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Probabilistic Automata*

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Probabilistic automata (p.a.) are a generalization of finite deterministic automata. We follow the formulation of finite automata in Rabin and Scott (1959) where the automata $\mathfrak{A}$ have two-valued output and thus can be viewed as defining the set $T(\mathfrak{A})$ of all tapes accepted by $\mathfrak{A}$. This involves no loss of generality. A p.a. is an automaton which, when in state s and when input is σ, has a probability $p_i(s, \sigma)$ of going into any state s_i. With any cut-point $0 \leq \lambda < 1$, there is associated the set $T(\mathfrak{A}, \lambda)$ of tapes accepted by $\mathfrak{A}$ with cut-point λ.

Here we develop a general theory of p.a. and solve some of the basic problems. Aside from the mathematical interest in pursuing this natural generalization of finite automata, the results also bear on questions of reliability of sequential circuits.

P.a. are, in general, stronger than deterministic automata (Theorem 2). By studying the way we may want to use p.a. we are led to introduce the concept of *isolated cut-point*. It turns out that every p.a. with isolated cut-point is equivalent to a suitable deterministic automaton (the Reduction Theorem 3). It is interesting to note that in passing from a minimal deterministic automaton to an equivalent p.a. we can sometimes save states (Section VII).

The Reduction Theorem is applied to prove the existence of an approximate calculation procedure for a calculation problem involving products of stochastic matrices (Section VIII). The problem is of a new kind in that there is no a-priori bound on the number of operations (matrix multiplications) which we may have to perform and therefore classical numerical estimates of round-off errors do not apply.

Actual automata (Definition 9) have the property, often existing in

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actual unreliable circuits, that all transition probabilities are strictly positive. Actual automata are proved to give only definite events. This points to the restrictions we may have to impose on a probabilistic sequential circuit if we want it to perform general tasks, namely, some transitions should be prohibited.

Finally we treat the important problem of stability. Is the operation of a p.a. stable (unchanged) under small enough perturbations of the transition probabilities? We have an affirmative answer to this question in the case of actual automata (Theorem 11) and we discuss the problem for the general case.

INTRODUCTION

Finite automata are mathematical models for systems capable of a finite number of states which admit at discrete time intervals certain *inputs* (incoming signals) and emit certain *outputs*. If the system is in state s and the input is σ then the system will move into a new state s_i which depends only on s and σ and will have an output which depends only (is a function of) on s_i. Thus the system will transform a sequence of inputs into a sequence of outputs and the relevant aspect of the system is this transformation. Sequential circuits, and even whole digital computers, provided the computer operates using only internal memory or just a fixed amount of tape, are systems which behave like finite automata. There is an extensive literature on finite automata. In this paper we follow the notations and use some of the results on automata contained in the paper by Rabin and Scott (1959). In particular the formulation given there amounts to assuming that the set of outputs contains just two elements. This is a convenient restriction which we follow also here but the results immediately extend to the general case of more than two outputs. Because of the restriction to two-valued outputs automata can be viewed as defining sets of sequences of inputs (tapes) and this point of view is adopted throughout this paper.

Finite automata exhibit a deterministic behavior. The state s and input σ determine the next state of the automaton. It is quite natural to consider automata with stochastic behavior. The idea is that the automaton, when in state s and when the input is σ, can move into any state s_i and the probability for moving into state s_i is a function $p_i(s, \sigma)$ of s and σ.

A practical motivation for considering probabilistic automata is that even the sequential circuits which are intended to be deterministic exhibit stochastic behavior because of random malfunctioning of components. This situation was first taken up by von Neumann (1956) who considered schemes for organizing combinatorial (and to some extent also sequential) circuits constructed with specific components so as to increase their reliability.

Though the generalization from the abstract deterministic automata to the abstract probabilistic automata (p.a.) lies near at hand, there are no general results about p.a. in the literature. In particular, it was not even known whether p.a. can do more than deterministic automata. In this paper we develop a general theory of p.a. and answer some of the basic questions about them.

It turns out that in general p.a. are stronger than deterministic automata. We introduce, however, a new concept of *isolated cut-point* and prove the fundamental Reduction Theorem 3 that *every p.a. with isolated cut-point is equivalent to a suitable deterministic automaton.*

In Section XI we define *actual automata* which are automata such that all their transition probabilities are strictly positive. These automata define a very limited class of regular events (Theorem 10). The results are of some significance for the theory of reliability. They indicate that if we want to synthesize general sequential circuits from unreliable components we must organize them so that *transitions between certain states are prohibited* (have probability zero), or else consider the circuit as having broken down if these transitions occurred.

Another problem bearing on theory of reliability is the *stability problem.* The probabilistic automaton is called *stable* if its behavior is not changed by small perturbations of the transition probabilities. In synthesizing circuits from unreliable components we surely want to get stable circuits. In Section XII we give a stability theorem for actual automata. We also discuss the general stability problem but leave it open.

I. FINITE AUTOMATA

In this section we give a brief resume of the basic definitions and some basic results which will be used in the sequel, from the theory of finite (deterministic) automata. The exposition follows closely that in Rabin and Scott (1959). By "automaton" we shall mean, throughout this section, deterministic automaton.

Let Σ be a finite nonempty set, to be called the *alphabet*. Letters τ, σ (with subscripts) will usually denote elements of Σ. The set of all finite sequences of elements of Σ will be denoted by Σ^*. The elements of Σ^* will be called *tapes*. The letters x, y, z, u, v (with subscripts) will always denote tapes. The empty tape (i.e., the sequence of length zero) will be denoted by Λ. Subsets of Σ^* (i.e., sets of tapes) will sometimes be called *events*.

If $x = \sigma_1 \cdots \sigma_k$ is a tape then the *length* $l(x)$ of x is $l(x) = k$. If x and y are tapes then xy will denote the concatenation of x and y. Note that Σ^* with this operation xy is a free semi-group with the elements of Σ as free generators.

DEFINITION 1. A *finite* (deterministic) *automaton over* Σ is a system $\mathfrak{A} = \langle S, M, s_0, F\rangle$ where S is a finite set (the set of *states*), M is a function from $S \times \Sigma$ into S (the *transition table*), $s_0 \in S$ (the *initial* state), and $F \subseteq S$ (the set of *designated final states*).

M can be extended to a function from $S \times \Sigma^*$ to S by, $M(s, \Lambda) = s$, $M(s, x\sigma) = M(M(s, x), \sigma)$ $(s \in S, x \in \Sigma^*, \sigma \in \Sigma)$. $M(s, x)$ is the state in which $\mathfrak{A}$ "gets off" the tape x if it started on x in state s.

DEFINITION 2. A tape x is said to be *accepted* by $\mathfrak{A}$ if and only if $M(s_0, x) \in F$. The set *defined* by $\mathfrak{A}$ is the set of all tapes accepted by $\mathfrak{A}$, and is denoted by $T(\mathfrak{A})$. An event $U \subseteq \Sigma^*$ is called a *regular event* if for some finite automaton $\mathfrak{A}$, $U = T(\mathfrak{A})$.

Every finite event is regular. If U and V are regular so are $U \cap V$, $U \cup V$ and $\Sigma^* - U$ (see Rabin and Scott, 1959).

In Rabin and Scott (1959) a necessary and sufficient condition for an event $T \subseteq \Sigma^*$ to be regular was given in terms of right equivalence relations.

DEFINITION 3. Let $T \subseteq \Sigma^*$, the *right-equivalence relation* $\equiv_T$ *generated* by T is defined as follows. For $x, y \in \Sigma^*$, $x \equiv_T y$ if and only if for all $z \in \Sigma^*$ we have: $xz \in T$ if and only if $yz \in T$.

It is easy to see that $\equiv_T$ is an equivalence relation on Σ^*. Note that $\equiv_T$ is *right-invariant* with respect to the multiplication of the semigroup Σ^*, i.e., for all $x, y, z \in \Sigma^*$, if $x \equiv_T y$ then $xz \equiv_T yz$.

THEOREM 1 (Rabin and Scott, 1959). *A set $T \subseteq \Sigma^*$ is a regular event if and only if the number of equivalence classes of Σ^* by the equivalence relation $\equiv_T$ is finite. If the number of equivalence classes is $e < \infty$ then for a suitable $\mathfrak{A}$, $T = T(\mathfrak{A})$ where the automaton $\mathfrak{A}$ has e states. No automaton with fewer than e states defines T.*

II. PROBABILISTIC AUTOMATA

We shall now define the basic concept of this investigation, namely the concept of probabilistic automata. It will be seen that probabilistic automata are like the usual automata except that now the transition table M assigns to each pair $(s, \sigma) \in S \times \Sigma$ certain transition probabilities.

DEFINITION 4. A *probabilistic automaton* (p.a.) over the alphabet Σ is a system $\mathfrak{A} = \langle S, M, s_0, F \rangle$ where $S = \{s_0, \ldots, s_n\}$ is a finite set (the set of *states*), M is a function from $S \times \Sigma$ into $[0, 1]^{n+1}$ [1] (the *transition probabilities table*) such that for $(s, \sigma) \in S \times \Sigma$

$$M(s, \sigma) = (p_0(s, \sigma), \cdots, p_n(s, \sigma)),$$

$$0 \leqq p_i(s, \sigma), \qquad \sum_i p_i(s, \sigma) = 1,$$

$s_0 \in S$ (the *initial state*), and $F \subseteq S$ (the set of *designated final states*).

Probabilistic automata are models for systems (such as sequential circuits) capable of a finite number of states $s_0, \ldots, s_n$. The system may receive inputs $\sigma \in \Sigma$. When in state s and if the input is σ then the system can go into any one of the states $s_i \in S$ and the probability of going into s_i is the $(i + 1)$th coordinate $p_i(s, \sigma)$ of $M(s, \sigma)$. These transition probabilities $p_i(s, \sigma)$ are assumed to remain fixed and be independent of time and previous inputs. Thus the system also has definite transition probabilities for going from state s to state s_i by a *sequence* $x \in \Sigma^*$ of inputs. These probabilities are calculated by means of products of certain stochastic matrices which we shall now define.

DEFINITION 5. For $\sigma \in \Sigma$ and $x = \sigma_1\sigma_2 \cdots \sigma_m$ define the $n + 1$ by $n + 1$ matrices $A(\sigma)$ and $A(x)$ by

$$A(\sigma) = [p_j(s_i, \sigma)]_{0 \leqq i \leqq n, 0 \leqq j \leqq n}$$

$$A(x) = A(\sigma_1)A(\sigma_2) \cdots A(\sigma_m) = [p_j(s_i, x)]_{0 \leqq i \leqq n, 0 \leqq j \leqq n}.$$

REMARK. An easy calculation (involving induction on m) will show the $(i + 1, j + 1)$ element $p_j(s_i, x)$ is the probability of $\mathfrak{A}$ for moving from state s_i to state s_j by the input sequence x.

DEFINITION 6. If $\mathfrak{A} = \langle S, M, s_0, F \rangle$ and $F = \{s_{i_0}, \ldots, s_{i_r}\}$, $I = \{i_0, \ldots, i_r\}$, define

$$p(x) = \sum_{i \in I} p_i(s_0, x).$$

$p(x)$ clearly is the probability for $\mathfrak{A}$, when started in s_0, to enter into a state which is member of F by the input sequence x.

[1] $[0, 1]$ is the closed unit interval $0 \leqq x \leqq 1$. $[0, 1]^{n+1}$ is the set of all $n + 1$-tuples $(x_0, \ldots, x_n)$ where $0 \leqq x_i \leqq 1$.

III. SETS OF TAPES DEFINED BY P.A.

A p.a. $\mathfrak{A}$ may be used to define sets of tapes in a manner similar to that of deterministic automata except that now the set of tapes will depend not just on $\mathfrak{A}$ but also on a parameter λ.

DEFINITION 7. Let $\mathfrak{A}$ be p.a. and λ be a real number, $0 \leqq \lambda < 1$. The set of tapes $T(\mathfrak{A}, \lambda)$ is defined by

$$T(\mathfrak{A}, \lambda) = \{x \mid x \in \Sigma^*, \lambda < p(x)\}.$$

If $x \in T(\mathfrak{A}, \lambda)$ we say that x is *accepted* by $\mathfrak{A}$ with *cut-point* λ. $T(\mathfrak{A}, \lambda)$ will also be called the set defined by $\mathfrak{A}$ with *cut-point* λ.

REMARK. Deterministic automata can be considered as a special case of p.a. Namely, if in Definition 1 $M(s, \sigma) = s_i$ then we can view this as if $\mathfrak{A}$ will enter state s_i with probability 1. Thus in rewriting the deterministic automaton as a p.a. the stochastic vectors $M(s, \sigma) = (p_0, \ldots, p_n)$ will have exactly one coordinate 1 and all the others 0. It is readily seen that in this case $p(x) = 1$, for $x \in \Sigma^*$, if and only if $x \in T(\mathfrak{A})$. Hence for any λ, $0 \leqq \lambda < 1$, we have $T(\mathfrak{A}) = T(\mathfrak{A}, \lambda)$. Thus every set definable by a deterministic automaton is trivially definable by some p.a. In the next section we shall see the converse is *not* true and that therefore p.a. give a strictly larger class of definable sets.

IV. PROBABILISTIC AUTOMATON DEFINING NONREGULAR EVENT

The following matrices were suggested by E. F. Moore.

$$P_0 = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \quad P_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix}.$$

It can be readily verified that if

$$P_{\delta_1} \cdot P_{\delta_2} \cdot \cdots \cdot P_{\delta_n} = \begin{bmatrix} m & p \\ q & r \end{bmatrix}, \qquad \delta_i \in \{0, 1\}$$

then $p = \cdot\, \delta_n \delta_{n-1} \cdots \delta_1$ where p is written in binary expansion.

THEOREM 2. *Let $\mathfrak{A} = \langle S, M, s_0, F \rangle$ be an automaton over $\Sigma = \{0, 1\}$ such that $S = \{s_0, s_1\}$, $A(0) = P_0$, $A(1) = P_1$, $F = \{s_1\}$. There exists a $0 \leqq \lambda < 1$ such that $T(\mathfrak{A}, \lambda)$ is not definable by a deterministic automaton (is not a regular event).*

PROOF: If $x = \delta_1 \delta_2 \cdots \delta_n \in \Sigma^*$, then by the above, $p(x) = \cdot\, \delta_n \delta_{n-1} \cdots \delta_1$.

The values $p(x)$ are dense in the whole interval [0, 1]. This implies that if $0 \leqq \lambda < \lambda_1 < 1$ then $T(\mathfrak{A}, \lambda_1) \subset T(\mathfrak{A}, \lambda)$ where the inclusion is proper. The sets $T(\mathfrak{A}, \lambda)$, $0 \leqq \lambda < 1$, therefore form a nondenumerable pairwise different collection of sets. But there is only a denumerable collection of regular events. Therefore there exists a λ such that $T(\mathfrak{A}, \lambda)$ is not regular.

Remark. The above argument is a pure existence proof. We can, however, present a specific λ such that $T(\mathfrak{A}, \lambda)$ is not regular. Namely, let $w_1, w_2, \ldots,$ be any enumeration of Σ^* then for $\lambda = .w_1w_2 \cdots$, $T(\mathfrak{A}, \lambda)$ is not regular; we omit the proof.

The above λ is irrational. It can in fact be shown that if λ is rational then $T(\mathfrak{A}, \lambda)$ is a regular event.

V. ISOLATED CUT-POINTS

Let $\mathfrak{A}$ be a p.a. and $0 \leqq \lambda < 1$. Given a tape $x \in \Sigma^*$ we devise the following probabilistic experiment E to test whether $x \in T(\mathfrak{A}, \lambda)$. We run x through $\mathfrak{A}$ a large number N of times and count the number $m(E)$ of times that $\mathfrak{A}$ ended in a state in F. If $\lambda < m(E)/N$ we accept x and otherwise we reject it. Because of the probabilistic nature of the experiment it is of course possible that we sometimes accept x even though $x \notin T(\mathfrak{A}, \lambda)$ or reject it even though $x \in T(\mathfrak{A}, \lambda)$. By the law of large numbers, however, there exist for each x such that $p(x) \neq \lambda$ and each $0 < \epsilon$ a number $N(x, \epsilon)$ such that

$$\Pr\left(E \mid \lambda \ldots < \frac{m(E)}{N(x, \epsilon)} \leftrightarrow x \in T(\mathfrak{A}, \lambda)\right) \geqq 1 - \epsilon.$$

That is, the probability of obtaining the *correct* answer by the experiment E (consisting of running x $N(x, \epsilon)$ times through $\mathfrak{A}$ and counting successes) is greater than $1 - \epsilon$.

To perform the above stochastic experiment we must know $N(x, \epsilon)$ which depends on $|p(x) - \lambda|$. Thus we have actually to know $p(x)$ in advance if we want to ascertain whether $x \in T(\mathfrak{A}, \lambda)$ with probability greater than $1 - \epsilon$ of being correct. Once we know $p(x)$, however, the whole experiment E is superfluous.

The way out is to consider values λ such that $|p(x) - \lambda|$ is bounded from below for all $x \in \Sigma^*$.

Definition 8. A cut-point λ is called *isolated* with respect to $\mathfrak{A}$ if there exists a $0 < \delta$ such that

$$\delta \leqq |p(x) - \lambda| \quad \text{for all} \quad x \in \Sigma^*. \tag{1}$$

REMARK. It is readily seen that there exists an integral valued function $N(\delta, \epsilon)$ such that for an isolated λ and *any* $x \in \Sigma^*$

$$\Pr\left(E \mid \lambda < \frac{m(E)}{N(\delta, \epsilon)} \leftrightarrow x \in T(\mathfrak{A}, \lambda)\right) \geqq 1 - \epsilon.$$

Thus the proposed stochastic experiment for determining whether $x \in T(\mathfrak{A}, \lambda)$ can be performed without any a-priori knowledge of $p(x)$. This fact makes it natural to consider isolated cut-points.

VI. THE REDUCTION THEOREM

THEOREM 3. *Let $\mathfrak{A}$ be a probabilistic automaton and λ be an isolated cut-point satisfying* (1). *Then there exists a deterministic automaton $\mathfrak{B}$ such that $T(\mathfrak{A}, \lambda) = T(\mathfrak{B})$. If $\mathfrak{A}$ has n states and F consists of just one state then $\mathfrak{B}$ can be chosen to have e states where*

$$e \leqq [1 + (1/\delta)^{n-1}.^2 \tag{2}$$

PROOF: Let the set of states S be $\{s_0, \ldots, s_{n-1}\}$ and $F = \{s_{n-1}\}$. For every tape x, $A(x)$ is an $n \times n$ matrix and $p(x)$ is the upper left corner element of $A(x)$.

Let $x_1, \ldots, x_k$ be tapes which are pairwise *inequivalent* by $\equiv_{T((\mathfrak{A},\lambda)}$ (cf. Definition 3). Thus for every $i \leqq k$, $j \leqq k$, $i \neq j$, there exists a tape y such that

$$x_i y \in T(\mathfrak{A}, \lambda), \qquad x_j y \notin T(\mathfrak{A}, \lambda) \tag{3}$$

or vice-versa. Let the first *row* of $A(x_i)$, $1 \leqq i \leqq n$, be $(\xi_1^{\,i}, \cdots, \xi_n^{\,i})$ and the last *column* of $A(y)$, for the particular y appearing above, be $(\eta_1, \cdots, \eta_n)$. From $A(x_i y) = A(x_i)A(y)$ and $A(x_j y) = A(x_j)A(y)$ it follows that

$$p(x_i y) = \xi_1^{\,i}\eta_1 + \cdots + \xi_n^{\,i}\eta_n, \qquad p(x_j y) = \xi_1^{\,j}\eta_1 + \cdots + \xi_1^{\,j}\eta_n.$$

Combining with (3) we get

$$\lambda < \xi_1^{\,i}\eta_1 + \cdots + \xi_n^{\,i}\eta_n, \qquad \xi_1^{\,j}\eta_1 + \cdots + \xi_n^{\,j}\eta_n \leqq \lambda. \tag{4}$$

Since λ is isolated and $\delta \leqq |\, p(x) - \lambda \,|$ for $x \in \Sigma^*$, (4) implies

$$2\delta \leqq (\xi_1^{\,i} - \xi_1^{\,j})\eta_1 + \cdots + (\xi_n^{\,i} - \xi_n^{\,j})\eta_n. \tag{5}$$

[2] If F contains r states then the bound is $e \leqq (1 + (r/\delta))^{n-1}$ and the proof is essentially the same.

Taking absolute values and observing that the η_i, as elements of a stochastic matrix, satisfy $|\eta_i| \leqq 1$, (5) leads to

$$2\delta \leqq |\xi_1^i - \xi_1^j| + \cdots + |\xi_n^i - \xi_n^j| \quad \text{for} \quad i \neq j. \tag{6}$$

An argument involving volumes in n-dimensional space will now be used to infer from (6) a bound on k. The n-tuples $(\xi_1, \cdots, \xi_n)$ will be considered as points of Euclidean n-space. Let σ_i, $1 \leqq i \leqq k$ be the set

$$\sigma_i = \{(\xi_1, \cdots, \xi_n) \mid \xi_j^i \leqq \xi_j, 1 \leqq j \leqq n, \sum_j (\xi_j - \xi_j^i) = \delta\}.$$

Each σ_i is a translate of the set

$$\sigma = \{(\xi_1, \cdots, \xi_n) \mid 0 \leqq \xi_j, 1 \leqq j \leqq n, \sum_j \xi_j = \delta\}.$$

The set σ is readily seen to be an $(n - 1)$-dimensional simplex which is a subset of the hyperplane $x_1 + \cdots + x_n = \delta$. The $n - 1$ dimensional volume $V_{n-1}(\sigma)$ of σ, expressed as a function of δ, is $c\delta^{n-1}$ where c is some constant not depending on δ.

From $\sum_j \xi_j^i = 1$ it follows that $(\xi_1, \cdots, \xi_n) \in \sigma_i$ implies

$$\sum_j \xi_j = 1 + \delta, \qquad 0 \leqq \xi_j, 1 \leqq j \leqq n.$$

Thus $\sigma_i \subseteq \tau$ where

$$\tau = \{(\xi_1, \cdots, \xi_n) \mid \sum_j \xi_j = 1 + \delta, 0 \leqq \xi_j, 1 \leqq j \leqq n\}.$$

Figure 1 shows the sets σ_i, $1 \leqq i \leqq k$, and τ, for the two-dimensional case $n = 2$ and for $k = 3$. The point with coordinates (ξ_1^i, ξ_2^i) is denoted by P_i, $1 \leqq i \leqq 3$.

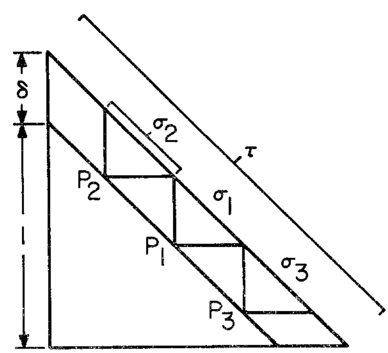

FIG. 1. The sets σ_i and τ for $n = 2$ and $k = 3$

A point $(\xi_1, \cdots, \xi_n) \in \sigma_i$ is an *interior point* of σ_i (in the topology of the hyperplane $x_1 + \cdots + x_n = 1 + \delta$) if and only if $0 < \xi_p - \xi_p^{\ i}$ for $1 \leqq p \leqq n$. Because of (6) σ_i and σ_j, $i \neq j$, have no interior points in common. For otherwise, if $(\xi_1, \cdots, \xi_n)$ is interior to both σ_i and σ_j, we would have $0 < \xi_p - \xi_p^{\ i}$, $0 < \xi_p - \xi_p^{\ j}$ and hence

$$| \xi_p^{\ i} - \xi_p^{\ j} | < | \xi_p - \xi_p^{\ i} | + | \xi_p - \xi_p^{\ j} |, \quad 1 \leqq p \leqq n.$$

Hence

$$\sum_p | \xi_p^{\ i} - \xi_p^{\ j} | < \sum_p | \xi_p - \xi_p^{\ i} | + \sum_p | \xi_p - \xi_p^{\ j} | = \delta + \delta,$$

contradicting (6).

Thus for $i \neq j$, σ_i and σ_j have no interior points in common. This implies

$$kc\delta^{n-1} = V_{n-1}(\sigma_1) + \cdots + V_{n-1}(\sigma_k) \leqq V_{n-1}(\tau) = c(1 + \delta)^{n-1}.$$

Hence $k \leqq [1 + (1/\delta)]^{n-1}$. Thus the number of e equivalence classes of the relation $\equiv_{T(\mathfrak{A},\lambda)}$ is at most $[1 + (1/\delta)]^{n-1}$. By Theorem 1, $T(\mathfrak{A}, \lambda)$ is definable by an automaton $\mathfrak{B}$ with e states.

VII. SAVING OF STATES

From the proof of the Reduction Theorem 3 and the estimate (2) given there, it seems possible that in passing from a p.a. $\mathfrak{A}$ to an equivalent deterministic automaton we may have to increase the number of states. In other words, the p.a. is more economical in terms of number of states. The following theorem shows that this does in fact happen in certain cases.

THEOREM 4. *There exists an automaton $\mathfrak{A}$ with just two states and a sequence λ_n, $1 \leqq n < \infty$, of isolated cut-points such that for each n, the automaton $\mathfrak{B}_n$ with the least number of states which satisfies $T(\mathfrak{A}, \lambda_n) = T(\mathfrak{B}_n)$ has at least n states.*

PROOF: Let $\Sigma = \{0, 2\}$, $S = \{s_0, s_1\}$, and $F = \{s_1\}$. Let the transition probabilities be such that

$$A(0) = \begin{bmatrix} 1 & 0 \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}, \quad A(2) = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ 0 & 1 \end{bmatrix}.$$

It is easy to verify that if $x = \delta_1\delta_2 \cdots \delta_n \in \Sigma^*$ then

$$p(x) = \frac{\delta_n}{3} + \frac{\delta_{n-1}}{3^2} + \cdots + \frac{\delta_1}{3^{n-1}}.$$

Remembering that $\delta_i \in \{0, 2\}$ we see that the topological closure C of the set $P = \{p(x) \mid x \in \Sigma^*\}$ is precisely Cantor's discontinuum.

Thus all the points λ, $0 \leqq \lambda < 1$, which satisfy $\lambda \notin C$ are isolated cut-points for $\mathfrak{A}$. Consider now the real number (written in *ternary* notation) $\lambda_n = \cdot 22 \cdots 211$ where the number of digits is $n + 1$. For $x \in \Sigma^*$ to satisfy $\lambda_n < p(x)$ it is necessary and sufficient that x have the form $x = x_1 22 \cdots 2$ where $x_1 \in \Sigma^*$ and the number of 2's is at least n. Thus the set $T(\mathfrak{A}, \lambda_n)$ is nonempty and if $x \in T(\mathfrak{A}, \lambda_n)$ then $n \leqq l(x)$. It follows from elementary theory of automata (see Rabin and Scott, 1959, Theorem 7) that the minimal deterministic automata $\mathfrak{B}_n$ for which $T(\mathfrak{A}, \lambda_n) = T(\mathfrak{B}_n)$ has at least $n + 1$ states (in fact, $\mathfrak{B}_n$ has exactly $n + 1$ states).

REMARK. An analysis of the possible values of $p(x)$ will show that for the λ_n as above

$$\liminf_{x \in \Sigma^*} | p(x) - \lambda_n | = \frac{1}{3^{n+1}} .$$

Thus in (2) we can take $\delta = 3^{-n-1}$ which gives for the number of states the bound $1 + 3^{n+1}$. In this case the bound turns out to be much too large. We do not know whether in other examples the bound is sharper or whether the bound in Theorem 3 can in fact be greatly improved.

VIII. APPROXIMATE CALCULATION OF MATRIX PRODUCTS

Let H be a finite set $H = \{P_1, \cdots, P_k\}$ of stochastic $n \times n$ matrices and let $0 < \epsilon$ be a given real number. The elements of all matrices $P \in H$ are assumed to be finite decimal fractions.[3] Consider the following computational task. At discrete time intervals $m = 1, 2, \cdots$, we are presented with matrices $P_{i_1} \in H$, $P_{i_2} \in H$, $\cdots$. Let $x_m = i_1 i_2 \cdots i_m$, $m = 1, 2, \cdots$. After each time m we wish to know, within ϵ, the element $p_{x_m}(1, n)$ of the product

$$\Pi_{x_m} = [p_{x_m}(i, j)]_{1 \leqq i \leqq n, 1 \leqq j \leqq n} = P_{i_1} P_{i_2} \cdots P_{i_m}$$

of the matrices given thus far. Since we are thinking here in terms of actual calculation (using, say, an actual computer with a fixed memory) it is not possible, in general, to solve our problem by calculating Π_{x_m}, at all times m, with complete accuracy. The elements $p_{x_m}(i, j)$ will have more and more decimals and recording and calculating with these

[3] This restriction on the matrices $P \in H$ is not essential and is included just in order that we can say that the matrices $P \in H$ are actually "given."

numbers will become impossible with increasing m. Nor is it possible to adopt a simple rounding-off procedure because the number m of matrix multiplications that we have to perform is not bounded in advance. Thus a fixed rounding-off procedure may result in a cumulative error which will become larger than ϵ.

We shall apply our reduction theorem (Theorem 3) to show that under certain conditions on H this problem of approximate calculation can be actually solved. The solution rests on the following:

THEOREM 5. *Let* $\Sigma = \{1, \cdots, k\}$ *and for* $x = i_1 i_2 \cdots i_m \in \Sigma^*$ *let* p_x *denote the* $(1, n)$ *element of the product* $P_{i_1} P_{i_2} \cdots P_{i_m}$. *Assume that* H *is such that* $V = \{p_x \mid x \in \Sigma^*\}$ *is nowhere dense in the interval* $[0, 1]$. *Then for every* $0 < \epsilon$ *there exists an integer* h, *real numbers* $\lambda_1, \cdots, \lambda_h$, *and deterministic automata* $\mathfrak{A}_1, \cdots, \mathfrak{A}_h$ *over* Σ *such that*

(i) $0 = \lambda_1 < \lambda_2 < \cdots < \lambda_h = 1$, $\lambda_{i+1} - \lambda_i < \epsilon$, $1 \leqq i < h$.

(ii) $\lambda_i \leqq p_x \leqq \lambda_{i+1}$ iff $x \in T(\mathfrak{A}_i) - T(\mathfrak{A}_{i+1})$, $1 \leqq i < h$.

PROOF: V nowhere dense means that the topological closure $\bar{V}$ does not contain any nontrivial interval. Thus there exists, for some integer h, a sequence λ_i, $1 \leqq i \leqq h$, satisfying (i) and also $\lambda_i \notin \bar{V}$ for $2 \leqq i \leqq h - 1$.

Consider the p.a. $\mathfrak{A}$ over Σ having the states $s_0, \cdots, s_{n-1}$, the set $\{s_{n-1}\}$ of designated final states, and transition probabilities such that the matrix corresponding to $i \in \Sigma$ is P_i. We have for $x \in \Sigma^*$, $p(x) = p_x$. The numbers λ_i, $2 \leqq i \leqq h - 1$ are isolated cut-points for $\mathfrak{A}$. Thus by Theorem 3 the set $T(\mathfrak{A}, \lambda_i)$, $2 \leqq i \leqq h - 1$, is definable by some deterministic automaton $T(\mathfrak{A}_i)$. Hence $\lambda_i < p_x$ for $2 \leqq i \leqq h - 1$ ($p_x = \lambda_i$ is not possible since λ_i is isolated) iff $x \in T(\mathfrak{A}_i)$. Let $\mathfrak{A}_1$ and $\mathfrak{A}_h$ be automata such that $T(\mathfrak{A}_1) = \Sigma^*$ and $T(\mathfrak{A}_h) = \phi$. The automata $\mathfrak{A}_i$, $1 \leqq i \leqq h$ satisfy (ii).

REMARK. The condition concerning V is satisfied, for example, by the set $H = \{A(0), A(2)\}$ of 2×2 matrices defined in Theorem 4. In this case $\bar{V}$ is Cantor's discontinuum. We do not have, however, a criterion for deciding whether a given H satisfies the condition.

The method for approximate calculation of $p_{x_m}(1, n)$ in the case that H satisfies the condition of Theorem 5 is now as follows. Given $0 < \epsilon$, let λ_i, $\mathfrak{A}_i$, $1 \leqq i \leqq h$, satisfy the conditions (i) and (ii) of Theorem 5. Using just a *fixed* amount of computer memory it is possible to simulate the automata $\mathfrak{A}_i$, $1 \leqq i \leqq h$. As the matrices P_{i_1}, P_{i_2}, $\cdots$, are given, the indexes $i_1, i_2, \cdots$, are fed into the simulated automata. At each time instant m the computer checks which automata $\mathfrak{A}_i$ accepted $x_m = i_1 i_2 \cdots i_m$. There exists precisely one j_m such that $x_m \in T(\mathfrak{A}_{j_m}) - T(\mathfrak{A}_{j_m+1})$. For λ_{j_m} we have $| p_{x_m}(1, n) - \lambda_{j_m} | < \epsilon$ and we take λ_{j_m} as the approximation for $p_{x_m}(1, n)$.

Thus we have proved the existence of an approximate calculation procedure which can actually be carried out by a computer. We do not know of a classical numerical-analysis method for obtaining this result. In fact, an example due to R. E. Stearns shows that without assumptions on H, a computational procedure need not exist.

IX. ACTUAL AUTOMATA

In certain actual situations it is natural to assume about an automaton $\mathfrak{A}$ that all transitions between states have strictly positive (though sometimes very small) probabilities. This motivates the following definition.

DEFINITION 9. A p.a, $\mathfrak{A}$ is called an *actual automaton* if for all $s \in S$, $s_i \in S$, and $\sigma \in \Sigma$ the transition probability $p_i(s, \sigma)$ of moving from state s to state s_i under input σ satisfies $0 < p_i(s, \sigma)$.

X. PRODUCTS OF POSITIVE STOCHASTIC MATRICES

It turns out that actual automata have very special properties. To study them we need some results about products of strictly positive stochastic matrices. The following Lemma 6 is a restatement, in our notation, of Theorem 4.1.3 of Kemeny and Snell (1960); the proof is included for the sake of completeness. Corollary 7 and Lemma 8 are closely related to Theorems 4.1.4–4.1.6 of Kemeny and Snell (1960) except that we treat products of several matrices instead of powers of a single matrix. The possibility of this generalization was pointed out by Mr. A. Paz.

DEFINITION 10. If $\alpha = [a_i]_{1 \leqq i \leqq n}$ is a column vector then $\| \alpha \|$ is defined as $\| \alpha \| = \max_i a_i - \min_i a_i$. If A is an $n \times n$ matrix having columns $\alpha_1, \cdots, \alpha_n$ then $\| A \|$ is defined by $\| A \| = \max_i \| \alpha_i \|$.

LEMMA 6. *If* $P = [p_{ij}]_{1 \leqq i, j \leqq n}$ *is a stochastic matrix and* $\Delta = min_{i,j}\, p_{ij}$ *and if* $\alpha = [a_i]_{1 \leqq i \leqq n}$ *is a column vector then*

$$\| P\alpha \| \leqq (1 - 2\Delta) \| \alpha \|.$$

PROOF: Let $P\alpha = [b_i]_{1 \leqq i \leqq n}$. We may assume, without loss of generality, that $b_1 = \max_i b_i$, $b_2 = \min_i b_i$, $a_1 = \max_i a_i$, and $a_2 = \min_i a_i$.

We have

$$b_1 = p_{11}a_1 + p_{12}a_2 + \cdots + p_{1n}a_n \leqq p_{11}a_1 + p_{12}a_2 + p_{13}a_1 + \cdots + p_{1n}a_1$$
$$= a_1 - p_{12}(a_1 - a_2).$$

Similarly, replacing in the sum for b_2 the a_1 by a_2, $b_2 \geqq a_2 + p_{21}(a_1 - a_2)$. Thus $\| P\alpha \| = b_1 - b_2 \leqq (a_1 - a_2)(1 - p_{12} - p_{21})$. But $a_1 - a_2 = \| \alpha \|$ and, since $\Delta \leqq p_{12}$ and $\Delta \leqq p_{21}$, we have $1 - p_{12} - p_{21} \leqq 1 - 2\Delta$. This establishes the lemma.

COROLLARY 7. If $H = \{P_1, \cdots, P_k\}$ where the matrices P_i, $1 \leqq i \leqq k$ are stochastic and all the elements of the P_i are greater than $0 < \Delta$ then for any $1 \leqq i_1, \cdots, i_m \leqq k$,

$$\| P_{i_1}P_{i_2} \cdots P_{i_m} \| \leqq (1 - 2\Delta)^{m-1}.$$

PROOF: The column vectors $\alpha_1, \cdots, \alpha_n$ of P_{i_m} satisfy $\| \alpha_i \| \leqq 1$. By Lemma 6, the columns $\beta_1, \cdots, \beta_m$ of $P_{i_{m-1}}P_{i_m}$ satisfy $\| \beta_i \| \leqq 1 - 2\Delta$. Repeating this argument $m - 1$ times we get the result.

For any $m \times n$ matrix $A = [a_{ij}]$ we define $|A| = \max_{i,j} |a_{ij}|$. This $|A|$ clearly has the usual properties of a norm.

LEMMA 8. If P is a stochastic $n \times n$ matrix and $\alpha = [a_i]_{1 \leqq i \leqq n}$ is a column vector then

$$|P\alpha - \alpha| \leqq \| \alpha \|.$$

PROOF: Let $(p_1, \cdots, p_n)$ be the first row of P and let b_1 be the first element of $P\alpha$. Then

$$|b - a_1| = |p_1a_1 + \cdots + p_na_n - a_1| \leqq p_2 |a_2 - a_1| + \cdots$$
$$+ p_n |a_n - a_1| \leqq \| \alpha \|.$$

The same applies to all the other elements of $P\alpha$.

COROLLARY 9. *If P is a stochastic $n \times n$ matrix and A is an $n \times n$ matrix then* $|PA - A| \leqq \| A \|$.

XI. ACTUAL AUTOMATA AND DEFINITE EVENTS

It will turn out that the sets accepted by actual automata are just those described in the following.

DEFINITION 11. A set $T \subseteq \Sigma^*$ is called a *definite event* if for some integer k the following holds. If $k \leqq l(x)$ then $x \in T$ if and only if $x = yz$ where $k = l(z)$ and $z \in T$.

In (Perles, Rabin, and Shamir, 1963) the properties of definite sets and the (deterministic) automata defining them are studied in detail.

THEOREM 10. *If $\mathfrak{A}$ is an actual automaton and λ is an isolated cut-point then $T(\mathfrak{A}, \lambda)$ is a definite set. Conversely, every definite set is definable by some actual automaton with isolated cut-point.*

PROOF: Let $0 < \delta \leqq |p(x) - \lambda|$ and assume that all elements of the stochastic matrices $A(\sigma)$, $\sigma \in \Sigma$ (see Definition 5) are greater than $0 < \Delta$. We assume that $\mathfrak{A}$ has just one designated final state, say s_{n-1}. The proof is essentially the same for the general case.

Let k be such that $(1 - 2\Delta)^{k-1} < 2\delta$. For any $z = \sigma_1 \cdots \sigma_k \in \Sigma^*$ of length k the matrix $A(z)$ equals $A(\sigma_1) \cdots A(\sigma_k)$ and thus, by Corollary 7, satisfies $\| A(z) \| \leqq (1 - 2\Delta)^{k-1} < 2\delta$.

Since $p(x)$, for $x \in \Sigma^*$, is the $(1, n)$ element of $A(x)$ we have, by Corollary 9, $|p(yz) - p(z)| \leqq |A(y)A(z) - A(z)| \leqq \| A(z) \|$. Thus for z satisfying $l(z) = k$, $|p(yz) - p(z)| < 2\delta$. Hence $yz \in T(\mathfrak{A}, \lambda)$ if and only if $z \in T(\mathfrak{A}, \lambda)$ which proves that $T(\mathfrak{A}, \lambda)$ is definite.

The converse is proved by explicit construction of the actual automaton defining the definite set T. We leave out the details.

XII. THE STABILITY PROBLEM

Consider a p.a. $\mathfrak{A}$ and an isolated cut-point λ. It is natural to ask whether the set $T(\mathfrak{A}, \lambda)$ remains unchanged (*stable*) under small perturbations of the transition probabilities of $\mathfrak{A}$. Results along this line we shall call stability theorems.

THEOREM 11. *Let $\mathfrak{A} = \langle S, M, s_0, F \rangle$ be an actual automaton and λ be an isolated cut-point. There exists an $0 < \epsilon$ such that for every automaton $\mathfrak{A}' = \langle S, M', s_0, F \rangle$ with transition probabilities differing from those of $\mathfrak{A}$ by less than ϵ, λ is an isolated cut-point of $\mathfrak{A}'$ and $T(\mathfrak{A}, \lambda) = T(\mathfrak{A}', \lambda)$.*

PROOF: Let $A(\sigma)$ and $A'(\sigma)$ be the matrices corresponding to $\sigma \in \Sigma$ in the automata $\mathfrak{A}$ and $\mathfrak{A}'$ respectively. Let Δ be the smallest element in the $A(\sigma)$ and Δ' be the smallest element in the matrices $A'(\sigma)$. We shall show that for every $0 < \delta_1$ we can find an $0 < \epsilon$ so that

$$|A(\sigma) - A'(\sigma)| < \epsilon, \qquad \sigma \in \Sigma \tag{7}$$

implies for all $x = \sigma_1 \cdots \sigma_m$

$$|A(x) - A'(x)| = |A(\sigma_1) \cdots A(\sigma_m) - A'(\sigma_1) \cdots A'(\sigma_m)| < \delta_1 \tag{8}$$

This, of course, implies the theorem.

Let k be such that $(1 - 2\Delta)^{k-1} < \delta_1/3$. We can choose $0 < \epsilon$ small enough so that (7) will imply (a) $(1 - 2\Delta')^{k-1} < \delta_1/3$, (this can be done because $\Delta = \lim_{\epsilon \to 0} \Delta'$); (b) for all z such that $l(z) \leqq k$, $|A(z) - A'(z)| < \delta_1/3$.

If $l(x) \leqq k$ then (8) holds trivially because of (b). If $k \leqq l(x)$ then $x = yz$ where $l(z) = k$. The matrix $A(z)$ is a product of k of the matrices $A(\sigma)$ and therefore, by Corollary 7, $\| A(z) \| \leqq (1 - 2\Delta)^{k-1} < \delta_1/3$. Similarly, using (a), we have $\| A'(z) \| < \delta_1/3$. Now

$$|A(x) - A'(x)| \leqq |A(y)A(z) - A(z)| + |A'(y)A'(z) - A'(z)| + |A(z) - A'(z)|.$$

By Corollary 9 and (b) each of the summands on the right is less than $\delta_1/3$.

Remark. Theorem 11 could not hold in full generality for arbitrary p.a. with isolated cut-point. For assume that $\mathfrak{A}$ is a p.a. with isolated cut-point λ such that $T(\mathfrak{A}, \lambda)$ is not a definite event. An automaton $\mathfrak{A}'$ may satisfy (7) and yet have only strictly positive transition probabilities. In this case either λ is not an isolated cut-point of $\mathfrak{A}'$, or $T(\mathfrak{A}', \lambda)$ is a definite event and hence $T(\mathfrak{A}', \lambda) \neq T(\mathfrak{A}, \lambda)$.

Thus a proposed formulation of a conjectured general stability theorem would be: If $\mathfrak{A}$ is a p.a. and λ is isolated cut-point then there exists an $0 < \epsilon$ such that for every automaton $\mathfrak{A}'$ with conditions as in Theorem 11 and such that the matrices $A'(\sigma)$ *have zeros where* $A(\sigma)$ *had zeros*, λ is an isolated cut-point of $\mathfrak{A}'$ and $T(\mathfrak{A}', \lambda) = T(\mathfrak{A}, \lambda)$. H. Kesten constructed a neat counterexample to this conjecture. Thus the problem of giving suitable extensions of Theorem 11 is completely open.

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