Ill-founded proofs for intuitionistic temporal logic

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Intuitionistic temporal logic Temporal logics such as Linear-time Temporal Logic (LTL) and Computation Tree Logic (CTL) are well studied logics in computer science. Advances in the proof theory of these logics have shown that ill-founded and cyclic proof calculi are particularly suitable for capturing the behaviour of their fixed point operators in a syntactic way (see e.g. [5, 3, 4, 1]). So far, the study of ill-founded proof systems for temporal logics has remained within the classical realm and their applicability to intuitionistic temporal logics largely unexplored. In this talk we present work towards filling this gap in the form of a sound and complete ill-founded sequent calculus for a fragment of intuitionistic LTL.

The known techniques for constructing ill-founded and cyclic systems for classical fixed point logics do not generalise to the intuitionistic case in a straightforward way, as the semantics of the intuitionistic logics is more involved. A standard way to present the semantics of intuitionistic logic is in terms of Kripke models $(W, \leq, V)$, where $\leq$ is a partial order on the set of worlds W and V a valuation that is monotone in $\leq$. A key property of this semantics is the monotonicity lemma: for all $s, t \in W$,

$$\text{if } s \leq t \text{ and } s \vDash \phi, \text{ then } t \vDash \phi.$$

The semantics of intuitionistic modal/temporal logics can be given in terms of intuitionistic Kripke models $(W, \leq, V)$ equipped with an additional, modal accessibility relation R on W. In order to satisfy the monotonicity lemma, which is generally taken as a minimal constraint on any intuitionistic version of a modal logic,[1] the relations R and $\leq$ need to satisfy some confluence property.

The intuitionistic temporal logic $\mathsf{ITL_F}$ was introduced by Boudou, Diéquez and Fernández-Duque [2] and given a complete Hilbert-style axiomatisation. Formulas of $\mathsf{ITL_F}$ are given by the following grammar:

$$\phi ::= \bot \mid p \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \to \phi \mid \mathsf{X}\phi \mid \mathsf{F}\phi$$

[1] See [9] for a discussion on what qualifies as 'the' intuitionistic version of a modal logic.

where p ranges over a set of propositional constants. Formulas $\mathsf{X}\phi$ and $\mathsf{F}\phi$ represent 'next ϕ' and 'eventually ϕ', respectively. The formulas of $\mathsf{ITL_F}$ are evaluated on *dynamic models*: an intuitionistic Kripke model $(W, \leq, V)$ equipped with a function $f : W \to W$ that maps each state to its temporal successor and that satisfies *forward confluence*:

$$\text{if } s \leq t, \text{ then } f(s) \leq f(t).$$

An ill-founded system for $\mathsf{ITL_F}$ We present a cut-free, ill-founded sequent calculus $\mathsf{ITL_F^{nest}}$ that is sound and complete for the intuitionistic linear temporal logic $\mathsf{ITL_F}$. To ensure completeness, the sequent calculus incorporates a simple form of nesting that enables formulas to be interpreted at different temporal positions. A nested formula, denoted by ϕ^n, is a tuple (ϕ, n) with ϕ a formula and $n < \omega$, whose interpretation is given by $\mathcal{I}(\phi^n) = \mathcal{I}(\mathsf{X}^n\phi)$, i.e., ϕ preceded by n-many Xs. The interpretation of a sequent $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite sets of nested formulas, is then

$$\mathcal{I}(\Gamma \Rightarrow \Delta) = \bigwedge_{\alpha \in \Gamma} \mathcal{I}(\alpha) \to \bigvee_{\beta \in \Delta} \mathcal{I}(\beta).$$

The sequent calculus $\mathsf{ITL_F^{nest}}$ contains the standard axioms and rules for multisuccedent intuitionistic propositional logic (see e.g. [7]), with the restriction that the right implication rule ($\to$R) may only be applied to formulas of nesting level 0; this restriction is necessary as unrestricted use of the $\to$R-rule enables one to prove $(\mathsf{X}\phi \to \mathsf{X}\psi) \to \mathsf{X}(\phi \to \psi)$, which is only valid on dynamic models that satisfy backward confluence.[2] In addition, $\mathsf{ITL_F^{nest}}$ contains the following modal rules:

$$\frac{\Gamma, \phi^n \Rightarrow \Delta \quad \Gamma, \mathsf{XF}\phi^n \Rightarrow \Delta}{\Gamma, \mathsf{F}\phi^n \Rightarrow \Delta}\ (\mathsf{FL}) \qquad \frac{\Gamma, \phi^{n+1} \Rightarrow \Delta}{\Gamma, \mathsf{X}\phi^n \Rightarrow \Delta}\ (\mathsf{XL}) \qquad \frac{\Gamma \Rightarrow \Delta}{\Sigma, \Gamma^{+1} \Rightarrow \Delta^{+1}, \Pi}\ (\mathrm{S})$$

$$\frac{\Gamma \Rightarrow \phi^n, \mathsf{XF}\phi^n, \Delta}{\Gamma \Rightarrow \mathsf{F}\phi^n, \Delta}\ (\mathsf{FR}) \qquad \frac{\Gamma \Rightarrow \phi^{n+1}, \Delta}{\Gamma \Rightarrow \mathsf{X}\phi^n, \Delta}\ (\mathsf{XR})$$

Here we define $\Gamma^{+1} = \{\phi^{n+1} : \phi^n \in \Gamma\}$. Note that the rules for F capture the equivalence $\mathsf{F}\phi \equiv \phi \vee \mathsf{XF}\phi$ of this fixed point operator, and that the S-rule ('shift') captures modal necessitation.

A *proof* of ϕ^n in $\mathsf{ITL_F^{nest}}$ is a labelled tree that is built according to the inference rules such that the root is labelled by $\Rightarrow \phi^n$, every leaf is labelled by an axiom and every infinite branch contains a formula trace that passes through the FL-rule infinitely often. The latter requirement captures that the interpretation of F is a least fixed point, thereby ensuring soundness.

Theorem. A nested $\mathsf{ITL_F}$ formula ϕ^n is provable in $\mathsf{ITL_F^{nest}}$ if and only if $X^n\phi$ is valid.

[2] A dynamic model $(W, \leq, V, f)$ satisfies backward confluence if whenever $t \geq f(s)$, there exists a $u \geq s$ with $f(u) = t$.

The proof of soundness proceeds with a standard argument on *signatures*, functions that map occurrences of eventually operators in formulas to natural numbers. A similar technique is used in [1]. The challenging part is the completeness proof, of which we give a sketch below.

Completeness We employ a game-theoretic argument similar to that of Niwiński and Walukiewicz [8]. Given a sequent σ we construct a two-player game, played by Prover (Prov) and Refuter (Ref), such that a winning strategy for Prov corresponds to the existence of a proof of σ and a winning strategy for Ref to the existence of a countermodel for σ. Martin's determinacy theorem [6] then implies that every valid formula must be provable.

The game is played on a *proof search tree*, which represents a search through possible derivations of σ in $\mathsf{ITL}_\mathsf{F}^{\mathsf{nest}}$. In the proof search tree, the two non-invertible rules $\to$R and S are replaced by the following C-rule, which represents the choices Prov can make when applying a non-invertible rule:

$$\frac{\Sigma, \Gamma^{+1}, \phi_0^0 \Rightarrow \psi_0^0 \quad \Sigma, \Gamma^{+1}, \phi_1^0 \Rightarrow \psi_1^0 \quad \cdots \quad \Sigma, \Gamma^{+1}, \phi_k^0 \Rightarrow \psi_k^0 \quad \Gamma \Rightarrow \Delta}{\Sigma, \Gamma^{+1} \Rightarrow (\phi_0 \to \psi_0)^0, (\phi_1 \to \psi_1)^0, \ldots, (\phi_k \to \psi_k)^0, \Delta^{+1}, \Pi} \text{ (C)}$$

The C-rule will only be applied to a sequent that is *saturated*, meaning that all invertible rules have been applied to a sufficient degree.

In the proof search tree, invertible rules represent choices for Ref. A winning strategy for Ref, also referred to as a *refutation*, is then a subtree of the proof search tree in which all branching is due to the C-rule. From such a refutation, we construct a countermodel for σ by treating the rightmost premise of C as temporal successor and all other premises as intuitionistic successors. The fact that the C-rule may only be applied to saturated sequents ensures that the intuitionistic order may be extended as to satisfy forward confluence without breaking monotonicity of the valuation. The nesting is crucial here, as it ensures that saturated sequents already contain the relevant information about future time steps.

Outlook As we are aware, the present work constitutes the first completeness proof for an intuitionistic fixed point logic via proof search. We expect that our approach can be extended to ITL_F interpreted on dynamic models that satisfy backward confluence, and to the intuitionistic fragment of LTL containing the next and until operator. Further, although the ill-founded system $\mathsf{ITL}_\mathsf{F}^{\mathsf{nest}}$ is analytic, in the sense that premises only contain formulas in the Fischer-Ladner closure of the conclusion, the nesting levels in a proof can be arbitrarily large. With the aim of obtaining a cyclic system, we are investigating whether a bound can be imposed on the nesting level.

References

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