Beyond Initial Algebras and Final Coalgebras

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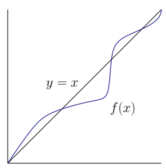
Abstract

We provide a construction of the fixed points of functors which may not be inital algebras or final coalgebras. For an endofunctor F, this fixpoint construction may be expressed as a pair of adjoint functors between F-coalgebras and F-algebras. We prove a version of the limit colimit coincidence theorem for these generalized fixed points.

1 Middle Fixed Points

The Knaster-Tarski theorem provides a construction of least and greatest fixed points for a monotone function $f : L \to L$ on a complete lattice L. Consider the following monotone function on the lattice $([0,1], \leq)$ of the interval of real numbers with the usual ordering.



The function f is overlayed with the function $y = x$. The intersection of the two curves indicate fixpoints of f. The least fixpoint of f is 0 and the greatest fixpoint is 1 but there are 3 other fixpoints in-between. These "middle" fixpoints have a similar construction to the least and greatest ones. Given a "pre-fixed point" i.e. a point $x \in [0,1]$ such that $x \leq f(x)$ we may find the first fixpoint above x as

$$\mu(x) = \sup\{x, f(x), f^2(x), f^3(x), \ldots\}$$

where the $\ldots$ indicate iteration to a sufficiently large ordinal. Similarly, given a "post-fixed point" $f(y) \leq y$, we may find the closest fixpoint below y

$$\nu(y) = \inf\{y, f(y), f^2(y), f^3(y), \ldots\}$$

. For a complete lattice L, let $Pre(f)$ be the suborder of L consisting of only the pre-fixed points $x \leq f(x)$. Similarly, let $Post(f)$ be the suborder of post fixed points $f(y) \leq y$. Then there is a Galois connection

$$Pre(f) \underset{\nu}{\overset{\mu}{\rightleftarrows}} Post(f) \quad \bot$$

Being a Galois connection means that

$$\mu(x) \leq y \iff x \leq \nu(y)$$

In this abstract we will generalized this Galois connection to fixpoints of functors rather than monotone functions. When generalizing from posets to categories we make the replacements shown in Table 1.

2 The Adjunction

In what follows, ω will be the category finite ordinals i.e. your favorite category with a countable number of objects and one generating morphism from each object to the next. Pictorally, ω is the category

$$\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \ldots$$

Theorem 2.1. *Suppose C is a category with colimits of shape ω and limits of shape ω^{op} and suppose that $F : C \to C$ preserves limits and colimits of these shape. Then there is an adjunction*

$$\mathsf{Coalg}(F) \underset{\nu}{\overset{\mu}{\rightleftarrows}} \mathsf{Alg}(F) \quad \bot$$

given by

$$\mu(b\colon B \to FB) =$$

$$\mathrm{colim}(\ B \xrightarrow{b} F(B) \xrightarrow{Fb} F^2(B) \xrightarrow{F^2(b)} \cdots\ )$$

$$\nu(a\colon FA \to A) =$$

$$\lim(\ A \xleftarrow{a} F(A) \xleftarrow{Fa} F^2(A) \xleftarrow{F^2(a)} \cdots\ )$$

and defined on morphisms using the universal property of limits and colimits.

Poset	Category
Monotone Function f	Functor F
Pre-fixed point of f	F-coalgebra
Post-fixed point of f	F-algebra
$\sup\{f(x), f^2(x), f^3(x), \ldots\}$	$\text{colim}(X \to F(X) \to F^2(X) \to F^3(X) \ldots)$
$\inf\{f(x), f^2(x), f^3(x), \ldots\}$	$\lim(X \leftarrow F(X) \leftarrow F^2(X) \leftarrow F^3(X) \ldots)$
Galois connection	Adjunction

Figure 1: Generalization of Posets to Categories

Proof. (Sketch) The adjunction isomorphism

$$\mathsf{Alg}(F)(\mu(b), a) \cong \mathsf{Coalg}(F)(b, \nu(a))$$

for algebras $a\colon FA \to A$ and coalgebras $b\colon B \to FB$ relies on the fact that both sets are naturally isomorphic to the set of coalgebra to algebra homomorphisms from b to a. Given a coalgebra to algebra homomorphism

$$\begin{array}{ccc} B & \xrightarrow{b} & F(B) \\ \downarrow f & & \downarrow F(f) \\ A & \xleftarrow[a]{} & F(A) \end{array}$$

we may iterate it countably many times to get a diagram

$$\begin{array}{ccccccc} B & \xrightarrow{b} & F(B) & \xrightarrow{Fb} & F^2(B) & \xrightarrow{F^2 b} & F^3(B)\cdots \\ f\downarrow & & \downarrow F(f) & & \downarrow F^2(f) & & \downarrow F^3(f)\cdots \\ A & \xleftarrow[a]{} & F(A) & \xleftarrow[Fa]{} & F^2(A) & \xleftarrow[F^3 a]{} & F^3(A)\cdots \end{array}$$

The colimit of the top row is $\mu(b)$ and the maps going down and left form a cocone over the diagram for $\mu(b)$. Therefore the universal property for colimits induces a morphism $\mu(b) \to a$ and we state without proof that this is an algebra homomorphism. Similarly, the limit of the bottom row is the coalgebra $\nu(a)$ and the morphisms going right and down form a cone. The universal property of limits supplies a morphism $b \to \nu(a)$. We state without proof that this morphism is a coalgebra homomorphism and that the correspondences described here are natural in both arguments. □

Let 1 be the terminal object of C and let 0 be the initial object. Then there is a unique algebra $1 : F1 \to 1$ and $\nu(1)$ is the terminal coalgebra. Similarly, the initial algebra is given by $\mu(0)$ for the unique coalgebra $0 : 0 \to F0$. The initial algebra and final coalgebra represent finite and infinite traces respectively [Rut00]. For a coalgebra $c : X \to FX$, the algebra $\mu(c)$, may be interpreted as a semantic object for c which represents neither finite or infinite traces. When F is a Set-functor, $\mu(c)$ contains elements of both finite traces and infinite traces. In the colimit for μ, the constants of F generate a copy of its inital algebra. For each element $x \in X$, there is an object of $\mu(c)$ representing its *orbit*. In the colimit for μ every object is identified with its successor, so in the algebra $\mu(c)$ there is one element for each equivalence class generated by the transitive closure of the sucessor relation. Note that the infinite trace semantics is given by the unique map $c \to \nu(1)$. Transferring this map accross the adjunction gives the unique morphism $\mu(c) \to 1$. To us, this suggests that the algebra $\mu(c)$ is somehow *precompiling* the final trace semantics of c. Before moving on to the next section we state a corollary.

Corollary 2.1. *The initial algebra for F is recursive and the final coalgebra for F is corecursive.*

Proof. A recursive algebra is one for which there is a unique coalgebra to algebra homomorphism into it for any coalgebra. Similarly, a corecursive coalgebra has a unique coalgebra to algebra morphism coming out of it for any algebra. The proof of the adjunction implies that $\mathsf{Coalg}(F)(c, \nu(1)) \cong \mathsf{CoAlgToAlg}(c, 1)$ where the latter set is the set of coalgebra to algebra homorphisms into the terminal algebra. This set has a unique element implying that $\nu(1)$ is corecursive. A similar proof holds for the dual statement. □

3 μ and ν Coincide With a Dagger

When coalgebras for a polynomial functor $F : \mathsf{Set} \to \mathsf{Set}$ are interpreted as F-shaped automata, the initial F-algebra serves as finite trace semantics and the terminal F-coalgebra gives an infinite trace semantics. When F is no longer a Set-functor this interpretation breaks down. For example if $F : \mathsf{Rel} \to \mathsf{Rel}$, where Rel is the category of sets and relations, then the initial algebra and terminal coalgebra coincide [SP82]. In [Kar19], it is shown that this holds more generally in any dagger category. With this coincidence, the initial algebra/final coalgebra gives a finite trace semantics instead of an infinite trace semantics. To obtain a semantics for infinite traces, Urabe and Hasuo construct an object which is weakly terminal among coalgebras and define the infinite trace semantics as the maximal map into this object [HU18]. Note that the limit colimit coincidence causes no issues when $\mu(c)$ is interpreted as a semantic object for c. However, A generalized limit colimit coincidence also holds for the fixed points generated by μ and ν.

Definition 3.1. *A dagger category* $(C, \dagger)$ *is a category equipped with a functor* $\dagger : C \to C^{\mathrm{op}}$ *such that* $\dagger^2 = id$.

Theorem 3.1. *Suppose that* $(C, \dagger)$ *is a dagger category with limits and colimits of countable chains and* $F : C \to C$ *is a dagger functor preserving such limits and colimits. Then there is an isomorphism*

$$\mu(c)^\dagger \cong \nu(c^\dagger)$$

for each coalgebra c*. Dually, for each algebra* a*, there is an isomorphism* $\nu(a)^\dagger \cong \mu(a^\dagger)$*.*

Proof. For a coalgebra $X \xrightarrow{c} FX$ we have

$$\begin{aligned}
\nu(c^\dagger) &\cong \lim(X \xleftarrow{c^\dagger} FX \xleftarrow{Fc^\dagger} F^2X \leftarrow \dots) \\
&\cong \mathrm{colim}_{C^{\mathrm{op}}}(X \xleftarrow{c^\dagger} FX \xleftarrow{Fc^\dagger} F^2X \leftarrow \dots) \\
&\cong \mathrm{colim}(X \xrightarrow{c} FX \xrightarrow{Fc} F^2X \to \dots) \\
&\cong \mu(c)^\dagger
\end{aligned}$$

The second isomorphism is because limits in C are colimits in C^{op} and the third isomorphism is because $\dagger$ preserves colimits because it is an equivalence. A similar proof holds for the dual statement. □

4 Conclusion

In this extended abstract, we have argued that *middle fixpoints* i.e. fixpoints which are neither initial or terminal are interesting enough to merit further study. The adjunction $\mu \vdash \nu$, is closely related to the concept of coalgebra to algebra homomorphisms. These have been studied in the case when they are unique through the notions or recursive algebras and corecursive coalgebras [CUV09]. However, in [H+15], the author argued that coalgebra to algebra morphisms also hold interest when they are not unique. Using examples in probability, dynamical systems, and a game theory, the authors showed how non-unique coalgebra to algebra morphisms often represent solutions to problems in these disciplines. A morphism out of $\mu(c)$ represents a coalgebra to algebra homomorphism originating in c and dually for $\nu(a)$. In future work, we hope to recast the examples given in [H+15] in terms of our adjunction μ and ν to further explore properties of these fixpoints as semantics for coalgebras and algebras.

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