On the computational expressivity of (circular) proofs with fixed points

Gianluca Curzi and Anupam Das

Cyclic proofs are an emerging topic of proof theory that is attracting increasing interest in the literature. This area originates (in its modern guise) in the context of the modal μ-calculus [NW96, DHL06], serving as an alternative framework to manipulate least and greatest fixed points, and hence to model inductive and coinductive reasoning as well as (co)recursion mechanisms.

Cyclic proof theory has been investigated in many settings, such as first-order inductive definitions [BS11, BT19], Kleene algebras [DP17, DP18], automata [KPP19, DBHS16, Dou17], continuous cut-elimination [FS13, BDS16], linear logic and proof nets [BDS16, DS19], arithmetic [Sim17, BT17, Das20b], Gödel's system T [Das20a, KPP21, Das21], and complexity [CD22a, CD22b].

In this paper we study the computational strength of $\mu\mathsf{LJ}$ and its circular presentation $\mathsf{C}\mu\mathsf{LJ}$, which are extensions of intuitionistic logic with least and greatest fixed points introduced by Clairambault in [Cla09, Cla13]. More specifically, we show that the number-theoretic functions representable in $\mu\mathsf{LJ}$ and $\mathsf{C}\mu\mathsf{LJ}$ are exactly those provably total in $\mu\mathsf{PA}$, a first-order arithmetic with generalised inductive definitions (see, e.g., [Mos08]). Our fundamental theorem will be established via a series of inclusions comparing the computational expressivity of $\mu\mathsf{LJ}$ and $\mathsf{C}\mu\mathsf{LJ}$ with various theories of arithmetic:

$$\mu\mathsf{PA} \overset{(i)}{\subseteq} \mu\mathsf{HA} \overset{(ii)}{\subseteq} \mu\mathsf{LJ} \overset{(iii)}{\subseteq} \mathsf{C}\mu\mathsf{LJ} \overset{(iv)}{\subseteq} \Pi^1_2\text{-}\mathsf{CA}_0 \overset{(v)}{\subseteq} \mu\mathsf{PA}$$

We first prove Π^0_2-conservativity of $\mu\mathsf{PA}$ over its intuitionistic version, $\mu\mathsf{HA}$, by standard double-negation translations (i). Secondly, we show that the provably total functions of $\mu\mathsf{HA}$ are representable in $\mu\mathsf{LJ}$ using standard realisability techniques (ii). Thirdly, we show a simulation result relating $\mu\mathsf{LJ}$ and $\mathsf{C}\mu\mathsf{LJ}$ (iii). The most relevant contribution of this paper is the inclusion (iv), where we formalise a totality argument for circular proofs in Π^1_2-CA_0, the subsystem of second-order arithmetic with Π^1_2-comprehension and set induction. In particular, the totality argument is based on hereditary recursive models. We conclude by leveraging on a recent result by Möllerfeld in [Mö03], who showed that Π^1_2-CA_0 is arithmetically conservative over $\mu\mathsf{PA}$ (v).

As a future work, we would like to extend the above methods to other fixed point logics, such as $\mu\mathsf{LL}$ (i.e., linear logic with least and greatest fixed points) [EJ21, EJS21] and its multiplicative-additive fragment $\mu\mathsf{MALL}$ [BM07, BDS16]. Also, we are planning to investigate the computational strength of notable subsystems of $\mu\mathsf{LJ}$, such as the those restricting fixed points to parameter-free formulas or to strictly positive formulas.

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