CONSTRUCTIVENESS AND RASIOWA-SIKORSKI FOR THE MODAL μ-CALCULUS

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The goal of this talk is to illustrate a general technique to obtain completeness theorems for the modal μ-calculi and variants, where least fixed points are axiomatized via the Park/Kozen induction rule. This technique, already in use in [4], allowed there to prove completeness of the alternation free fragment of the modal μ-calculus with respect to the class of *complete* μ-modal algebras.

Only recently we realized that the same technique allows to obtain completeness with respect to the class of complex modal algebras, that is, power sets of Kripke frames. Otherwise said, the technique yields a completeness theorem for the alternation free fragment of the modal μ-calculus with respect to the class of Kripke frames.

We sketch next this technique, that relies on algebraic and order-theoretic ideas. A *modal μ-algebra* (see [4, Section 3]) is an algebraic model of the modal μ-calculus. That is, it is a modal algebra $\mathcal{B} = (B, \top, \wedge, \bot, \vee, \neg, \Box)$—therefore, with $(B, \top, \wedge, \bot, \vee, \neg)$ a Boolean algebra and $\Box$ a unary operator that distributes with finite meets—which is complete enough to interpret μ-terms of the modal μ-calculus, as expected from the Park/Kozen axiomatization of least fixed points:

$$\| \mu_x.t \|_v := \bigwedge \{ b \in B \mid \mathbf{t}(b) \leq b \} ,$$

where $\mathbf{t}$ is the monotone map sending $b \in B$ to $\| t \|_{v,x \mapsto b}$ and $v, x \mapsto b$ is the valuation equal to v except that the propositional variable x takes b as value. We say that a modal μ-algebra is *constructive* if, for each μ-term of the form $\mu_x.t$, we have

$$\| \mu_x.t \| = \bigvee_{\alpha \text{ an ordinal}} \mathbf{t}^\alpha .$$

Above, the approximants $\mathbf{t}^\alpha$ are defined as usual by ordinal induction from the bottom of the lattice:

$$\mathbf{t}^0 := \bot , \quad \mathbf{t}^{\alpha+1} := \mathbf{t}(\mathbf{t}^\alpha) , \quad \text{and, for a limit ordinal } \beta, \quad \mathbf{t}^\beta := \bigvee_{\alpha < \beta} \mathbf{t}^\alpha .$$

A complete modal algebra is always constructive. However, Lindenbaum algebras of a logic, which we need to consider when proving completeness results, are seldom complete: they are such only if they turn out to be finite (for example, for the locally tabular modal logics).

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The following proposition allows to reduce, for the modal μ-calculus, completeness to constructiveness of its Lindenbaum algebras:

Proposition. *Let $\mathcal{B}$ be a countable constructive modal μ-algebra. Then $\mathcal{B}$ has an embedding into the complex algebra of a Kripke frame.*

The completeness strategy worked out in [4] amounts exactly to show that Lindenbaum algebras (that is, free modal μ-algebras generated by a countable set of propositional variables) are constructive.

The embedding mentioned in the Proposition is the Stone-Rasiowa-Sikorski embedding into the powerset of prime Q-filters of $\mathcal{B}$, see [1] and [5] for applications in modal logic. This embedding is parametrised by Q, a countable set of possibly infinite infima and suprema that we aim to preserve. Formally, Q is a pair of collections $\{ X_n \}_{n \geq 0}$ and $\{ Y_{n \geq 0} \}$, with $X_n, Y_n \subseteq B$, such that $\bigwedge X_n, \bigvee Y_n$ both exist in $\mathcal{B}$. A prime filter (or ultrafilter) F of $\mathcal{B}$ is a Q-filter if $X_n \subseteq F$ implies $\bigwedge X_n \in F$, and $\bigvee Y_n \in F$ implies $Y_n \cap F \neq \emptyset$. Then, the Stone map sending $b \in B$ to the set $\{ F \subseteq B \mid F \text{ a } Q\text{-filter}, b \in F \}$ is a Boolean homomorphism from B to the powerset of the set of Q-filters, which preserves the meets $\bigwedge X_n$ and joins $\bigvee Y_n$; moreover, it is an embedding, thus leading to completeness results when $\mathcal{B}$ is a Lindenbaun algebra.

If $\mathcal{B}$ is a constructive μ-algebra, then we aim to preserve the suprema $\mathbf{t}^{\beta}$, for each μ-term $\mu_x.t$. Since $\mathcal{B}$ is countable, then so are the suprema that we need to preserve w.r.t. $\mu_x.t$. Since all the μ-terms of the form $\mu_x.t$ are countable, all the suprema we aim to preserve are countable. It follows then, in a way similar to [4, Lemma 3.5], that the interpretation of all the μ-terms is preserved from $\mathcal{B}$ to its Stone-Rasiowa-Sikorski extension.

Let us summarize what we can achieve, at the moment of writing, with this technique:

Proposition. *Each countably generated free modal μ-algebra has a modal algebra homomorphism into the complex algebra of a frame that preserves all the μ-terms in the alternation free fragment.*

The result above, yielding a completeness theorem for the alternation free fragment of the modal μ-calculus, is independent of existing proofs of completeness. However, using the completeness and the finite model theorems, we know that a countably generated free modal μ-algebra is constructive. Using this, recalling that modal μ-algebras have an equational axiomatization [3], and relying on the description of congruences given in [2], we can transfer constructiveness to finitely presented algebras:

Proposition. *Every finitely presented modal μ-algebra is constructive, thus it has an embedding into the complex algebra of some frame.*

It is tempting to study how constructiveness transfers from free modal μ-algebras, which algebraically code the modal μ-calculus, the fixed point extensions of the modal logic **K**, to free modal μ-algebras coding other fixed point extensions of modal logics over **K**, such as for example **T**, **B**, **S4**. Except for trivial cases, these free modal μ-algebras are not finitely presentable. If any kind of transfer theorem can be derived, then the completeness issue is reduced to the to the well-known canonicity problem in modal logic.

References

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