = 1/\Gamma, \Gamma) \overset{\Gamma \gg \Gamma_0}{\approx} (5e-10)/4\ e^{-\Lambda/\Gamma}\ . \tag{71}$$

Hence, we find - taking the limits as explained above - that it vanishes for sufficiently large $\Gamma \gg \Gamma_0$ what justifies (ii).

Concerning point (iii) we argue that couplings between third neighbours are only created when a field, say h_i, is decimated, see (80) and (83), and then are of value $\frac{J^{(2)}_{i-2,i} J^{(2)}_{i,i+2}}{h_i}$ what is much smaller than all other energies in the environment of the ancient site i. So generated couplings between sites of distance ≥ 3 should be irrelevant.

In the above argumentation we came to see that initially small next nearest neighbour couplings renormalise in such a way that they fulfill (57) better and better. They evolve to energies considerably smaller and smaller than the energies of the other parts - fields and nearest neighbour couplings - present in the system and so the system renormalises to the fixed point (12). As a consequence, such next nearest neighbour couplings do not modify the critical behavior of the system and thus correspond to an irrelevant perturbation.

We now can generalise the considerations of this section to the case of present nearest and small next nearest neighbour couplings both of arbitrary signs. We can show by an explicit calculation that in this case next nearest neighbour couplings remain irrelevant. A less involved way is to use once again the gauge transformation (32) to map the general Hamiltonian including positive and negative nearest and next nearest neighbour couplings to the Hamiltonian (56). Then one can apply the renormalisation scheme as in this section to the new Hamiltonian and recover thus the irrelevance of small next nearest neighbour couplings of arbitrary signs.

5.3 Arbitrary large next nearest neighbour couplings

We now advance to regimes where next nearest neighbour couplings are not any more negligible, so we forget the conditions (57). In this case the application

[9] In the case of a finite chain, renormalisation stops having reached a chain of length 1 and Γ_f has to be smaller than the well-defined strength of the magnetic field associated to the last remaining effective spin.

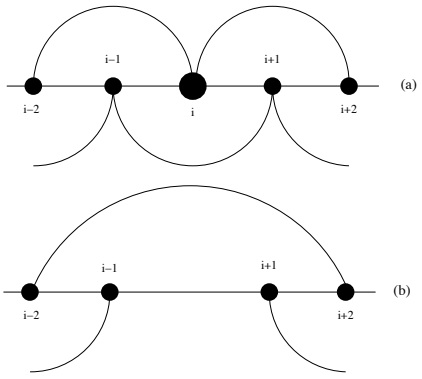

Figure 8: Decimation of a field h_i at site i in the next nearest neighbour chain. There is created a coupling between new third nearest neighbours.

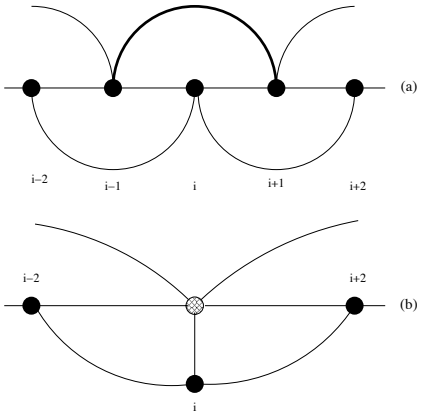

Figure 9: Decimation of a next neighbour coupling $J^{(2)}_{i-1,i+1}$ between sites $i-1$ and $i+1$. This makes us leave the chain-topology of the lattice.

of the renormalisation procedure, that has been that useful up to now, gets problematic because the Hamiltonian (56) has no more a closed form. On the one hand we will encounter non negligible couplings between third and higher nearest neighbours and on the other hand we will even leave the topology of a spin chain. We describe the basic processes causing these effects.

We still use the results of subsection A, equations (80) and (83). Let us first consider the decimation of a field, e. g. h_i, and follow the procedure illustrated in figure 8. We see that out of the two couplings $J^{(2)}_{i-2,i}$ and $J^{(2)}_{i,i+2}$ a new third nearest neighbour coupling of value $\frac{J^{(2)}_{i-2,i}J^{(2)}_{i,i+2}}{h_i}$ is created. Once third neighbour couplings created, further field decimations will make arise couplings between sites of distance four and so on. So in fact, we now have to take in account couplings between sites of any distance, e. g. we have to trace an infinity of probability distributions, that is one for all couplings between sites of a fixed distance.

Next we consider the decimation of a next nearest neighbour coupling - a process that we did not allow in the discussion of subsection 5.1. We show a picture of that in figure 9, here the next nearest neighbour coupling $J^{(2)}_{i-1,i+1}$ is decimated. It appears now a part of the shape of a "T" in the chain and we leave the initial linear topology of the lattice. Further decimations of couplings between sites of distances ≥ 2 can produce a very complicated topology.

It is not clear how to deal with this case. We do not see how to establish a manageable renormalisation scheme for this system.

6 Final Flow Diagram

Let us summarise our considerations of the quantum Ising spin glass (1) and try to draw a diagram reflecting the renormalisation group flows near the identified

fixed points.

We consider three "directions" in the functional space of the probability distributions for magnetic fields $\tilde{\mathcal{P}}^{(h)}(h,\Omega)$, positive and negative nearest neighbour couplings $\tilde{\mathcal{P}}^{(1+)}(J,\Omega)$ and $\tilde{\mathcal{P}}^{(1-)}(J,\Omega)$ as well as next nearest neighbour couplings $\tilde{\mathcal{P}}^{(2)}(J,\Omega)$ given by the parameters

$$\begin{aligned}
\delta &:= [\ ln|h|\ ]_{av} - \Big[\ ln|J^{(1)}|\ \Big]_{av} := \\
&= \int_{-\infty}^{\infty} \mathrm{d}h\ \tilde{\mathcal{P}}^{(h)}(h,\Omega) \ln|h| - \int_0^{\infty} \mathrm{d}J\ \left(\tilde{\mathcal{P}}^{(1-)}(J,\Omega) + \tilde{\mathcal{P}}^{(1+)}(J,\Omega)\right)\ \ln J \\
\kappa &:= \| \tilde{\mathcal{P}}^{(1+)}(J,\Omega) \| - \| \tilde{\mathcal{P}}^{(1-)}(J,\Omega) \| := \\
&= \int_0^{\infty} \mathrm{d}J\ \tilde{\mathcal{P}}^{(1+)}(J,\Omega) - \int_0^{\infty} \mathrm{d}J\ \tilde{\mathcal{P}}^{(1-)}(J,\Omega) \\
\lambda &:= \Big[\ |J^{(2)}|\ \Big]_{av} := \int_{-\infty}^{\infty} \mathrm{d}J\ \tilde{\mathcal{P}}^{(2)}(J,\Omega)\ |J|. \qquad (72)
\end{aligned}$$

The parameter δ is an appopriate measure for the distance from the fixed point in the "direction" of the "proportion of forces" of the fields and nearest neighbour couplings in their struggle about order and disorder, see [2]. For $\delta < 0$ the system is in the ordered phase and for $\delta > 0$ in the disordered one, $\delta = 0$ corresponds to the phase transistion associated to the fixed point (12).

We use κ to measure the distribution of the signs of the nearest neighbour couplings. In subsection 4.3 we found the "ferromagnetic" fixed point (39) and the "spin glass" fixed point (40) and examined their stability. We do not know if there exist further fixed points, the following diagram assumes that there are no other ones.

Finally, λ measures the presence of next nearest neighbour couplings. In subsection 5.2 we found that every fixed point on the axis $\delta = \lambda = 0$ will be stable with respect to their introduction.

We thus get the schematic flow diagram in figure 6. The "spin glass" fixed point (40) found in 4.3 is the most important attractor.

7 Conclusions

In this work we considered the quantum Ising spin glass in one dimension. We have been interested in its possible phases and phase transitions at zero temperature.

Our most important tool for handling disorder has been a functional renormalisation group transformation developed by Dasgupta and Ma. Fisher later has shown that this method can be applied with great succes to the examination of the random ferromagnetic quantum Ising chain and other models.

Following this renormalisation scheme we established a set of differential equations describing the flow of the probability distributions of fields and couplings during renormalisation. We found a fixed point in these probability distributions where positive and negative nearest neighbour couplings are equally distributed. We associate this fixed point to the quantum phase transition of the spin glass chain.

We then concentrated on the examination of this fixed point. We have shown that it is stable with respect to perturbations caused by the introduction of

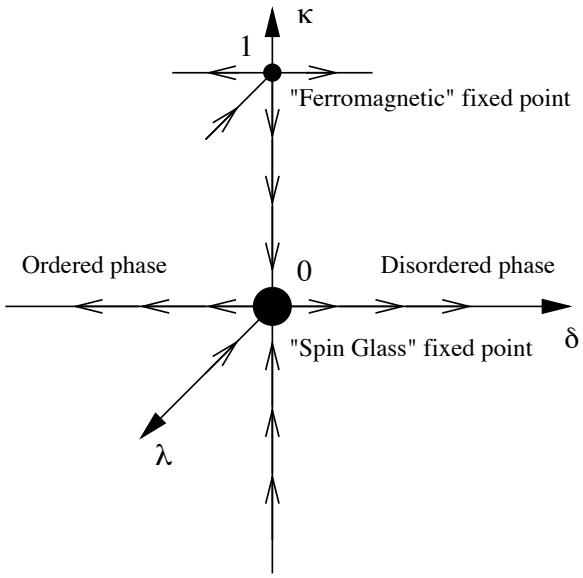

Figure 10: Schematic flow diagram for the Quantum Ising Spin Glass in one dimension.

next nearest neighbour couplings as well as perturbations deforming the equal distribution of positive and negative nearest neighbour couplings. Thus, the fixed point turned out to be the most important attractor. We were then able to find the behavior of some of the magnetic properties of this fixed point.

It will be worthwhile to retrace the results found in this work by establishing the renormalisation scheme numerically.

Furthermore, after having characterised the fixed point representing the zero temperature phase transition in the spin glass chain it will be an interesting project to examine now the two different phases that it separates. One would like to reveal their magnetic properties and especially to identify an appropriate order parameter.

J'aimerais remercier mes maîtres de stage, Pierre Pujol et David Carpentier, pour tout le temps qu'ils ont pris pour moi.

A Simplification of the spin chain by perturbation theory

We describe how perturbation theory can be used to find a low energy approximation for some given spin chain. It prooves useful to consider the general Hamiltonian

$$H = -\sum_i \left(\sum_{l \geq 1} J^{(l)}_{i,i+l} \sigma^z_i \sigma^z_{i+l} + h_i \sigma^x_i + h\mu_i \sigma^z_i \right) \tag{73}$$

involving couplings between all sites and - to measure susceptibility - including an arbitrary small constant field h in z-direction and associated magnetic moments μ_i to each spin.

In what follows we treat the decimation of couplings and fields. We use perturbation theory of degenerated states carried up to second order. We recall the formulae for the energy corrections in first and second order $E^{(1)}$ and $E^{(2)}$: Let us consider the perturbation H_1 to a system H_0 and denote $\mathcal{D}_0 \subset \mathcal{H}$ the corresponding degenerate subspace of interest (spanned by eigenvectors of H_0) in our Hilbert space $\mathcal{H}$, $\mathcal{D} := \mathcal{H} \setminus \mathcal{D}_0$ the complementary space. Then the energy corrections are given by the eigenvalues of the following matrices

$$\begin{aligned} &< \varphi^i_{\mathcal{D}_0} | H_1 | \varphi^j_{\mathcal{D}_0} > \text{ yields } \{E^{(1)}_{ij}\} \\ &< \varphi^i_{\mathcal{D}_0} | \sum_{\substack{|\varphi^k_{\mathcal{D}}> \in \\ \text{basis of } \mathcal{D}}} \frac{H_1 |\varphi^k_{\mathcal{D}}>< \varphi^k_{\mathcal{D}}| H_1}{E^{(0)} - E^k_{\mathcal{D}}} |\varphi^j_{\mathcal{D}_0}> \text{ yields } \{E^{(2)}_{ij}\} \end{aligned} \tag{74}$$

for the vectors $|\varphi^i_{\mathcal{D}_0}>$ and $|\varphi^k_{\mathcal{D}}>$ of an arbitrary basis of $\mathcal{D}_0$ and $\mathcal{D}$, respectively, and $E^{(0)}$ the H_0-eigenvalue in the degenerate subspace $\mathcal{D}_0$.

A.1 Decimation of a coupling $J^{(L)}_{i,i+L}$ between sites i and $i+L$

Let us suppose that one of the random couplings, say $J^{(L)}_{i,i+L}$, is responsable for the largest energy gap in the spin chain, that means all fields and couplings in the chain are smaller than it and (at least) those in its environment are even much smaller. For this reason we decimate it. With the help of second order perturbation theory we will find a new, simpler spin chain as a low energy appoximation for the original chain. The calculations for different L are the same and that is why we immediately treat the general case of arbitrary L.

For simplicity we consider exclusively the subsystem of the chain described by the parts of the Hamiltonian (73) that involve the sites i or $i+L$ (the other parts do not enter the perturbational treatment) given by $H^J = H^J_0 + H^J_1$ with

$$\begin{aligned} H^J_0 &= -J^{(L)}_{i,i+L} \sigma^z_i \sigma^z_{i+L}, \quad H^J_1 = H^J_{1,x} + H^J_{1,z} \\ H^J_{1,x} &= -h_i \sigma^x_i - h_{i+L} \sigma^x_{i+L} \\ H^J_{1,z} &= -\sum_{l \notin \{i,i+L\}} \left(J^{(|i-l|)}_{l,i} \sigma^z_l \sigma^z_i + J^{(|i+L-l|)}_{l,i+L} \sigma^z_l \sigma^z_{i+L} \right) - h(\mu_i \sigma^z_i + \mu_{i+L} \sigma^z_{i+L}) \end{aligned} \tag{75}$$

For H^J_0 there are two eigenvalues, $-J^{(L)}_{i,i+L}$ and $+J^{(L)}_{i,i+L}$, corresponding to two degenerate subspaces $\mathcal{D}_0$ and $\mathcal{D}$ of the Hilbert space $\mathcal{H}$ and we are always going to choose $\mathcal{D}_0$ as the subspace corresponding to the smaller eigenvalue, so it is

$$\begin{aligned} \mathcal{D}_0 &:= \text{span}\{|\uparrow_i\uparrow_{i+L}>, |\downarrow_i\downarrow_{i+L}>\} \otimes \mathcal{H}^{\text{remaining sites}} \\ \mathcal{D} &:= \text{span}\{|\uparrow_i\downarrow_{i+L}>, |\downarrow_i\uparrow_{i+L}>\} \otimes \mathcal{H}^{\text{remaining sites}} \end{aligned} \tag{76}$$

for $J^{(1)}_{i,i+L} > 0$ and vice versa in the other case. Here $|\uparrow_i>, |\downarrow_i>$ design the eigenstates of σ^z_i to characterise the quantum state on site i.

We neglect the states in $\mathcal{D}$, which are the high energy states, throwing them out. We then rewrite approximatively the action of H_1^J in $\mathcal{D}_0$ by making use of the perturbation theory. The matrix elements mentionned above are calculated to be

$$E_{ij}^{(1)} =< \varphi_{\mathcal{D}_0}^i | H_1^J | \varphi_{\mathcal{D}_0}^j >=< \varphi_{\mathcal{D}_0}^i | H_{1,z}^J | \varphi_{\mathcal{D}_0}^j > \tag{77}$$

and

$$\begin{aligned} E_{ij}^{(2)} &= < \varphi_{\mathcal{D}_0}^i | \sum_{\substack{|\varphi_{\mathcal{D}}^k> \in \\ \text{basisof } \mathcal{D}}} \frac{H_1^J |\varphi_{\mathcal{D}}^k >< \varphi_{\mathcal{D}}^k| H_1^J}{E^{(0)} - E_{\mathcal{D}}^k} |\varphi_{\mathcal{D}_0}^j > \\ &= < \varphi_{\mathcal{D}_0}^i | \sum_{\substack{|\varphi_{\mathcal{D}}^k> \in \\ \text{basisof } \mathcal{D}}} \frac{H_{1,x}^J |\varphi_{\mathcal{D}}^k >< \varphi_{\mathcal{D}}^k| H_{1,x}^J}{\mp 2 J_{i,i+L}^{(L)}} |\varphi_{\mathcal{D}_0}^j > \\ &= < \varphi_{\mathcal{D}_0}^i | \sum_{\substack{|\varphi_{\mathcal{D}}^k> \in \\ \text{basisof } \mathcal{D}_0 \cup \mathcal{D}}} \frac{H_{1,x}^J |\varphi^k >< \varphi^k| H_{1,x}^J}{\mp 2 J_{i,i+L}^{(L)}} |\varphi_{\mathcal{D}_0}^j > \\ &= \frac{< \varphi_{\mathcal{D}_0}^i | \left(H_{1,x}^J\right)^2 |\varphi_{\mathcal{D}_0}^j >}{-2|J_{i,i+L}^{(L)}|} \end{aligned} \tag{78}$$

where here and in the following the signs $\pm$ are chosen $+(-)$ if $J_{i,i+L}^{(L)} > (<)0$. Now in $\mathcal{D}_0$ we have the identity $\sigma_i^z = \pm\sigma_{i+L}^z$ and we use it to define a new effective spin $\tilde{\underline{\sigma}}_{i,i+L}$ by

$$\begin{aligned} &\tilde{\sigma}_{i,i+L}^x := \sigma_i^x \sigma_{i+L}^x \\ &\tilde{\sigma}_{i,i+L}^z := \begin{cases} +\sigma_i^z & \text{if } (J_{i,i+L}^{(L)} > 0) \text{ or } (J_{i,i+L}^{(L)} < 0 \text{ and } \mu_i > \mu_{i+L}) \\ -\sigma_i^z & (J_{i,i+L}^{(L)} < 0 \text{ and } \mu_i < \mu_{i+L}) \end{cases} \end{aligned} \tag{79}$$

This reflects the fact that in $\mathcal{D}_0$ (and in $\mathcal{D}$ too) the two spins act like only one spin in a certain sense. The definition of $\tilde{\sigma}_{i,i+L}^z$ is chosen in a way that the magnetic moment of the new effective spin is non negative, $\tilde{\mu}_{i,i+L} = |\mu_i \pm \mu_{i+L}|$.

So we find (in $\mathcal{D}_0$) the approximation

$$\begin{aligned} H^J &= H_0^J + H_{1,z}^J - \frac{\left(H_{1,x}^J\right)^2}{2|J_{i,i+L}^{(L)}|} + o(\left(\frac{1}{J_{i,i+L}^{(L)}}\right)^2) \\ &= -\Big\{ |J_{i,i+L}^{(L)}| + \frac{h_i^2 + h_{i+L}^2}{2|J_{i,i+L}^{(L)}|} + \frac{h_i h_{i+L}}{|J_{i,i+L}^{(L)}|} \tilde{\sigma}_{i,i+L}^x + h\ |\mu_i \pm \mu_{i+L}|\ \tilde{\sigma}_{i,i+L}^z + \\ &\quad + \sum_{j \notin \{i, i+L\}} \left(J_{i,j}^{(|i-j|)} \pm J_{i+L,j}^{(|i+L-j|)} \right) \sigma_j^z \sigma_i^z + o(\left(\frac{1}{J_{i,i+L}^{(L)}}\right)^2) \Big\} \end{aligned} \tag{80}$$

where the σ_i^z occuring in the last sum can be expressed in terms of the new spin variable by inversion of (79).

A.2 Decimation of a transverse field h_i

Let now one of the random transverse fields, say h_i, be responsable for the largest energy gap in the spin chain: We suppose that all fields and couplings in the chain are smaller than h_i and (at least) those in its environment are even much smaller.

In this case we deal with the subsystem

$$H^h = -\left\{ h_i \sigma_i^x + \sum_{l \neq i} J_{i,l}^{(|l-i|)} \sigma_i^z \sigma_l^z + h\mu_i \sigma_i^z \right\} \tag{81}$$

and identify

$$\begin{aligned} \mathcal{D}_0 &:= \text{span}\left(\frac{|\uparrow_i> + |\downarrow_i>}{\sqrt{2}} \right) \otimes \not{0}^{\text{remaining sites}} \\ \mathcal{D} &:= \text{span}\left(\frac{|\uparrow_i> - |\downarrow_i>}{\sqrt{2}} \right) \otimes \not{0}^{\text{remaining sites}} \end{aligned} \tag{82}$$

as the subspaces to eigenvalues $-h_i$ and $+h_i$, respectively.

We calculate the matrix elements of the perturbative operator as in the precedent case and then find an effective Hamiltonian

$$\begin{aligned} H^h &= -\Bigg\{ h_i + \frac{\sum_{l \neq 0} \left(J_{i,i+l}^{(|l|)} \right)^2 + (h\mu_i)^2}{2h_i} + \\ &\quad + \frac{h\mu_i}{h_i} \sum_{l \neq i} J_{i,l}^{(|l-i|)} \sigma_l^z + \frac{1}{2h_i} \sum_{\substack{l,n \neq i \\ n \neq l}} J_{i,l}^{(|l-i|)} J_{i,n}^{(|n-i|)} \sigma_l^z \sigma_n^z \Bigg\}. \end{aligned} \tag{83}$$

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